Integrable Hamiltonian Systems: Problems 3

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Universiteit Antwerpen – Autumn 2018
(Dated: October 16, 2018)

The problems marked with $*$ are due at the beginning of the class on Tuesday 23 October.

Problem 3.1. Show that any symplectic manifold is a Poisson manifold.

Problem 3.2. Find the action-angles coordinates for the Hamiltonian function

$$H(q,p) = \frac{1}{2}(p^2 + \lambda q^2) \quad (q,p) \in \mathbb{R} \times \mathbb{R}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

Problem 3.3*. **(20 points)** The motion of n point vortices of strength $\gamma_1, \ldots, \gamma_n$ on the plane with positions $\mathbf{r}_i = (x_i, y_i)$ is governed by the Hamiltonian function

$$H = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \gamma_i \gamma_j \ln |\mathbf{r}_i - \mathbf{r}_j| \qquad \mathbf{r}_i \neq \mathbf{r}_j.$$

(i) Write down the Hamiltonian equations $\dot{x}_k = \{x_k, H\}$ and $\dot{y}_k = \{y_k, H\}$ with respect to the Lie-Poisson bracket

$$\{F,G\} = \sum_{j=1}^{n} \frac{1}{\gamma_j} \left(\frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j} \frac{\partial G}{\partial x_j} \right). \qquad \textbf{(7 points)}$$

(ii) Show that $F_1 = \sum_{i=1}^{n} \gamma_i y_i$ and $F_2 = -\sum_{i=1}^{n} \gamma_i x_i$ are two integral of motions. What is their physical meaning? Give the brackets $\{F_1, H\}, \{F_2, H\}$ and $\{F_1, F_2\}$. **(6 points)**

(iii) Show that for a system of 2 point vortices the vector field X_H is Liouville integrable. Solve the equations of motion in this case. **(7 points)**

Problem 3.4*. **(5 points)**
On $\mathbb{R}^2$ with the coordinates (x,y) we consider the bracket

$$\{F,G\} = y \left(\frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial G}{\partial x} \right) \qquad F, G \in C^\infty(\mathbb{R}^2).$$

Verify that it defines a Lie-Poisson bracket. Show that this bracket does not define a symplectic structure on $\mathbb{R}^2$.