Integrable Hamiltonian Systems: Problems 5

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The problems marked with $*$ are due at the beginning of the class on Tuesday 6 November.

Problem 5.1*. **(10 points)** The Kepler Hamiltonian is

$$H = \frac{1}{2}\|\mathbf{p}\|^2 - \frac{\nu}{\|\mathbf{q}\|}, \qquad \mathbf{q} \in \mathbb{R}^3 \setminus \{0\}, \quad \mathbf{p} \in \mathbb{R}^3, \quad \nu > 0. \tag{1}$$

(i) Write down the Hamiltonian equations and derive the Kepler's equations

$$\ddot{\mathbf{q}} = -\nu \frac{\mathbf{q}}{\|\mathbf{q}\|^3}. \qquad \textbf{(2 points)}. \tag{2}$$

(ii) Let $(\mathbf{q}_0, \mathbf{p}_0) \in \mathbb{R}^3 \setminus \{\mathbf{0}\} \times \mathbb{R}^3$ be an initial condition of the Hamiltonian equations and let $\pi = \mathrm{span}(\mathbf{q}_0, \mathbf{p}_0)$ in $\mathbb{R}^3$. Show that the unique integral curve $(\mathbf{q}(t), \mathbf{p}(t))$ through $(\mathbf{q}_0, \mathbf{p}_0)$ remains in $\pi \setminus \{\mathbf{0}\} \times \pi$. **(2 points)**

(iii) The angular momentum is the bivector $\mathbf{C} = \mathbf{q} \wedge \mathbf{p} \in \Lambda^2(\mathbb{R}^3)$. Verify that it is constant along the motion, that is $\dot{\mathbf{C}} = \mathbf{0}$. **(1 points)**

(iv) The linear map $\iota_{\mathbf{p}} : \Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$ denotes the contraction by $\mathbf{p} \in \mathbb{R}^3$. Applying it to the bivector $\mathbf{C} \in \Lambda^2(\mathbb{R}^3)$ yields

$$\iota_{\mathbf{p}}\mathbf{C} = \iota_{\mathbf{p}}(\mathbf{q} \wedge \mathbf{p}) = \iota_{\mathbf{p}}(\mathbf{q} \otimes \mathbf{p} - \mathbf{p} \otimes \mathbf{q}) = (\mathbf{q} \cdot \mathbf{p})\mathbf{p} - \|\mathbf{p}\|^2\mathbf{q}.$$

In particular the *Runge-Lenz vector* $\mathbf{L} = \iota_{\mathbf{p}}\mathbf{C} + \frac{\nu\mathbf{q}}{\|\mathbf{q}\|}$ is a vector of $\mathbb{R}^3$. Show that $\dot{\mathbf{L}} = \mathbf{0}$ and that it satisfies the relation

$$\|\mathbf{L}\|^2 = 2H\mathbf{C}^2 + \nu^2$$

where $\mathbf{C}^2 = \|\mathbf{q}\|^2\|\mathbf{p}\|^2 - (\mathbf{q} \cdot \mathbf{p})^2$. **(4 points)**

(v) Let $(\mathbf{q}_0, \mathbf{p}_0)$ and π as in *(ii)*. Let $(\mathbf{q}, \mathbf{p})$ be the unique integral curve through $(\mathbf{q}_0, \mathbf{p}_0)$. Argue that $\mathbf{L}$ remains in π along this solution. **(1 points)**

Problem 5.2*. **(10 points)** In celestial mechanics the motion of two massive bodies relative to an inertial frame of reference and subject to the mutual forces of gravitation is governed by the Hamiltonian function

$$H = \frac{\|\mathbf{p}_1\|^2}{2m_1} + \frac{\|\mathbf{p}_2\|^2}{2m_2} - \frac{Gm_1m_2}{\|\mathbf{q}_1 - \mathbf{q}_2\|}, \qquad \mathbf{q}_1 \neq \mathbf{q}_2.$$

where G is the universal gravitational constant, m_1, m_2 are the masses of the bodies, $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^3$ are their vector positions, and $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$ are their respective momenta.

(i) Write down the Hamiltonian equations. **(2 points)**

(ii) Verify that the vector position of the center of mass

$$\mathbf{Q} = \frac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2}{m_1 + m_2}$$

has zero acceleration. **(2 points)**

(iii) Show that the vector of the relative positions $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ satisfies Kepler's equations (2) with $\nu = G(m_1 + m_2)$. **(3 points)**

(iv) Show that in the center of mass moving frame with $\mathbf{Q} = \mathbf{0}$ the motions of the bodies can be written in terms of $\mathbf{q}$:

$$\mathbf{q}_1 = \frac{m_2}{m_1 + m_2}\mathbf{q} \qquad \text{and} \qquad \mathbf{q}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{q}. \qquad \textbf{(3 points)}$$