Integrable Hamiltonian Systems: Problems 5

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*The problems marked with $*$ are due on **Tuesday, November** 5 at the beginning of the class.*

Problem 5.1. Recall from Problem sheet 2 that

$$TS^2 = \left\{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|q\|^2 = 1,\ q \cdot p = 0\right\}.$$

The Hamiltonian of the spherical pendulum $H : TS^2 \to \mathbb{R}$ is given by $H(q,p) = \frac{1}{2}\|p\|^2 - \Gamma \cdot q$ with $\Gamma = (0,0,1)$.

(i) Show that the restriction to TS^2 of the standard symplectic form on $\mathbb{R}^3 \times \mathbb{R}^3$ is non-degenerate.

(ii) Verify that the Hamiltonian equations are given by

$$\begin{cases} \dot{q} &= p \\ \dot{p} &= \Gamma - \left((q \cdot \Gamma) + \|p\|^2\right) q \end{cases}$$

(iii) Let $R_t \in SO(3)$ be the matrix of rotation about the z-axis in $\mathbb{R}^3$ with angle t. Show that the map $\varphi_t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ given by $\varphi_t(q,p) = (R_t q, R_t p)$ leaves TS^2 invariant and that its restriction to TS^2 is a symplectomorphism for every t. Verify that φ is the flow of the Hamiltonian vector field X^K whose Hamiltonian function is $K(q,p) = (q \wedge p) \cdot \Gamma$.

(iv) Show that $F = (H,K) : TS^2 \to \mathbb{R}^2$ is a completely integrable system and find all the critical points of F. Show that the boundary of the image $F(TS^2) \subset \mathbb{R}^2$ is parametrised by the (piecewise smooth) curve

$$\gamma(\lambda) = \left(\frac{\lambda^4 - 3}{2\lambda^2},\ \lambda - \frac{1}{\lambda^3}\right), \quad \lambda^2 \geq 1.$$

(v) Describe the level sets of F.

Problem 5.2*. **(20 points)** The motion of n point vortices of strength $\gamma_1, \dots, \gamma_n$ on the plane with positions $\mathbf{r}_i = (x_i, y_i)$ is governed by the Hamiltonian function

$$H = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \gamma_i \gamma_j \ln |\mathbf{r}_i - \mathbf{r}_j| \qquad \mathbf{r}_i \neq \mathbf{r}_j.$$

(i) Write down the Hamiltonian equations $\dot{x}_k = \{x_k, H\}$ and $\dot{y}_k = \{y_k, H\}$ with respect to the Lie-Poisson bracket

$$\{F, G\} = \sum_{j=1}^{n} \frac{1}{\gamma_j} \left(\frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j} \frac{\partial G}{\partial x_j} \right). \qquad \textbf{(7 points)}$$

(ii) Show that $F_1 = \sum_{i=1}^{n} \gamma_i y_i$ and $F_2 = -\sum_{i=1}^{n} \gamma_i x_i$ are two integral of motions. What is their physical meaning? Give the brackets $\{F_1, H\}, \{F_2, H\}$ and $\{F_1, F_2\}$. **(6 points)**

(iii) Show that for a system of 2 point vortices the vector field X_H is Liouville integrable. Solve the equations of motion in this case. **(7 points)**

Problem 5.3*. **(5 points)** On $\mathbb{R}^2$ with the coordinates (x, y) we consider the bracket

$$\{F, G\} = y \left(\frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial G}{\partial x} \right) \qquad F, G \in C^{\infty}(\mathbb{R}^2).$$

Verify that it defines a Lie-Poisson bracket. Show that this bracket does not define a symplectic structure on $\mathbb{R}^2$.