Floer homology for homoclinic tangles

Von der Fakultät für Mathematik und Informatik
der Universität Leipzig
angenommene

DISSERTATION

zur Erlangung des akademischen Grades

DOKTOR RERUM NATURALIUM
(Dr. rer. nat.)

im Fachgebiet Mathematik

vorgelegt

von Dipl. Math. Sonja Hohloch

geboren am 6.5.1978 in Freiburg i. Br.

Die Annahme der Dissertation haben empfohlen:

(1) Prof. Dr. Matthias Schwarz, Universität Leipzig
(2) Prof. Dr. Alberto Abbondandolo, Universität Pisa
(3) Prof. Dr. Karl-Friedrich Siburg, Universität Dortmund

Die Verleihung des akademischen Grades erfolgt auf Beschluß des Rates der Fakultät für Mathematik und Informatik vom 18.2.2008 mit dem Gesamtprädikat *magna cum laude*.

Meinen Eltern

(To my parents)

Summary

In this work we will point out a relation between two important topics of symplectic dynamical systems — homoclinic points and Lagrangian Floer homology. Based on this we will construct a new symplectic invariant for homoclinic tangles:

Primary homoclinic Floer homology

Let (M, ω) be $(\mathbb{R}^2, dx \wedge dy)$ or a symplectic closed two-dimensional manifold with genus $g \geq 1$ and let φ be a symplectomorphism with hyperbolic fixed point x. For symplectomorphisms the (un)stable manifolds $L_0 := W^u(x, \varphi)$ and $L_1 := W^s(x, \varphi)$ are *Lagrangian* submanifolds. Thus the set of *homoclinic* points $\mathcal{H} := L_0 \pitchfork L_1$ can be seen as the intersection set associated to the *noncompact Lagrangian intersection problem* (L_0, L_1). This motivates the construction of (Lagrangian) Floer homology for homoclinic tangles.

There is a $\mathbb{Z}$-action on $\mathcal{H}$. For transversely intersecting $L_0 \pitchfork L_1$ the set $\mathcal{H}/\mathbb{Z}$ is still infinite. This prevents the well-definedness of the usual Floer differential on $\mathcal{H}$. Moreover the action filtration admits neither finite sup- nor finite sublevel sets (mod $\mathbb{Z}$).

Nevertheless there is a natural subset of $\mathcal{H}$ on which the Floer differential is well-defined. Denote by $[p, q]_i$ the segment between p and q in L_i for $i \in \{0, 1\}$. We call p contractible if the loop $[p, x]_0 \cup [p, x]_1$ is contractible and denote by $\mathcal{H}_{[x]} \subset \mathcal{H}$ the set of contractible homoclinic points. Then

$$\mathcal{H}_{pr} := \{p \in \mathcal{H}_{[x]} \mid]p, x[_0 \cap\,]p, x[_1 \cap \mathcal{H}_{[x]} = \emptyset\}$$

is the set of *primary* homoclinic points. $\tilde{\mathcal{H}}_{pr} := \mathcal{H}/\mathbb{Z}$ is finite and we denote the equivalence class of $p \in \mathcal{H}_{pr}$ by $\langle p \rangle$. The Maslov index μ induces a grading on $\tilde{\mathcal{H}}_{pr}$ and we define, analogously to classical Lagrangian Floer homology,

$$C_k := C_k(x, \varphi) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p \rangle) = k}} \mathbb{Z}\langle p \rangle,$$

$$\partial \langle p \rangle := \sum_{\substack{\langle q \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q \rangle) = \mu(\langle p \rangle) - 1}} m(\langle p \rangle, \langle q \rangle) \langle q \rangle,$$

$$H_* := H_*(x, \varphi) := \frac{\ker \partial}{\operatorname{Im} \partial}.$$

The well-definedness of ∂ and the proof of $\partial \circ \partial = 0$ are tricky combinations of dynamical and combinatorial arguments.

H_* is invariant under so called *contractibly strongly intersecting (symplectic) isotopies.* The proof has to combine analytical and combinatorial arguments. Note that a primary point $p \in \mathcal{H}_{pr}$ might vanish (analogously arise) in two ways:

- p vanishes as intersection point,
- p persists as intersection point, but is no longer primary.

The invariance implies an existence and bifurcation criterion for homoclinic points and the fixed point. In the two-dimensional situation H_* also can be defined for nonsymplectic diffeomorphisms, but there is no natural invariance. Thus H_* is an symplectic invariant.

H_* is invariant under conjugacy. Moreover we compare $H_*(x, \varphi)$ and $H_*(x, \varphi^n)$. *Chaotic primary homoclinic Floer homology* takes also the chaos near a homoclinic tangle into account and gives rise to a symplectic zeta function. Moreover we define the action spectrum and action filtration of primary homoclinic Floer homology and investigate their properties.
Then we analyse the problems which prevent differential graded algebras or $\mathcal{A}_\infty$-structures based on (primary) homoclinic points.

Finally we sketch a stronger invariance theorem and applications to Birkhoff invariants. Moreover we briefly discuss the problems arising on higher dimensional manifolds.

H_* is the first invariant which takes the *algebraic* interaction of homoclinic points into account. Moreover H_* simultanously is a *semi-global and semi-local* invariant: On the one hand the branches and homoclinic points can lie anywhere on the manifold, but on the other hand we are bound to contractible points. Thus the topology of the manifold enters only indirectly: If $H_*(x, \varphi) = 0$ then either L_0 and L_1 do not intersect or there are no *contractible* homoclinic points. There is obviously no direct way to relate $H_*(x, \varphi)$ to the topology of M or L_0 and L_1.

Acknowledgements

I wish to thank my supervisor Prof. Matthias Schwarz for his support and encouragement during the progress of this work. He interested me in the fascinating topic of homoclinic points. I am much obliged to Gregor Noetzel for many helpful discussions and proofreading. I thank Prof. Zhihong Jeff Xia for explanations and references and Prof. Edi Zehnder for references. Moreover I am indebted to Peter Albers and Felix Schlenk for explanations in the very beginning of this work.

I wish to thank my parents, my sister and my friends and collegues of the Graduiertenkolleg 'Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften' for the moral support and friendly atmosphere during the last years.

The author was supported by the Graduiertenkolleg 'Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften' and by the DFG-grant SCHW892/2-2.

Contents

CHAPTER 1

Introduction

We will give a brief overview over the history of homoclinic points and Lagrangian Floer homology. Then we will turn to the topic of this thesis and introduce and present Floer homology for homoclinic tangles.

1. Homoclinic points

Given a diffeomorphism φ with hyperbolic fixed point x we call the intersection points of the stable and unstable manifold of x *homoclinic points (of x)*. Homoclinic points approach the fixed point in backward and forward iteration. Analogously we define homoclinic solutions for flows. For integrable systems like

$$\dot{q} = p \qquad \text{and} \qquad \dot{p} = q - q^2$$

induced by the Hamiltonian $H(q,p) := \frac{1}{2}p^2 - \frac{1}{2}q^2 + \frac{1}{3}q^3$ the existence of homoclinic solutions γ with $\lim_{t\to\pm\infty}\gamma(t) = (0,0)$ was well known for a long time. The associated phase portrait is sketched in figure 1.1 (a). Here whole halfspaces of the unstable and stable manifold coincide.
Poincaré's [**Po1, Po2**] important discovery was the existence of systems where the stable and unstable manifold intersect, but do not coincide. He studied the n-body problem when he noticed around 1889 the existence of motions which

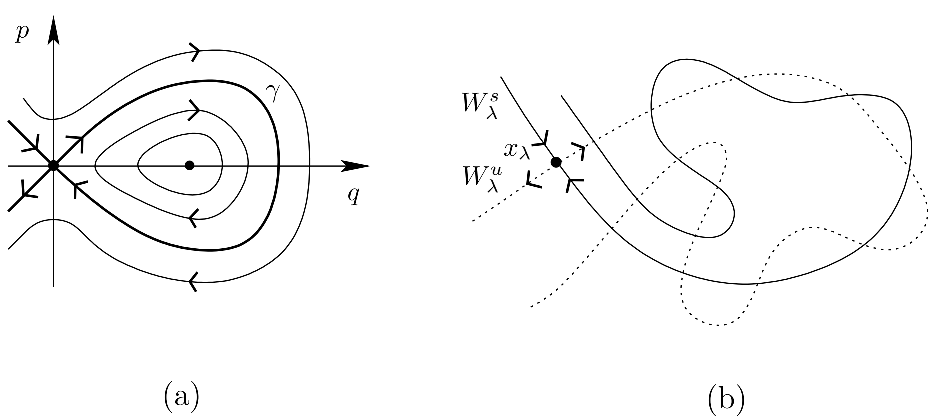


FIGURE 1.1. Homoclinic solutions

could not be presented by trigonometric series due to their lack of convergence. This phenomenon arises if we perturb the above integrable system slightly by a 1-periodic time dependent term:

$$\dot{q} = p \qquad \text{and} \qquad \dot{p} = q - q^2 + \lambda \sin(2\pi t)$$

is induced by the Hamiltonian $H_\lambda(t, q, p) := \frac{1}{2}p^2 - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \lambda \sin(2\pi t)q$ with $\lambda > 0$ sufficiently small. For $\lambda = 0$ we obtain the integrable system. Denote the time dependent flow of the perturbed system by φ_t^λ and its time-1 map by φ_1^λ. The latter has a hyperbolic fixed point x_λ near $(0, 0)$ and the stable and unstable manifolds $W^s := W^s_\lambda(x_\lambda, \varphi_1^\lambda)$ and $W^u := W^u_\lambda(x_\lambda, \varphi_1^\lambda)$ intersect, but do not coincide as sketched in figure 1.1 (b).
The set $W^s \cap W^u$ carries a $\mathbb{Z}$-action induced by iterating φ_1^λ. If we try to sketch higher iterates of figure 1.1 (b) we obtain a rather complicated picture, compare figure 5.2: Since the system is volume preserving the contraction and expansion near the hyperbolic fixed point forces the loops to become thinner and longer, accumulate on itself and intersect each other. This picture is called the **homoclinic tangle** of W^s and W^u.

In 1935 the next important result about homoclinic solutions was announced by Birkhoff [**Bi**] who proved that near a homoclinic orbit there is an intricate amount of (mostly high)periodic orbits.

In 1963 the dynamical structure of a homoclinic tangle was formally described by Smale [**Sm1, Sm2**]. He devised his famous horseshoe which relates the dynamics of the homoclinic tangle to the dynamics of the shift operator on the space of bi-infinite sequences of two symbols.

After this break-through (un)stable manifolds and homoclinic points were studied under genericity and stability aspects by Smale, Kupka, Robinson, Palis, Takens and others (see for instance [**Ku**], [**Ro**], [**Pa**], [**Ta**]). Smale and Kupka showed that generically (un)stable manifolds intersect transversely. Moreover Smale showed that the existence of homoclinic points not necessary prevents stability of the map.
In 1972 Takens proved for volume preserving maps on compact two-dimensional manifolds with the C^1-topology that a hyperbolic fixed point generically has a homoclinic point. The generalization to higher dimensions took until 1996 by Xia [**Xia1**]. On closed surfaces and with C^r-topology, $r > 1$, it was proven by Oliveira [**Ol**] under certain conditions on the symplectomorphism in 2000. In 2006 Xia [**Xia3**] proved it for symplectomorphisms isotopic to the identity on closed surfaces.
In the 1970's *paths* of diffeomorphism and the homoclinic bifurcation behaviour came into focus with important works by Palis, Newhouse, Takens and others

(see for instance [**Ne1**], [**NePT**], [**PaT1**], [**PaT2**]), but many questions are still open.

There are good survey articles by Newhouse [**Ne2**] and Shil'nikov [**Sh**] covering this period. Many results have been proven independently in the western and Russian mathematical community.
In 1973 Zehnder [**Ze**] proved the generic existence of homoclinic points in the KAM-picture around an elliptic fixed point (see Arnold & Avez [**ArA**], Moser [**Mo**]).

In 1963 Melnikov [**Me**] deduced by means of perturbation theory a criterion which controlled the existence of homoclinic points of slightly time dependent systems arising from homoclinic loops of time independent systems. This approach was extended and improved by many mathematicians and physicists, see for instance Kirchgraber & Stoffer [**KiS**], Kuznetsov & Zaslavsky [**KZ**], Zaslavsky [**Za**], Rom-Kedar [**RK1, RK2**], Wiggins [**W**] and others.
Sometimes the splitting of the homoclinic loop of the time independent system is to small to be detected by Melnikov's method. For certain of those cases Lazutkin [**Laz**] and Gelfreich [**Ge1**] presented an invariant which has been extended by Lazutkin, Gelfreich and Simo and others (see [**Ge2**], [**GeL**], [**GeS**]).

Since the break-though by Rabinowitz [**Ra2**] the calculus of variations is a strong tool for existence results of homoclinic points. Usually under certain convexity assumtions on the Hamiltonian Rabinowitz, Bolotin, Coti Zelati, Ekeland, Séré and others (see for instance [**Ra1**], [**CZES**], [**Sér**]) deduce many important results. Cieliebak & Séré [**CiS**] combined variational technics and pseudo-holomorphic curves. Lisi [**Li**] generalized [**CZES**] using Lagrangian embedding technics. And finally Ambrosetti & Badiale [**AmB**] devised an approach which encompasses the Melnikov method and the calculus of variations.

There is also an combinatorial and numerical approach to homoclinic tangles. Easton [**E**] devised a 'structure index' as invariant of a homoclinic tangle. Rotation numbers of homoclinic points appear in Hocket & Holmes [**HH**]. Rom-Kedar [**RK1, RK2**] combined combinatorial and approximation methods in order to deduce the 'topological approximation method' which analyses mixing and transport in a homoclinic tangle under iteration. Contopoulos & Polymilis [**CP**] investigated the long-time behaviour of homoclinic points using high-precision computer programs.

2. Lagrangian Floer homology

In the 1960s Arnold [**Ar1**] announced several important conjectures concerning symplectic topology and Hamiltonian diffeomorphisms (time-1 maps of a time dependent Hamiltonian flows). He claimed that on compact symplectic manifolds

(M, ω) the number of fixed points of a nondegenerate Hamiltonian diffeomorphisms $\varphi : M \to M$ is greater or equal to the sum of the Betti numbers:

$$\#\,\mathrm{Fix}(\varphi) \geq \sum_k \mathrm{rk}\, H_k(M).$$

Since the diagonal $\triangle$ and the graph of φ are Lagrangian submanifolds of $(M \times M, \omega \oplus (-\omega))$ the above question can be transformed into the *Lagrangian intersection problem*

$$\#(\triangle \pitchfork \mathrm{graph}\,\varphi) \geq \sum_k \mathrm{rk}\, H_k(M). \tag{1.1}$$

$\pitchfork$ stands for 'transversely intersecting'. Floer [**Fl1, Fl2, Fl3**] proved the conjecture in case $\pi_2(M) = 0$ using the 'L^2 gradient flow' of the action functional. The general case has been proven successively by several authors, compare for example [**Sa**], [$\mathbf{FO_3}$].
We briefly sketch Floer's approach. Consider a compact Lagrangian submanifold $L \subset M$ with $\pi_2(M, L) = 0$ and set $L_0 := L$ and $L_1 := \varphi(L)$. Floer defined a relative Maslov index $\mu(p, q) = \mu(p) - \mu(q)$ for p, $q \in L_0 \pitchfork L_1$. Using an ω-compatible almost complex structure J he defined $\mathcal{M}(p, q) := \mathcal{M}(p, q, L_0, L_1, J)$ to be the space of maps $u : \mathbb{R} \times [0, 1] \to M$ such that

$$\begin{cases} \partial_s u + J\partial_t u = 0 \quad \text{'}J\text{-holomorphic'}, \\ u(s, 0) \in L_0,\ u(s, 1) \in L_1 \text{ for all } s \in \mathbb{R}, \\ \lim\limits_{s\to-\infty} u(s, \cdot) = p, \quad \lim\limits_{s\to+\infty} u(s, \cdot) = q. \end{cases} \tag{1.2}$$

For generic J the space $\mathcal{M}(p, q)$ is a smooth manifold of dimension $\mu(p) - \mu(q)$. It carries the $\mathbb{R}$-action $(\sigma.u)(s, t) := u(s + \sigma, t)$ such that $\widehat{\mathcal{M}}(p, q) := \mathcal{M}(p, q)/\mathbb{R}$ is of dimension $\mu(p) - \mu(q) - 1$. For $\mu(q) = \mu(p) - 1$ holds $\#\widehat{\mathcal{M}}(p, q) < \infty$ since $\widehat{\mathcal{M}}(p, q)$ is compact. Thus $m_2(p, q) := \#\widehat{\mathcal{M}}(p, q) \bmod 2$ is well-defined. For $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$-coefficients Floer defined a chain complex via

$$\begin{aligned} C_k(L_0, L_1) &:= \bigoplus_{\substack{p \in L_0 \pitchfork L_1 \\ \mu(p) = k}} \mathbb{Z}_2 p, \\ \mathfrak{d} : C_* \to C_{*-1}, \quad \mathfrak{d}(p) &:= \sum_{\substack{q \in L_0 \pitchfork L_1 \\ \mu(q) = \mu(p) - 1}} m_2(p, q) q \end{aligned} \tag{1.3}$$

and proved $\mathfrak{d} \circ \mathfrak{d} = 0$ using the so called 'gluing' and 'breaking' constructions. The homology

$$FH_*(L_0, L_1, J) := \frac{\ker \mathfrak{d}}{\mathrm{Im}\, \mathfrak{d}}$$

is called **Lagrangian Floer homology** and Floer proved independence of the chosen J and invariance under Hamiltonian deformations. Thus $FH_*(L_0, L_1, J) = FH_*(L)$ which he could identify with the Morse homology and thus the singular homology of L.

Fukaya & Oh & Ohta & Ono [$\mathbf{FO_3}$] devised an obstruction theory for Lagrangian Floer homology if L_1 is not a Hamiltonian deformation of L_0.

Floer's proof consists of nontrivial Fredholm analysis and elliptic regularity theory. Nevertheless for $\dim M = 2$ there is an equivalent combinatorial approach by de Silva [**dS**], Gautschi & Robbin & Salamon [**GauRS**] and Robbin [**R**]. For two transverse embedded noncontractible nonisotopic closed curves L_0 and L_1 they find

$$\dim HF(L_0, L_1) = \min\{\#(L_0 \pitchfork L_1') \mid L_1' \text{ embedded and isotopic to } L_1\}.$$

Closely related to the combinatorics of two-dimensional Lagrangian Floer homology are differential graded algebras (DGAs) of Legendrian knots, compare Chekanov [**Che**], Ng [**Ng**], Etnyre & Ng & Sabloff [**ENS**]. DGAs of Legendrian knots can be seen as application of contact homology and and symplectic field theory (SFT) devised by Eliashberg & Givental & Hofer [**EGH**].

3. Floer homology for homoclinic tangles

Let (M, ω) be a symplectic manifold with $\pi_2(M) = 0$ and consider a symplectomorphism $\varphi : M \to M$ having a hyperbolic fixed point x. Note that (un)stable manifolds of symplectomorphisms are in fact Lagrangians. Homoclinic points can be seen as intersection set of the *Lagrangian intersection problem* $W^s(x, \varphi) \cap W^u(x, \varphi)$. Thus Floer's approach to (1.1) inspires Lagrangian Floer homology for $L_0 := W^u(x, \varphi)$ and $L_1 := W^s(x, \varphi)$ which we will discuss now.

Consider transversely intersecting $\mathcal{H} := L_0 \pitchfork L_1$. Floer's definition of the Maslov index μ carries over, but apart from that there are striking differences to the classical situation:

(1) The Lagrangians L_0 and L_1 are noncompact.
(2) $L_0 \pitchfork L_1$ carries a $\mathbb{Z}$-action induced by φ.
(3) $(L_0 \pitchfork L_1)/\mathbb{Z}$ is countably infinite.

If $\dim M > 2$ there is no combinatorial approach. For given p and q the noncompactness of L_0 and L_1 turns the Fredholm analysis associated to the dimension formula and compactness discussion of $\widehat{\mathcal{M}}(p, q)$ into a nearly hopeless task. Thus we first try to understand the two-dimensional situation where we can resort to combinatorics.

From now on (M,ω) is a closed **two-dimensional** manifold with genus $g \geq 1$ or $(\mathbb{R}^2, dx \wedge dy)$. Note $\pi_2(M) = 0$ in both cases. L_0 and L_1 are one-dimensional injectively immersed submanifolds and for p, $q \in \mathcal{H}$ we define $[p,q]_i$ to be the unique (unoriented) segment between p and q in L_i. Moreover we abbreviate $p^n := \varphi^n(p)$ for $n \in \mathbb{Z}$. The loop c_p starting in p, running through $[p,x]_0$ to x and back through $[x,p]_1$ to p defines a homotopy class $[p] := [c_p] \in \pi_1(M,x)$. For sake of $[p] = [p^n]$ we restrict us from now on to the *contractible* points $\mathcal{H}_{[x]} := \{p \in \mathcal{H} \mid [p] = [x]\}$. Then we can define $\mu(p) := \mu(p,x)$ and obtain a grading $\mathcal{H}_{[x]} = \bigoplus_{n\in\mathbb{Z}} \{p \in \mathcal{H}_{[x]} \mid \mu(p) = n\}$. If we adjust the segments $[p,x]_0$ and $[p,x]_1$ to meet orthogonally in p and x then $\mu(p)$ is the sum of twice the winding number of the segments using the parametrization of c_p.

Let D, D_b, D_c be the standard 2-gons from figure 3.2. The combinatorial approach (see the previous section) allows to redefine $\mathcal{M}(p,q)$ for $\mu(p,q) = 1$ as space of smooth *immersed di-gons* $u : D \to M$ satisfying

(1) u is **orientation preserving**,
(2) $u(B_0) \subset L_0$ and $u(B_1) \subset L_1$,
(3) $u((-1,0)) = p$ and $u((1,0)) = q$.

Dividing by the group of orientation preserving diffeomorphisms of D fixing the vertices yields $\widehat{\mathcal{M}}(p,q)$. Analogously define for $\mu(p,q) = 2$ the space $\mathcal{N}(p,q)$ and $\widehat{\mathcal{N}}$ based on $u : D_b \to M$ and $u : D_c \to M$.
The huge obstacle in higher dimensions turns out to be trivial for $\dim M = 2$: We always have

$$\#\widehat{\mathcal{M}}(p,q) \in \{0,1\}.$$

We call φ L-orientation preserving if φ is orientation preserving on L_0 and L_1 and otherwise L-orientation reversing. For L-orientation preserving φ we can define $\mathbb{Z}$-valued signs $m(p,q) \in \{\pm\#\widehat{\mathcal{M}}(p,q)\}$. But as long as we deal with $\mathcal{H}_{[x]}$ there are in the L-orientation reversing case only $\mathbb{Z}_2$-signs possible.

Is now (1.3) well-defined in our setting? Set $p \sim q$ if and only if $q = p^n$ for some $n \in \mathbb{Z}$ and define $\tilde{\mathcal{H}} := \mathcal{H}/_\sim$. Let $\langle p \rangle$ denote the equivalence class of p w.r.t. $\sim$. The well-definedness of (1.3) is tied to the following questions: Does for given $p \in \mathcal{H}_{[x]}$ hold

$$(1.4) \qquad \begin{cases} \#\{q \in \mathcal{H}_{[x]} \mid m(p,q) \neq 0\} < \infty \ ? \\ \#\{n \in \mathbb{Z} \mid m(p,q^n) \neq 0\} < \infty \text{ for } q \in \mathcal{H} \ ? \\ \#\{\langle q \rangle \in \tilde{\mathcal{H}} \mid m(p,q^n) \neq 0 \text{ for some } n \in \mathbb{Z}\} < \infty \ ? \end{cases}$$

Unfortunately all three sets can be infinite: For x in figure 1.2 holds $m(x,p^n) \neq 0$ for all $n \in \mathbb{Z}$. There are also tangles with p, $q \in \mathcal{H}$ and $m(p,q^n) \neq 0$ for $n \in \mathbb{Z}^{>n_o}$ or $n \in \mathbb{Z}^{<n_0}$ for some $n_0 \in \mathbb{Z}$. And for p in figure 1.2 holds $m(p,s_n) \neq 0 \neq m(p,r_n)$ and $\langle s_n \rangle$ and $\langle r_n \rangle$ are all mutually distinct for $n \in \mathbb{N}$.

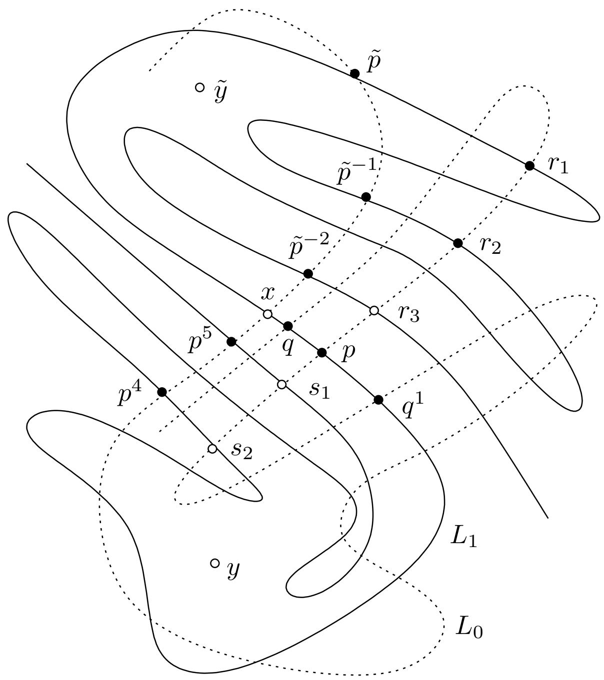


FIGURE 1.2. The differential of p is infinite mod $\mathbb{Z}$ with converging action

For a *single* $p \in \mathcal{H}$ nevertheless (1.3) makes geometrically sense: The sum might be infinite, but *geometrically* we have $\mathfrak{d}\mathfrak{d}p = 0$ due to the so called *gluing and cutting construction*: Call the two connected components of $L_i \backslash \{x\}$ the *branches* of L_i for $i \in \{0, 1\}$ and call L_0 and L_1 *strongly intersecting* if each branch of L_0 intersects each branch of L_1. This is in fact a generic property on compact manifolds. We obtain

THEOREM 1.5 (Gluing and Cutting). *(1) Let p, q, $r \in \mathcal{H}$ with $[p] = [q] = [r]$ and $\mu(p, q) = 1 = \mu(q, r)$. Let $u \in \widehat{\mathcal{M}}(p, q)$ and $v \in \widehat{\mathcal{M}}(q, r)$. Then the gluing procedure $\#$ for u and v yields an immersed heart $w := v\#u \in \widehat{\mathcal{N}}(p, r)$.*

(2) Let L_0 and L_1 be transversely and strongly intersecting. Let p, $r \in \mathcal{H}$ with $[p] = [r]$ and $\mu(p, r) = 2$ and $w \in \mathcal{N}(p, r)$. Then there are unique q_0, $q_1 \in \mathcal{H}$ with $\mu(p, q_i) = 1 = \mu(q_i, r)$ and $u_i \in \mathcal{M}(p, q_i)$, $v_i \in \mathcal{M}(q_i, r)$ such that $v_i\#u_i = w$ for $i \in \{0, 1\}$. Moreover

$$m(p, q_0) \cdot m(q_0, r) = -m(p, q_1) \cdot m(q_1, r). \tag{1.6}$$

The proof relies on the contraction and expansion property of the homoclinic tangle near x, more precisely on the so called 'λ-lemma'. Now (1.6) implies $\mathfrak{d}\mathfrak{d}p = 0$ for a single $p \in \mathcal{H}$.

Now we try to solve the dilemma (1.4). The natural idea is a *filtration* $(\mathcal{H}^c)_{c\in\mathbb{R}} \subset \mathcal{H}_{[x]}$ satisfying

(1) $\mathcal{H}^c \subset \mathcal{H}^{c'}$ for $c < c'$ and $\varphi(\mathcal{H}^c) = \mathcal{H}^c$ and $\#\tilde{\mathcal{H}}^c < \infty$.
(2) Theorem 1.5 holds within $\mathcal{H}^c$ for all c.

This turns out to be a rather tricky task. The most natural idea are the sublevel set of the (well-defined) *action functional*

$$\mathcal{A}(p) := \mathcal{A}(p, x) := \int_u \omega$$

where $p \in \mathcal{H}_{[x]}$ and $u : [0,1]^2 \to M$ is smooth with $u(0,\cdot) = p$, $u(1,\cdot) = x$, $u(s,i) \in L_i$ for all $s \in [0,1]$ and $i \in \{0,1\}$. If $\mu(p,q) = 1$ and $m(p,q) \neq 0$ then

$$\mathcal{A}(p) > \mathcal{A}(q)$$

such that sublevel sets of $\mathcal{A}$ are naturally compatible with Theorem 1.5. Unfortunately we notice for p, $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ in figure 1.2 (more detailed in Chapter 9)

$$\mathcal{A}(s_n) < -2\mathcal{A}(p) < \mathcal{A}(r_n) \quad \text{and} \quad \lim_{n\to\infty} \mathcal{A}(s_n) = -2A(p) = \lim_{n\to\infty} \mathcal{A}(r_n).$$

Thus the action filtration fails — and so does (nearly) every attempt, compare Chapter 9 !

Fortunately there is a natural subset of $\mathcal{H}$ for which the sets of (1.4) are finite and Theorem 1.5 holds: The set of *primary* points

$$\mathcal{H}_{pr} := \{p \in \mathcal{H}_{[x]} \mid]p, x[_0 \cap]p, x[_1 \cap \mathcal{H}_{[x]} = \emptyset\}$$

is finite mod $\mathbb{Z}$ since for $p \in \mathcal{H}_{pr}$ every $p \neq q \in \mathcal{H}_{pr}$ has exactly one iterate in $[p, p^1]_0 \pitchfork [p, p^1]_1$. Nonprimary points we call secondary. We define $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/_\sim$, $[\langle p\rangle] := [p]$, $\mu(\langle p\rangle, \langle q\rangle) := \mu(p,q)$ and $\mu(\langle p\rangle) := \mu(p)$ for primary p and q. For $\langle p\rangle$, $\langle q\rangle \in \tilde{\mathcal{H}}_{pr}$ set

$$m(\langle p\rangle, \langle q\rangle) := \sum_{n\in\mathbb{Z}} m(p, q^n).$$

For *primary* points it is possible to define signs which admit also in the L-orientation reversing case $\mathbb{Z}$-coefficients.

Definition 1.7. *We define*

$$C_m := C_m(x, \varphi; \mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle) = m}} \mathbb{Z}\langle p\rangle,$$

$$\partial_m : C_m \to C_{m-1}, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle) = \mu(\langle p\rangle) - 1}} m(\langle p\rangle, \langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity.

We obtain

THEOREM 1.8. *(C_*, ∂_*) is a chain complex, i.e.*

$$\partial \circ \partial = 0,$$

and the **primary homoclinic Floer homology of φ in x**

$$H_m := H_m(x, \varphi; \mathbb{Z}) := \frac{\ker \partial_m}{\operatorname{Im} \partial_{m+1}}.$$

is welldefined. Analogously primary homoclinic Floer **cohomology** *is defined.*

The the well-definedness of Definition 1.7 and the proof of Theorem 1.8 rely on Theorem 1.5 and on classifications of $\mathcal{M}(p,q)$ and $\mathcal{N}(p,q)$ for p, $q \in \mathcal{H}_{pr}$ which turn out to be embeddings on the universal cover of M. This allows to show $\#\{n \in \mathbb{Z} \mid m(p, q^n) \neq 0\} < \infty$ which implies the well-definedness of ∂. Note that L_0 and L_1 actually need not to be strongly intersecting in order to show the cutting procedure for primary points.

We compute several examples and give a rough classification for systems with exactly two primary equivalence classes in each pair of intersecting branches. Note that for primary points p the Maslov index has only values $\mu(p) \in \{\pm 1, \pm 2, \pm 3\}$.

Primary homoclinic Floer homology is on **two-**dimensional manifolds also defined for diffeomorphisms. It is invariant under conjugation of the symplectomorphism or diffeomorphism by a homeomorphisms, but the 'symplectic aspects' are only preserved under symplectic conjugacy.

Nevertheless the invariance discussion shows that primary homoclinic Floer homology is a *symplectic invariant*:

DEFINITION 1.9.

(1) *Let φ, $\psi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ both hyperbolic. An* **isotopy (between (x, φ) and (y, ψ))** *is a smooth path $\Phi : [0,1] \to \mathrm{Diff}_\omega(M)$, $\tau \mapsto \Phi(\tau) =: \Phi_\tau$ with $\Phi_0 = \varphi$, $\Phi_1 = \psi$, $x_0 = x$ and $x_1 = y$ and $x_\tau \in \mathrm{Fix}(\Phi_\tau)$ as continuation for all $\tau \in [0,1]$ between x and y. Φ is called* **Hamiltonian** *if Φ_τ is Hamiltonian for all $\tau \in [0,1]$.*

(2) *Let $\varphi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic. (x, φ) is called* **contractibly strongly intersecting (csi)** *if L_0 and L_1 are strongly intersecting and if each pair of branches has contractible homoclinic points. An isotopy Φ is csi if (x_τ, Φ_τ) is csi for all $\tau \in [0,1]$.*

Attaching τ to a symbol associates it to Φ_τ, i.e. $\mathcal{H}^\tau_{pr}$ denotes the set of primary points of Φ_τ etc.

THEOREM 1.10. *Let (M,ω) be a closed symplectic two-dimensional manifold with genus $g \geq 1$. Let φ, $\psi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$. Let (x,φ) and (y,ψ) be csi and let all primary points of φ and ψ be transverse. Assume there is a csi isotopy Φ from (x,φ) to (y,ψ). Then*

$$H_*(x,\varphi) \simeq H_*(y,\psi).$$

We will give a brief sketch of the proof of Theorem 1.10 after the next two statements. Note that in contrast to classical Floer theory invariance needs existence of certain intersection points. The proof carries over to compactly supported symplectomorphisms on $\mathbb{R}^2$.

THEOREM 1.11. *Let φ, $\psi \in \mathrm{Diff}_{dx\wedge dy}(\mathbb{R}^2)$ be compactly supported with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$. Let (x,φ) and (y,ψ) be strongly intersecting and let all primary points of φ and ψ be transverse. Assume there is a compactly supported strongly intersecting isotopy Φ from (x,φ) to (y,ψ). Then*

$$H_*(x,\varphi) \simeq H_*(y,\psi).$$

As applications we obtain from the above theorems the following existence and bifurcation criterion:

COROLLARY 1.12. *Assume the conditions of Theorem 1.10 resp. Theorem 1.11 for (M,ω), (x,φ) and (y,ψ), but $H_*(x,\varphi) \neq H_*(y,\psi)$. Then (x,φ) and (y,ψ) cannot be joint by a csi (resp. compactly supported) isotopy.*
Thus if there is a path $(\Phi_\tau)_{\tau\in[0,1]} \in \mathrm{Diff}_\omega(M)$ between φ and ψ then

(1) *either Φ is no isotopy, i.e. there is $\tau_0 \in [0,1]$ where x_{τ_0} vanishes or undergoes a bifurcation,*
(2) *or if Φ is a (compactly supported) isotopy there has to be a pair of branches and some $\tau_0 \in [0,1]$ where all contractible homoclinic points vanish, i.e. there are homoclinic bifurcations.*
(3) *or Φ is no* compactly supported *isotopy.*

In particular symplectomorphisms φ and ψ admitting homoclinic tangles like figure 5.2 resp. figure 5.3 cannot be joint by a (compactly supported) csi isotopy.

Now we sketch the proof of Theorem 1.10. The modern 'homotopy of homotopies' approach is not available since the methods in the proof of Theorem 1.8 do not carry over to the parameter dependent case. Thus we generalize and modify Floer's original idea to our setting with $\mathbb{Z}$-action, $\mathbb{Z}$-coefficients and combinatorial features:

(1) Csi is stable under small perturbations in the set of symplectomorphisms (not in $\mathrm{Diff}(M)$!). Thus generic perturbations make sense. Note that primary homoclinic Floer homology is already determined by *compact* segments centered around the fixed point. Perturb Φ such that

the bifurcations relevant for primary homoclinic Floer homology can be modeled as sequence of 'moves' similarly to knot theory. Since the (un)stable manifolds do not admit self-intersections only the 'second Reidemeister move' appears.

(2) A primary point arises (analogously vanishes) if
- it arises as intersection point,
- the intersection point was secondary and becomes primary ('primary-secondary flip').

If a move generates (analogously destroys) two points the following combinations are possible:
- both primary ('primary move'),
- one primary and one secondary ('mixed move'),
- both secondary ('secondary move').

A mixed move might also flip a certain number of primary points secondary although they do not participate in the move itself!

(3) Show invariance of primary homoclinic Floer homology under primary, mixed and secondary moves.

We believe Theorem 1.10 also true for Hamiltonian diffeomorphisms with Hamiltonian isotopies, see Chapter 9. Moreover as long as the combinatorics of the proof can be justified we have invariance in a 'combinatorial sense'.

Now we compare $H_*(x,\varphi)$ and $H_*(x,\varphi^n)$. Denote by $\langle p_1\rangle,\dots,\langle p_k\rangle$ the generators of $C_*(x,\varphi)$ and set $p_i^j := \varphi^j(p_i)$. Then $C_*(x,\varphi^n)$ is generated by $\langle p_1^0\rangle,\dots,\langle p_k^0\rangle,\langle p_1^1\rangle,\dots,\langle p_k^{n-1}\rangle$. φ induces a $\mathbb{Z}_n$-action on $C(x,\varphi^n)$ which passes to $H_*(x,\varphi^n)$. For even $n\in\mathbb{N}$ let $\mathbb{K}=\mathbb{Q}$ and for odd n let $\mathbb{K}=\mathbb{Z}_2$ and define

$$f: C_*(x,\varphi^n;\mathbb{K}) \simeq C^{-*}(x,\varphi^{-n},\mathbb{K}) \to C_*(x,\varphi;\mathbb{K}), \qquad f(\langle p_i^j\rangle) := \langle p_i\rangle,$$

$$g: C_*(x,\varphi;\mathbb{K}) \to C_*(x,\varphi^n;\mathbb{K}) \simeq C^{-*}(x,\varphi^{-n};\mathbb{K}), \qquad g(\langle p_i\rangle) := \frac{1}{n}\sum_{j=0}^{n-1}\langle p_i^j\rangle$$

which are chain maps and we compute $f\circ g = \mathrm{Id}_{C_*(x,\varphi;\mathbb{K})}$. Denote by g_* and f_* the induced maps on the (co)homology. If we use those signs which allow in the L-orientation reversing case $\mathbb{Z}$-coefficients we can replace $\mathbb{Z}_2$ by $\mathbb{Q}$.

Theorem 1.13. *Let $\varphi\in\mathrm{Diff}(M)$ and $x\in\mathrm{Fix}(\varphi)$ hyperbolic satisfy the conditions of Theorem 7.1 if φ is not symplectic.*

(1) Let φ be L-orientation preserving and $n\in\mathbb{N}_0$. Then g_* is injective and f_* surjective. Thus

$$\dim H_*(x,\varphi;\mathbb{Q}) \le \dim H_*(x,\varphi^n;\mathbb{Q}) = \dim H^{-*}(x,\varphi^{-n};\mathbb{Q})$$

and the difference is measured by the long exact sequence

$$\cdots \to H_l(\ker f;\mathbb{Q}) \to H_l(x,\varphi^n;\mathbb{Q}) \to H_l(x,\varphi;\mathbb{Q}) \to H_{l-1}(\ker f,\mathbb{Q}) \to \cdots$$

(2) *Let φ be L-orientation reversing then φ^2 is L-orientation preserving and the first item applies for φ^2 and $\varphi^{2n} = (\varphi^2)^n$. For $n \in \mathbb{N}_0$ odd g_* is injective and f_* surjective. Thus*

$$\dim H_*(x, \varphi; \mathbb{Z}_2) \leq \dim H_*(x, \varphi^n; \mathbb{Z}_2) = \dim H^{-*}(x, \varphi^{-n}; \mathbb{Z}_2)$$

and the difference is measured by the long exact sequence

$$\cdots \to H_l(\ker f; \mathbb{Z}_2) \to H_l(x, \varphi^n; \mathbb{Z}_2) \to H_l(x, \varphi; \mathbb{Z}_2) \to H_{l-1}(\ker f, \mathbb{Z}_2) \to \cdots$$

In all explicitly computed examples we obtained $H_*(x, \varphi) = H_*(x, \varphi^n)$ for all $n \in \mathbb{N}$.

Whereas the action functional $\mathcal{A}$ did not yield an accessable filtration on $\mathcal{H}$ it is clearly compatible with the definiton of primary homoclinic Floer homology. The *action spectrum* of (x, φ) is the set of its critical values

$$\Sigma_{x,\varphi} := \{\mathcal{A}(\langle p \rangle) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}(x, \varphi)\}.$$

We define the *primary radius* $r = r(L_0, L_1)$ which can be estimated by

$$\sqrt{\tfrac{2}{\pi}\mathcal{A}(p, q)} \geq r$$

for ‘adjacent’ primary points p and q. There is a filtered chain complex for $a \in \mathbb{R}$

$$C_k^a := C_k^a(x, \varphi, \mathbb{Z}) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(p) = k \\ \mathcal{A}(p) \leq a}} \mathbb{Z}\langle p \rangle$$

which gives rise for $-\infty \leq a < b < c \leq \infty$ to the long exact sequence

$$\cdots \to H_{k+1}^{]b,c]} \to H_k^{]a,b]} \xrightarrow{i_*} H_k^{]a,c]} \xrightarrow{j_*} H_k^{]b,c]} \to H_{k-1}^{]a,b]} \to \ldots$$

of **filtered primary homoclinic Floer homology groups** which is invariant under conjugation by a symplectomorphism.
Unfortunately there is no analogon of the constructions done by Schwarz [**Sch3**] and Leclercq [**Le**] who assign to a given homology class of M resp. L a section of the action spectrum bundle. Again invariance properties are diffcult to obtain since the homotopy argument is not at our disposal. Only for *Melnikov* and *Lazutkin systems* we can estimate the relative action $\mathcal{A}(p) - \mathcal{A}(q)$ for ‘adjacent’ points depending on the isotopy parameter and obtain estimates for the primary radius. Nevertheless it is possible to distinguish certain ‘invariant’ levels realized by homology classes in the filtration.

Apart from the importance of the invariance properties and their implications for existence and bifurcations of fixed and homoclinic points primary homoclinic Floer is interesting as algebraic structure:

It is the first invariant which measures the *algebraic* interaction within a homoclinic tangle. Instead of investigating the *chaos* produced by the iterations $H_*(x, \varphi)$ tries to find *order*. All other invariants or technics known to the author (see the first section of this chapter) try to analyse the iteration behaviour, predict or prevent the existence of homoclinic points or are confined to a small neighbourhood of the fixed point.
Primary homoclinic Floer homology simultanously is a *semi-global and semi-local* invariant: On the one hand the branches and homoclinic points can lie anywhere on the manifold, but on the other hand we are bound to contractible points. Thus the topology of the manifold enters only indirectly: If $H_*(x, \varphi) = 0$ then either L_0 and L_1 do not intersect or there are no *contractible* homoclinic points. There is obviously no direct way to relate $H_*(x, \varphi)$ to the topology of M or L_0 and L_1.
Moreover we can define a version of primary homoclinic Floer homology which recovers parts of the chaos close to the homoclinic tangle. *Chaotic primary homoclinic Floer homology* is best seen as sequence $n \mapsto H_*^{\mathrm{Fix}}(x, \varphi^n)$. Its chain groups coincide with them of $H_*(x, \varphi^n)$, but the boundary operator ignores all immersions which contain fixed points of φ^n in their ranges. This yields a quite unpredictable sequence which can be used in order to define a symplectic zeta function.

It is not possible to define a differential graded algebra or an A_∞-structure based on primary points since the cutting construction is not well-defined within the set of *primary* points. Within the set of homoclinic points, cutting and gluing is possible, but problems similar to those in (1.4) arise.

4. Overview

Chapter 1: We give an survey over homoclinic points and Lagrangian Floer homology. Then we motivate, state and briefly explain the results of this thesis.
Chapter 2: We recall the necessary properties of (symplectic) dynamical systems, define homotopy properties of homoclinic points and the Maslov index.
Chapter 3: We introduce the spaces $\mathcal{M}(p, q)$ and $\mathcal{N}(p, q)$, discuss their properties and prove the general cutting and gluing construction. Then we define 'coherent' signs (valid for arbitrary homoclinic points) which allow $\mathbb{Z}$- resp. $\mathbb{Z}_2$-coefficients in case of L-orientation preserving resp. reversing symplectomorphisms.
Chapter 4: We define and analyse primary homoclinic points, define primary homoclinic Floer (co)homology and prove well-definedness.
Chapter 5: We calculate some examples and give a rough classification.

Chapter 6: We prove the invariance of primary homoclinic Floer homology under certain symplectic isotopies and deduce an existence and bifurcation criterion.
Chapter 7: We define primary homoclinic Floer homology for diffeomorphisms. Then we prove invariance under topological and symplectic conjugacy. Moreover we compare $H_*(x,\varphi)$ and $H_*(x,\varphi^n)$ and define chaotic primary homoclinic Floer homology. Furthermore we introduce new signs which are only valid for *primary* points, but admit also in the L-orientation reversing case $\mathbb{Z}$-coefficients. Finally we points out why differential graded algebras and A_∞-structures cannot be defined using primary points.
Chapter 8: We discuss the action spectrum and action filtration for primary homoclinic Floer homology.
Chapter 9: We sketch a generalization of the invariance theorem and point out an application to Birkhoff invariants. Then we discuss the generalization of primary homoclinic Floer homology to nonprimary points and to higher dimensional manifolds.
Appendix A: We sketch Melnikov's perturbation method.
Appendix B: We sketch Lazutkin's approach and its generalizations.

CHAPTER 2

Foundations

In this chapter we fix some notion and recall some facts from the theory of (symplectic) dynamical systems, in particular about (un)stable manifolds. Then we give the definition of homoclinic points and define homotopy classes and a Maslov index for them.

1. Symplectic dynamical systems

Let (M, ω) be a symplectic manifold of dimension $2n$ and $H : \mathbb{R} \times M \to \mathbb{R}$ be a 1-periodic time dependent Hamiltonian function, i.e. $H(t, \cdot) = H(t+1, \cdot)$ for all $t \in \mathbb{R}$. Setting $H_t := H(t, \cdot)$ the **Hamiltonian vector field** $X_t := X(t, \cdot) := X^H(t, \cdot)$ is given by

$$\omega(X_t, \cdot) = -dH_t(\cdot)$$

which reads in local coordinates $(q, p) \in M$

$$\begin{cases} \dot{q} = & H_p(t, q, p) \\ \dot{p} = & -H_q(t, q, p). \end{cases}$$

Its **time dependent flow** is given by

$$\dot{\varphi}_t = X_t(\varphi_t) \quad \text{with} \quad \varphi_0 = \mathrm{Id}$$

and we call φ_1 the **time-1 map**. The 1-periodicity in time of H implies $X(t+1, x) = X(t, x)$ for all $x \in M$ and $t \in \mathbb{R}$ which leads to a $\mathbb{Z}$-action on the solution space given by $n.x(t) := x(t+n)$ for a solution x and $n \in \mathbb{Z}$.

Instead of considering the dynamics of the flow φ_t we can consider the discrete dynamical system induced by iterating the time-1 map φ_1. For instance, in this way a 1-periodic solution x corresponds to the fixed point $x_0 := x(0)$ of the time-1 map.

In this work we will focus on discrete dynamical systems for which we will now fix some notation.

For a diffeomorphism φ we denote by

$$\mathrm{Fix}(\varphi) := \{x \in M \mid \varphi(x) = x\}$$

its **fixed point set** and we call $x \in \mathrm{Fix}(\varphi)$ **hyperbolic** if $D\varphi(x)$ has no eigen values of modulus 1. The following well-known definition and theorem can be found for example in the appendix of Palis & Takens [**PaT2**].

Definition and Theorem 2.1 (Invariant manifold theorem). *Let φ be a C^k-diffeomorphism with $k \geq 1$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic. We call*

$$W^s := W^s(x) := W^s(x, \varphi) := \{q \in M \mid \lim_{k\to\infty} \varphi^k(q) = x\},$$
$$W^u := W^u(x) := W^u(x, \varphi) := \{q \in M \mid \lim_{k\to-\infty} \varphi^k(q) = x\}.$$

the **stable** *and* **unstable manifolds** *of φ in x. They are injectively immersed submanifolds of the same differentiability class as φ and $\dim W^s$ resp. $\dim W^s$ equals the number of eigenvalues of $D\varphi|_x$ with modulus smaller resp. larger than 1. Thus in particular $\dim W^s(x, \varphi) + \dim W^u(x, \varphi) = \dim M$.*

In order to appreciate the (un)stable manifolds from the symplectic and Hamiltonian point of view we introduce the following definitions.

If (V, ω) is a symplectic vector space and $W < V$ a subspace we define $W^\omega := \{v \in V \mid \omega(v, w) = 0 \,\forall\, w \in W\}$. W is called **isotropic** if $W < W^\omega$, i.e. $\omega|_W = 0$, and **Lagrangian** if $W = W^\omega$. This implies for the dimensions of V, W and W^ω the relations $\dim V = \dim W + \dim W^\omega$ and for W isotropic $\dim W \leq \dim W^\omega$ and for W Lagrangian $\dim W = \frac{1}{2} \dim V$.
Now consider a submanifold $L \subset (M, \omega)$ and call L **isotropic** resp. **Lagrangian** if $T_xL < T_xM$ is isotropic resp. Lagrangian for all $x \in L$. In particular the dimension of a Lagrangian submanifold equals half of the dimension of the underlying manifold.

A diffeomorphism φ of M is called **symplectic** or **symplectomorphism** if $\varphi^*\omega = \omega$. In particular note that the diffeomorphisms $\varphi_t : M \to M$ induced by the flow of a Hamiltonian system are symplectic.

We prove now that the (un)stable manifolds of a hyperbolic fixed point of a symplectomorphism are Lagrangian submanifolds.

Proposition 2.2. *Let (M^{2n}, ω) be a symplectic manifold and x a hyperbolic fixed point of a symplectomorphism φ. Then the stable and unstable manifold $W^s := W^s(x, \varphi)$ and $W^u := W^u(x, \varphi)$ are Lagrangian.*

Proof : φ is contracting resp. expanding along W^s resp. W^u, i.e. for $p \in W^s$ and $v \in T_pW^s$ holds $\lim_{n\to\infty} d\varphi^n(p).v = 0$ and thus $\lim_{n\to\infty} \omega(d\varphi^n(p).v, d\varphi^n(p).w) = 0$ for all $p \in W^s$ and v, $w \in T_pW^s$. But since $\varphi^*\omega = \omega$ we also have $\omega(d\varphi^n(p).v, d\varphi^n(p).w) = \omega(v, w)$ for all $n \in \mathbb{N}$. Therefore ω has to vanish along W^s, i.e. W^s is isotropic and $\dim T_xW^s \leq \dim(T_xW^s)^\omega$. Analogously we deduce W^u isotropic. Now suppose W^s or W^u are not Lagrangian.

Since x is hyperbolic we have $\dim W^s + \dim W^u = \dim M = 2n$ and we estimate

$$\begin{aligned} 2n &= \dim T_x W^s + \dim T_x W^u \\ &< \dim (T_x W^s)^\omega + \dim (T_x W^u)^\omega \\ &= 2n - \dim T_x W^s + 2n - \dim T_x W^u \\ &= 2n \quad \lightning \end{aligned}$$

■

2. (Un)stable manifolds and homoclinic points

Within this section M is assumed to be either $\mathbb{R}^2$ or a closed two-dimensional manifold with genus $g \geq 1$, but some of the following statements and definitions are also true in higher dimensions.

Let $\varphi : M \to M$ be a diffeomorphism with hyperbolic fixed point x and one-dimensional (un)stable manifolds $W^s(x,\varphi)$ and $W^u(x,\varphi)$. In accordance to the notation in Floer theory and Lagrangian intersection theory we abbreviate

$$L_0 := W^u(x,\varphi) \qquad \text{and} \qquad L_1 = W^s(x,\varphi).$$

The connected components of $L_i \backslash \{x\}$ are denoted by L_i^+ and L_i^- for $i \in \{0,1\}$ and are called the **branches** of the (un)stable manifolds. The set

$$\mathcal{H} := \mathcal{H}(x,\varphi) := L_0 \cap L_1$$

is called the **set of homoclinic points** of x under φ. It consists out of those points p who approach x under backward and forward iteration of φ (note that we include the fixed point in the definition of $\mathcal{H}$). φ induces a $\mathbb{Z}$-action

$$\mathbb{Z} \times \mathcal{H} \to \mathcal{H}, \quad (n,p) \mapsto \varphi^n(p)$$

which is free on $\mathcal{H} \backslash \{x\}$. For the remainder of this section fix parametrizations $\gamma_i : \mathbb{R} \to L_i$ for $i \in \{0,1\}$ as assured by Theorem 2.1. Since L_0 and L_1 are 1-dimensional we can define

NOTATION 2.3. *The parametrization $\gamma_i : \mathbb{R} \to L_i$ induces an* **ordering** $<_i$ *resp.* $\leq_i$ *on L_i via*

$$\gamma_i(t) <_i \gamma_i(s) \;\Leftrightarrow\; t < s \quad \textit{resp.} \quad \gamma_i(t) \leq_i \gamma_i(s) \;\Leftrightarrow\; t \leq s.$$

By abuse of notation we can say that p, $q \in L_i$ induce an ordering on L_i via setting $p <_i q$ resp. $p \leq_i q$.

For $i \in \{0,1\}$ consider p, $q \in L_i$ and set $t_i^p = \gamma_i^{-1}(p)$, $t_i^q := \gamma_i^{-1}(q)$, $t_i^- := \min\{t_i^p, t_i^q\}$ and $t_i^+ := \max\{t_i^p, t_i^q\}$. We call

$$[p,q]_0 := \gamma_0([t_0^-, t_0^+]) \quad \text{resp.} \quad [p,q]_1 := \gamma_1([t_1^-, t_1^+])$$

the **segments in L_0 resp. L_1 between p and q**. The segments are independent of the chosen parametrization and a priori just sets of points without additional information like parametrization, orientation etc. Thus $[p,q]_i = [q,p]_i$. Analogously we define the open and half-open segments $]p,q[_i$ and $[p,q[_i$.

We have to distinguish carefully between the *intrinsic topology* of the (un)stable manifolds induced by the immersions γ_i and the *manifold topology* induced by the topology of M. The $\gamma_i : \mathbb{R} \to L_i$ are generally no homeomorphisms onto their images in the manifold topology. Now a continuous bijective map from a quasicompact space (i.e. every open covering admits a finite subcovering) to a Hausdorff space is a homeomorphism. Thus for $K \subset \mathbb{R}$ compact the restriction $\gamma_i|_K : K \to \gamma_i(K)$ is a homeomorphism. We define the path spaces

$$\mathcal{P}(L_0, L_1) := \{\beta : [0,1] \to M \mid \beta(0) \in L_0,\ \beta(1) \in L_1\},$$
$$\Lambda_x M := \{\beta : [0,1] \to M \mid \beta(0) = x = \beta(1)\}.$$

Provide $C([0,1]; M)$ with the compact-open topology and $\Lambda_x M \subset C([0,1]; M)$ with the induced one. On $\mathcal{P}(L_0, L_1)$ we use the smallest topology such that the following three maps are continuous:

$$\mathcal{P}(L_0, L_1) \to C([0,1]; M), \quad \beta \mapsto \beta,$$
$$\mathcal{P}(L_0, L_1) \to \mathbb{R}, \quad \beta \mapsto \gamma_0^{-1}(\beta(0)),$$
$$\mathcal{P}(L_0, L_1) \to \mathbb{R}, \quad \beta \mapsto \gamma_1^{-1}(\beta(1)).$$

We define the concatenation $\#$ of two paths α, $\beta : [0,1] \to M$ with $\alpha(1) = \beta(0)$ to be

$$\beta\#\alpha(t) := \begin{cases} \alpha(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

For $p \in L_i$ let $t_i^p := \gamma_i^{-1}(p)$ and define

$$\beta_0^p(t) := \gamma_0(t_0^x + (t_0^p - t_0^x)t) \quad \text{and} \quad \beta_1^p(t) := \gamma_1(t_1^p + (t_1^x - t_1^p)t).$$

Lemma 2.4. *$\mathcal{P}(L_0, L_1)$ and $\Lambda_x M$ are homotopy equivalent and $\pi_0(\mathcal{P}(L_0, L_1)) \simeq \pi_1(M, x)$.*

Proof : *For the first claim* consider $\beta \in \mathcal{P}(L_0, L_1)$ and abbreviate $\beta_0 := \beta_0^{\beta(0)}$ and $\beta_1 := \beta_1^{\beta(1)}$. Define the maps

$$f : \mathcal{P}(L_0, L_1) \to \Lambda_x M, \quad \beta \mapsto \beta_1\#(\beta\#\beta_0),$$
$$g : \Lambda_x M \to \mathcal{P}(L_0, L_1), \quad \beta \mapsto \beta$$

and calculate $f(g(\beta)) = \beta_1\#(\beta\#\beta_0)$ where $\beta_0 \equiv x \equiv \beta_1$. Then

$$H : [0,1] \times \Lambda_x M \to \Lambda_x M,$$
$$H(\tau, \beta)(t) := \beta_1\#(\beta\#\beta_0)((-\tfrac{3}{4}\tau + 1)t + \tfrac{1}{4}\tau)$$

satisfies $H(0,\beta)(t) = \beta_1\#(\beta\#\beta_0)(t)$ and $H(1,\beta)(t) = \beta(t)$ and therefore $f \circ g \simeq \mathrm{Id}_{\Lambda_x M}$.
In order to show $g \circ f \simeq \mathrm{Id}_{\mathcal{P}(L_0,L_1)}$ calculate $g(f(\beta)) = \beta_1\#(\beta\#\beta_0)$. Consider

$$\begin{aligned} &H : [0,1] \times \mathcal{P}(L_0, L_1) \to \mathcal{P}(L_0, L_1), \\ &H(\tau,\beta)(t) := \beta_1\#(\beta\#\beta_0)((-\tfrac{3}{4}\tau + 1)t + \tfrac{1}{4}\tau) \end{aligned}$$

which fulfils $H(0,\beta)(t) = \beta_1\#(\beta\#\beta_0)(t)$ and $H(1,\beta)(t) = \beta(t)$ and therefore $g \circ f \simeq \mathrm{Id}_{\mathcal{P}(L_0,L_1)}$.
For the second claim recall from algebraic topology (see Spanier [**Sp**] § 1.6) that the *suspension* S of a sphere S^k is homeomorphic to S^{k+1}, i.e. $S(S^k) \simeq S^{k+1}$ for $k \geq 0$. Moreover, if X and Y are pointed topological spaces and if $\Omega(Y)$ denotes the pointed loop space of Y then the mapping classes $[S(X), Y] \simeq [X, \Omega(Y)]$ are equivalent. In our context this yields $\pi_{k+1}(M, x) \simeq \pi_k(\Lambda_x M)$ for $k \geq 0$ and therefore we obtain using the first claim $\pi_0(\mathcal{P}(L_0, L_1)) \simeq \pi_0(\Lambda_x M) \simeq \pi_1(M, x)$. ∎

Now we want to assign to each $p \in \mathcal{H}$ a **homotopy class** in $\pi_0(\mathcal{P}(L_0, L_1)) \simeq \pi_1(M, x)$.

Definition 2.5. *Let $p \in \mathcal{H}$ and denote by $c_p : [0,1] \to L_0 \cup L_1$ the curve with $c_p(0) = x = c_p(1)$ which runs through $[x,p]_0$ to p and through $[p,x]_1$ back to x. Set $[p] := [c_p] \in \pi_1(M,x)$ and $[-p]$ for the path with the inverse parametrization and define for $\gamma \in \pi_1(M,x)$*

$$\mathcal{H}_\gamma := \{p \in \mathcal{H} \mid [p] = \gamma\}.$$

Later on we will mostly use the following class of homoclinic points:

Definition 2.6. *$p \in \mathcal{H}$ is called* **contractible** *if $[p] = [x]$.*

Since there is the natural $\mathbb{Z}$-action by φ on $\mathcal{H}$ we have to ask if and under which conditions some of the homotopy classes or even the whole decomposition

$$\mathcal{H} = \bigcup_{\gamma \in \pi_1(M,x)} \mathcal{H}_\gamma$$

stay invariant under the action.

Lemma 2.7.
(1) *For all diffeomorphisms φ the class of contractible homoclinic points $\mathcal{H}_{[x]}$ is invariant under the action of φ.*
(2) *Let $\varphi = \varphi_1$ be the time-1 map of a flow. Define $\xi : S^1 \to M$, $\xi(t) := \varphi_t(x)$ and assume ξ contractible or $\pi_1(M,x)$ abelian. Then $[p] = [\varphi_1^n(p)]$ for all $p \in \mathcal{H}$ and $n \in \mathbb{Z}$, i.e. the decomposition $\mathcal{H} = \bigcup_{\gamma \in \pi_1(M,x)} \mathcal{H}_\gamma$ is compatible with the action.*

Proof : *The first claim* is clear.
The rough idea *for the second claim* is to consider c_p from Definition 2.5 and $\tau \mapsto \varphi_\tau(c_p)$ and adjust the endpoints in order to obtain a homotopy within $\Lambda_x M$. Thus for a path $c \in \Lambda_x M$ and $\tau \in [0,1]$ define $\beta^c_\tau \in \Lambda_x M$ via

$$\beta^c_\tau(t) := \begin{cases} \varphi_{4t\tau}(x) & \text{for } t \in [0, \frac{1}{4}], \\ \varphi_\tau(c(4(t - \frac{1}{4}))) & \text{for } t \in [\frac{1}{4}, \frac{1}{2}], \\ \varphi_{2\tau(1-t)}(x) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that by change of parametrization β^c_0 is homotopic to c and that $\beta^c_1 = \bar{\xi}\#(c\#\xi)$ where $\bar{\xi}(t) := \xi(1-t)$. Since L_i is invariant under φ_1 we obtain $\varphi_1([x,p]_i) = [x, \varphi_1(p)]_i$ for $i \in \{0,1\}$ and therefore $\varphi_1(c_p) = c_{\varphi_1(p)}$.
This yields $[c_p] = [\bar{\xi}] * [c_{\varphi_1(p)}] * [\xi] = [\xi]^{-1} * [c_{\varphi_1(p)}] * [\xi] \in \pi_1(M,x)$. If $[\xi]$ is trivial or if $\pi_1(M,x)$ is commutative we get $[c_p] = [c_{\varphi_1(p)}] \in \pi_1(M,x)$ and thus $[p] = [\varphi_1(p)]$ and the claim follows by induction. ∎

To get a feeling about the geometric meaning of those homotopy classes consider

Example 2.8. *(1) For a homoclinic tangle in $\mathbb{R}^2$ like in figure 2.2 the decomposition $\mathcal{H} = \bigcup_{\gamma \in \pi_1(\mathbb{R}^2,x)} \mathcal{H}_\gamma$ is trivial since $\pi_1(\mathbb{R}^2,x) = 0$.*

(2) Consider the discrete dynamical system generated by the time-1 map φ_1 of a Hamiltonian system on an annulus or cylinder. Since its fundamental group equals $\mathbb{Z}$ we get the decomposition $\mathcal{H} = \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n$ where $\mathcal{H}_n$ consists of those homoclinic points p whose trajectories $t \mapsto \varphi_t(p)$ travel n times around the cylinder resp. the center of the annulus.

3. Maslov index and Maslov grading

Denote by $\mathcal{L}(n)$ the **space of Lagrangian subspaces** of $(\mathbb{R}^{2n}, \omega_0)$ with $\omega_0 := \sum_{i=1}^n dx_i \wedge dy_i$. The following can be found for instance in McDuff & Salamon [**McS2**], §2.2, §2.3.
The unitary group $U(n)$ is a maximal compact subgroup of the symplectic group $Sp(2n)$ and the quotient $Sp(2n)/U(n)$ is contractible. Since the determinant $\det : U(n) \to S^1$ induces an isomorphism of fundamental groups we get $\pi_1(Sp(2n)) \simeq \mathbb{Z}$. And since $\mathcal{L}(n)$ is naturally isomorphic to $U(n)/O(n)$ there is also $\pi_1(\mathcal{L}(n)) \simeq \mathbb{Z}$. Explicit isomorphisms are given by the Maslov index which we will define now.

First consider $Sp(2n)$. Each $\Psi \in Sp(2n)$ can be uniquely decomposed as $\Psi = P \circ Q$ where $P := (\Psi \circ \Psi^T)^{-\frac{1}{2}}$ is symmetric and positive definite and $Q := (\Psi \circ \Psi^T)^{-\frac{1}{2}} \circ \Psi \in Sp(2n) \cap O(2n)$ is orthogonal and can be written as $Q =: \left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right)$. Setting

$$\rho : Sp(2n) \to S^1, \quad \rho(\Psi) := \det(X + iY)$$

we define for a **loop** $\Psi : \mathbb{R}/\mathbb{Z} \to Sp(2n)$ the **Maslov index of loops of symplectic matrices** to be

$$\mu(\Psi) := \deg(\rho \circ \Psi)$$

where deg is the mapping degree of $\rho \circ \Psi : \mathbb{R}/\mathbb{Z} \to S^1$. For a lift $\alpha : \mathbb{R} \to \mathbb{R}$ of $\rho \circ \Psi$, i.e. $\det(X(t) + iY(t)) = e^{2\pi i \alpha(t)}$, holds

$$\mu(\Psi) = \alpha(1) - \alpha(0).$$

Now consider $\mathcal{L}(n)$*:* Represent $\Lambda \in \mathcal{L}(n)$ by $\Lambda = \binom{X}{Y}$ where $U := X + iY \in U(n)$ and define

$$\rho : \mathcal{L}(n) \to S^1, \quad \rho(\Lambda) := \det(U \circ U)$$

where the square of U is needed since we consider **unoriented** Lagrangian subspaces. For a **loop** $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$ define the **Maslov index of loops of Lagrangian subspaces** by

$$\mu(\Lambda) := \deg(\rho \circ \Lambda)$$

where deg denotes the mapping degree of $\rho \circ \Lambda : \mathbb{R}/\mathbb{Z} \to S^1$. If $\alpha : \mathbb{R} \to \mathbb{R}$ is a lift of $\rho \circ \Lambda$, i.e. if

$$\det(X(t) + iY(t)) = e^{i\pi\alpha(t)}$$

we get

$$\mu(\Lambda) = \alpha(1) - \alpha(0).$$

The Maslov index of Lagrangian subspaces has the following properties:

THEOREM 2.9. *The Maslov index is the unique isomorphism $\mu : \pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ which satisfies*

(1) **Homotopy:** *Two loops in $\mathcal{L}(n)$ are homotopic if and only if they have the same Maslov index.*
(2) **Product:** *If $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$ and $\Psi : \mathbb{R}/\mathbb{Z} \to Sp(2n)$ then*

$$\mu(\Psi \circ \Lambda) = \mu(\Lambda) + 2\mu(\Psi).$$

In particular, for $\Psi \equiv \Psi_0$ constant we have $\mu(\Psi \circ \Lambda) = \mu(\Lambda)$; and for a constant loop $\Lambda(t) \equiv \Lambda_0$ holds $\mu(\Lambda) = 0$.
(3) **Direct sum:** *For $n = n' + n''$ identify $\mathcal{L}(n') \oplus \mathcal{L}(n'')$ as a submanifold of $\mathcal{L}(n)$. Then $\mu(\Lambda' \oplus \Lambda'') = \mu(\Lambda') + \mu(\Lambda'')$.*
(4) **Normalization:** $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$, $t \mapsto e^{i\pi t}\mathbb{R} \subset \mathbb{R}^2 \simeq \mathbb{C}$ *has Maslov index 1.*

In two dimensions we have $Sp(2) = SL(2)$ and every 1-dimensional subspace is Lagrangian. Therefore we can assign a Maslov index to a smooth loop $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ with $\dot{\gamma} \neq 0$ by defining it as the Maslov index of $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(1)$, $t \mapsto \Gamma_t := \mathrm{Span}_{\mathbb{R}}\{\dot{\gamma}(t)\}$. In fact the Maslov index of γ equals twice its tangent winding number.

Now consider a symplectomorphism φ with fixed point x. We assume all homoclinic points with which we deal in the remainder of this section to be **transverse** intersection points of L_0 and L_1. Since φ is symplectic L_0 and L_1 are Lagrangian by Proposition 2.2.

We want to define a (relative) Maslov index for p, $q \in \mathcal{H}$. Consider p and q as constant paths in $\mathcal{P}(L_0, L_1)$. Since the following construction requires a path u connecting p and q in $\mathcal{P}(L_0, L_1)$ it works only for points in the same connected component of $\mathcal{P}(L_0, L_1)$. The following approach is due to Viterbo [**V**] and Floer [**Fl1**].

Fix $\alpha \in \mathcal{P}(L_0, L_1)$ and denote its connected component by $\mathcal{P}_\alpha(L_0, L_1)$. Consider the constant paths p, $q \in \mathcal{P}_\alpha(L_0, L_1)$. (In dimension two this means $[p] = [q] = [\alpha]$ in the light of Definition 2.5.) Let $u : [0,1] \to \mathcal{P}(L_0, L_1)$ with $u(0) \equiv p$ and $u(1) \equiv q$ and see it as map $u : [0,1]^2 \to M$ via $u(s,t) := u(s)(t)$. The square $[0,1]^2$ is contractible and we can find a trivialization $\Phi := \Phi_u : u^*TM \to [0,1]^2 \times \mathbb{R}^{2n}$ with

- the symplectic form on the fibers is mapped to the standard ω_0 on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$,
- Φ is constant on $\{0\} \times [0,1]$ and on $\{1\} \times [0,1]$,
- $\Phi(T_pL_1) = i\Phi(T_pL_0)$ and $\Phi(T_qL_1) = i\Phi(T_qL_0)$.

Denote by $\partial[0,1]^2$ the boundary of $[0,1]^2$ and define the loop $\Lambda_u : \partial[0,1]^2 \to \mathcal{L}(n)$ starting in $(0,0)$ and running through $(1,0)$, $(1,1)$ and $(0,1)$ back to $(0,0)$ piecewise via

$$\begin{aligned}(s,0) &\mapsto \Phi(T_{u(s,0)}L_0),\\ (1,t) &\mapsto e^{\frac{i\pi t}{2}}\Phi(T_qL_0),\\ (s,1) &\mapsto \Phi(T_{u(s,1)}L_1),\\ (0,t) &\mapsto e^{\frac{i\pi(t-1)}{2}}\Phi(T_pL_1).\end{aligned}$$

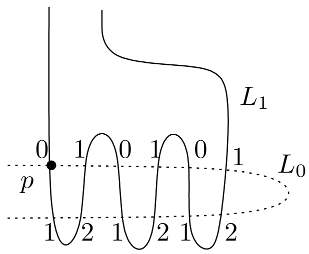
Definition 2.10. *Under the above conventions we define the* **relative Maslov index for** p**,** $q \in \mathcal{H}$ *via* $\mu(p,q) := \mu(\Lambda_u)$.

Since $\pi_2(M) = 0$ and therefore $c_1|_{\pi_2(M)} = 0$ (where c_1 denotes the first Chern class) the construction is independent from the choosen path u and the trivialization Φ. Therefore $\mu(p,q)$ is well-defined.

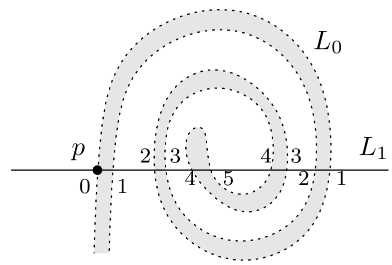
For the remainder of this section we assume (M, ω) to be two-dimensional. We obtain for the contractible homoclinic points:

Proposition 2.11. *Consider* p, $\tilde{p}$, $q \in \mathcal{H}$ *with* $[p] = [\tilde{p}] = [q] = [x]$. *Then*

(1) $\mu(q,p) = -\mu(p,q)$ *and* $\mu(p,q) + \mu(q,\tilde{p}) = \mu(p,\tilde{p})$.



(a) ‘n-legged camel’



(b) ‘cycle’

FIGURE 2.1. The Maslov index $\mu_p(q) := \mu(q,p)$

(2) $\mu(p,q) = \mu(\varphi^n(p), \varphi^n(q))$ *for* $n \in \mathbb{Z}$, *i.e. the (relative) Maslov index of* p *and* q *is invariant under the* $\mathbb{Z}$*-action of* φ *on* $\mathcal{H}$*.*
(3) $\mu(p, \varphi^n(p)) = 0$ *for all* $n \in \mathbb{Z}$*.*
(4) $\mu(p,q) = \mu(p, \varphi^n(q))$ *for* $n \in \mathbb{Z}$*.*

PROOF : The *first* item follows from the concatenation of the paths.
For the *second* one choose for p and q a smooth $u : [0,1]^2 \to M$ and a trivialization $\Phi : u^*TM \to [0,1]^2 \times \mathbb{R}^{2n}$ as used in Definition 2.10 in order to define $\mu(p,q)$. Setting $\Psi := \Phi \circ (D\varphi)^{-1}$ yields a trivialization $\Psi : (\varphi \circ u)^*TM \to [0,1]^2 \times \mathbb{R}^2$ which satisfies the hypothesis of Definition 2.10 and yields $\Lambda_u = \Lambda_{\varphi\circ u}$. This implies $\mu(p,q) = \mu(\varphi(p), \varphi(q))$ and the claim follows by induction.
The *third* item is true since $[x] = [p] = [\varphi^n(p)]$ by assumption and by Lemma 2.7 such that we can decompose and write using the first and second item

$$\begin{aligned}\mu(p, \varphi^n(p)) &= \mu(p,x) + \mu(x, \varphi^n(p)) = \mu(p,x) - \mu(\varphi^n(p), x)\\ &= \mu(p,x) - \mu(\varphi^n(p), \varphi^n(x)) = \mu(p,x) - \mu(p,x)\\ &= 0.\end{aligned}$$

The *fourth* item follows from the third item and $\mu(p, \varphi^n(q)) = \mu(p,q) + \mu(q, \varphi^n(q)) = \mu(p,q)$. ■

Now we want to induce a grading by the (relative) Maslov index on the set of contractible points in $\mathcal{H}$.

DEFINITION 2.12. *For* $p \in \mathcal{H}$ *with* $[p] = [x]$ *we set* $\mu(p) := \mu_x(p) := \mu(p,x)$ *which defines a* **grading** $\mu : \mathcal{H}_{[x]} \to \mathbb{Z}$ *such that*

$$\mathcal{H}_{[x]} = \bigcup_{n\in\mathbb{Z}} \mathcal{H}^n_{[x]} \quad \textit{for} \quad \mathcal{H}^n_{[x]} := \{p \in \mathcal{H}_{[x]} \mid \mu(p) = n\}.$$

Now we describe and sketch some examples for the Maslov index and the induced grading.

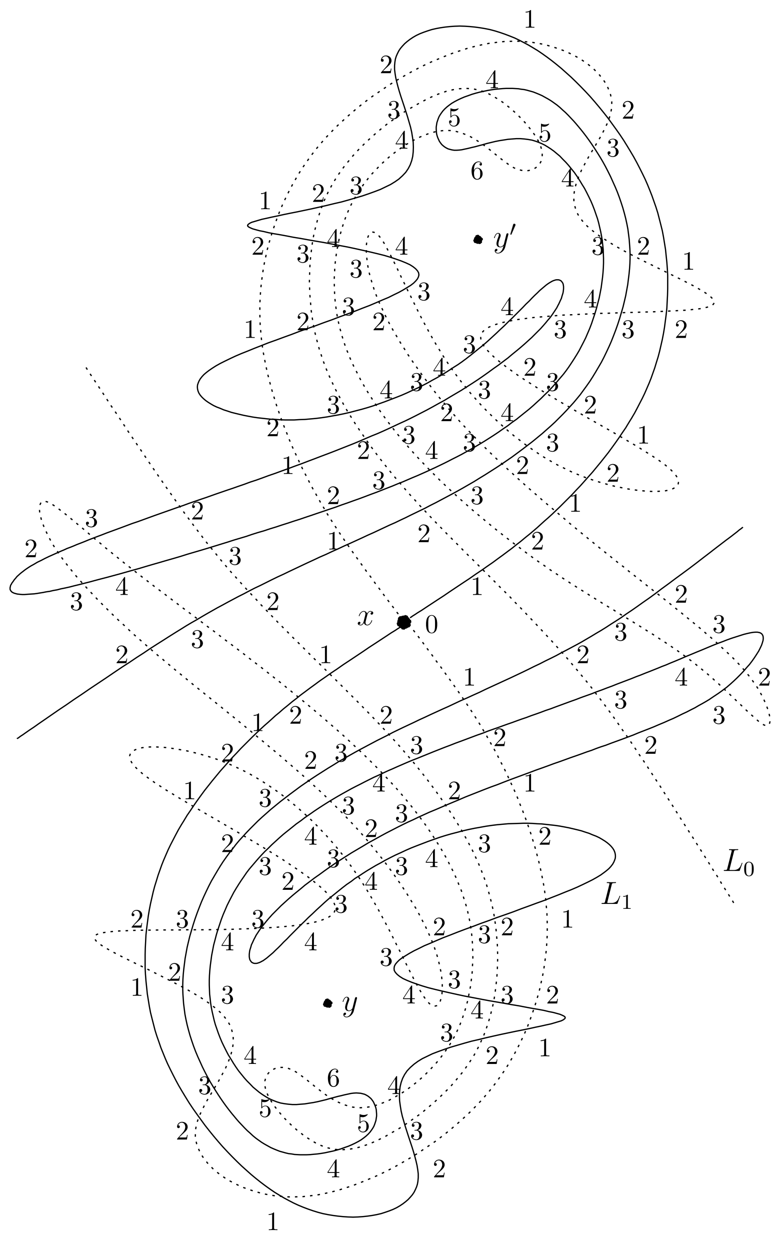


FIGURE 2.2. Homoclinic tangle with Maslov grading $\mu(p) := \mu(p, x)$

'Cycles' or 'n-legged camels' (see figure 2.1) can produce strictly increasing or alternating sequences of Maslov indices. 'Camels' and 'cycles' definitively can occur in homoclinic tangles, see Contopoulos & Polymilis [**CP**].

In figure 2.2 the Maslov index $\mu(p) := \mu(x, p)$ is computed for (parts of) a homoclinic tangle in $\mathbb{R}^2$. Since $0 = \pi_1(\mathbb{R}^2, x) = \pi_0(\mathcal{P}(L_0, L_1))$ the path space $\mathcal{P}(L_0, L_1)$ consists of one connected component. Obvioulsy $\mu(p)$ can become arbitrarily high.

CHAPTER 3

Immersions, gluing and cutting

In this chapter we define and discuss the spaces of immersed 2-gons needed for the definition of the Floer differential later on. Then we present the gluing and cutting construction on which the vanishing of the square of the Floer differential relies. Finally we introduce coherent orientations.

Within this chapter the symplectic manifold (M, ω) is either a closed surface with genus $g \geq 1$ or $(\mathbb{R}^2, \omega)$. $\varphi : M \to M$ is a symplectomorphism with hyperbolic fixed point x and (un)stable manifolds $L_0 := W^u(x, \varphi)$ and $L_1 := W^s(x, \varphi)$ and $\mathcal{H} := L_0 \cap L_1$. The connected components of $L_i \backslash \{x\}$ are denoted by L_i^+ and L_i^- for $i \in \{0, 1\}$ and are called the branches of the (un)stable manifolds. We assume all appearing homoclinic points to be transverse intersection points.
There is a $\mathbb{Z}$-action of φ on $\mathcal{H}$ given by $\mathbb{Z} \times \mathcal{H} \to \mathcal{H}$, $(n, p) \mapsto \varphi^n(p)$ which is free on $\mathcal{H}\backslash\{x\}$. Our convention for the Maslov index is $\mu(p) := \mu(p, x)$ and therefore

$$\mu(p, q) = \mu(p, x) + \mu(x, q) = \mu(p, x) - \mu(q, x) = \mu(p) - \mu(q).$$

1. Di-gons and hearts

In this section we will define and discuss the moduli spaces which we need for the definition of the coefficients in the Floer differential.

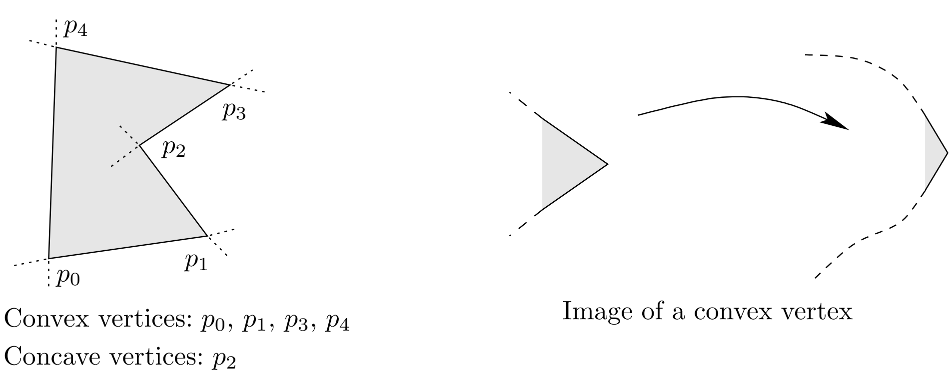


FIGURE 3.1. Convex and concave vertices

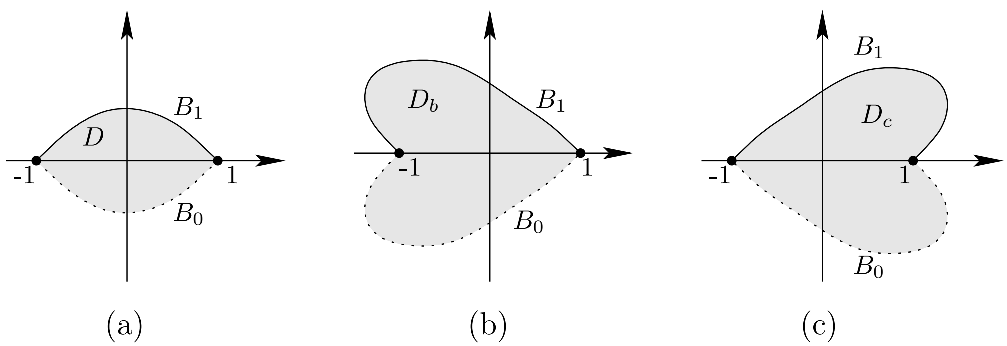


FIGURE 3.2. Di-gon and heart

We call a vertex of a polygon in $\mathbb{R}^2$ **concave** if the imaginary continuations of the two edges which meet in the vertex pass through the interior of the polygon after meeting in the vertex. Otherwise the vertex is called **convex**. The situation is sketched in figure 3.1 (a).

In the following we need a special kind of polygons, namely

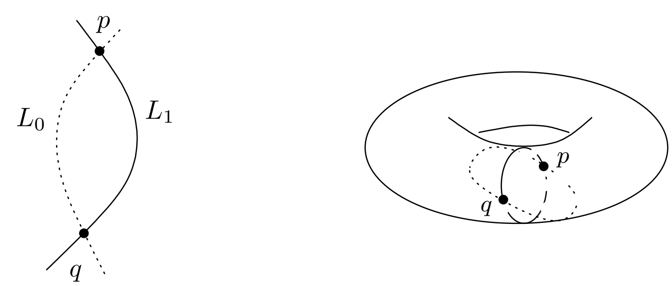
DEFINITION 3.1.
(1) *A* **di-gon** *is the polygon $D \subset R^2$ with two convex vertices at $(-1, 0)$ and $(1, 0)$ sketched in figure 3.2 (a). Denote its upper boundary (edge) by B_1 and its lower boundary by B_0.*
(2) *A* **heart shaped polygon** *or briefly a* **heart** *is either the polygon D_b of figure 3.2 (b) or the polygon D_c of figure 3.2 (c). A heart is characterised by two vertices at $(-1, 0)$ and $(1, 0)$ where one is convex and one concave. Denote their upper boundaries by B_1 and their lower boundaries by B_0.*

Di-gons also appear in the literature as **2-gons**, **lunes** or **half-moons**, see Chekanov [**Che**], de Silva [**dS**], Gautschi & Robbin & Salamon [**GauRS**] or Robbin [**R**].

We recall that a **smooth immersion** is a smooth map with injective differential. **If the domain of definition is a polygon we require the map to be an immersion also on the boundary, i.e. also in the vertices**. This requirement implies the following important fact:

REMARK 3.2. *The image of a small neighbourhood of a convex (concave) vertex of a polygon under an orientation preserving immersion is a wedge-shaped region with angle smaller (larger) than π, see figure 3.1 (b).*

Now we are able to define the moduli spaces which we need for the definition of the Floer differential.



(a) 1 disk (b) 2 disks

FIGURE 3.3

DEFINITION 3.3. *Let D be the di-gon and p, $q \in \mathcal{H}$ with $\mu(p,q) = 1$. We define $\mathcal{M}(p,q)$ to be the* **space of smooth, immersed di-gons** *$u : D \to M$ satisfying*

(1) u is **orientation preserving**,
(2) $u(B_0) \subset L_0$ and $u(B_1) \subset L_1$,
(3) $u((-1,0)) = p$ and $u((1,0)) = q$.

Denote by $G(D)$ the group of orientation preserving diffeomorphisms of D which preserve the vertices and call $\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/G(D)$ the **space of unparametrized immersed di-gons**.

Recall that for p, $q \in \mathcal{H}$ there is exactly one segment $[p,q]_i$, $i \in \{0,1\}$ joining them (see figure 3.3 (a)) and that $\pi_2(M) = 0$ since M is a closed two-dimensional manifold with genus $g \geq 1$ or R^2. Therefore it is easy to see that $\#\widehat{\mathcal{M}}(p,q) \in \{0,1\}$ for p and q with $\mu(p,q) = 1$. Note that for *closed* L_0 and L_1 there might be two connecting segments between p and q in each of the L_i and maybe two unparametrized immersions with $\mu(p,q) = 1$ and the correct boundary conditions, see figure 3.3 (b).

For the gluing and cutting procedure later on we need the following moduli spaces.

DEFINITION 3.4. *Consider the hearts D_b and D_c and p, $q \in \mathcal{H}$ with $\mu(p,q) = 2$. We define $\mathcal{N}_b(p,q)$ resp. $\mathcal{N}_c(p,q)$ to be the* **space of smooth immersed hearts** *$u : D_b \to M$ resp. $u : D_c \to M$ satisfying*

(1) u is **orientation preserving**,
(2) $u(B_0) \subset L_0$ and $u(B_1) \subset L_1$,
(3) $u(-1,0) = p$ and $u(1,0) = q$.

We set $\mathcal{N}(p,q) := \mathcal{N}_b(p,q) \dot\cup \mathcal{N}_c(p,q)$. Denote by $G(D_b)$ resp. $G(D_c)$ the group of orientation preserving diffeomorphisms of D_b resp. D_c which preserve the vertices and let $\widehat{\mathcal{N}}_b(p,q) := \mathcal{N}_b(p,q)/G(D_b)$ resp. $\widehat{\mathcal{N}}_c(p,q) := \mathcal{N}_c(p,q)/G(D_c)$ and

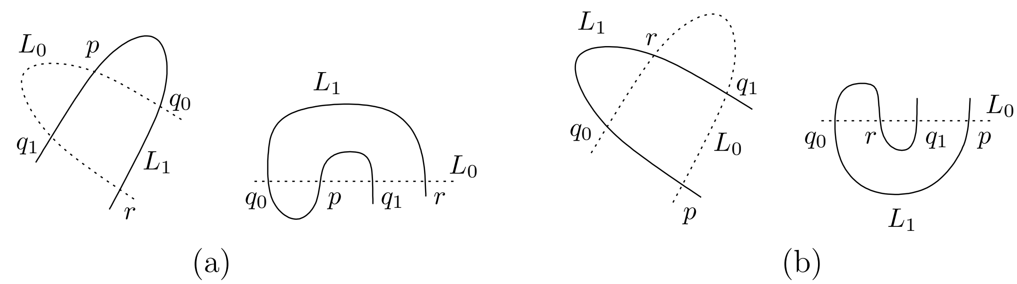


FIGURE 3.4. Shapes for immersions with $\mu(p) = \mu(q) + 1 = \mu(r) + 2$

$\widehat{\mathcal{N}}(p,q) := \widehat{\mathcal{N}}_b(p,q) \,\dot{\cup}\, \widehat{\mathcal{N}}_c(p,q)$ *be the* **spaces of unparametrized immersed hearts**.

The decomposition $\mathcal{N}(p,q) = \mathcal{N}_b(p,q) \,\dot{\cup}\, \mathcal{N}_c(p,q)$ is a disjoint union due to Remark 3.2.

Now we are interested in the geometric relation of $\mathcal{M}(p,q)$, $\mathcal{M}(q,r)$ and $\mathcal{N}(p,r)$ for $\mu(p,q) = 1 = \mu(q,r)$. Drawing one element of $\mathcal{M}(p,q)$ and one of $\mathcal{M}(q,r)$ under the condition $\mu(p,q) = 1 = \mu(q,r)$ yields two possible configurations and within each two possible choices for q, namely the left sketches in figure 3.4 (a) and (b). The gluing procedure Theorem 3.14 will recognize them as elements of $\mathcal{N}_b(p,r)$ resp. $\mathcal{N}_c(p,r)$. The intersection point q is labeled q_i for $i \in \{0,1\}$ if it arises as intersection point of the continuation of $[p,r]_i$ after passing the concave vertex through the 'interior' of the heart.
Sometimes it is more convenient to draw the 'heart-shaped' configurations in the left part of figure 3.4 (a) and (b) with one edge mapped to the horizontal axis as it is exemplarily done in the right part of figure 3.4 (a) and (b). We summarize our considerations to

REMARK 3.5. *Consider p, q, $r \in \mathcal{H}$ with $[p] = [q] = [r]$ and $\mu(p,q) = 1 = \mu(q,r)$ and let $u \in \mathcal{M}(p,q)$ and $v \in \mathcal{M}(q,r)$. Geometrically this configuration can be realized only by the left sketches in figure 3.4 (a) and (b) with $q \in \{q_0, q_1\}$.*

In the following we will briefly use **immersion** or more explicitly **immersion of (relative) index 1 resp. 2** for immersed di-gon or immersed heart and analogously for **embedding**. Remark 3.2 justifies to call a vertex of an immersion **convex (concave)** if it is the image of a convex (concave) vertex of the di-gon or heart.

2. The winding number

If we work with the spaces $\mathcal{M}(p,q)$ and $\mathcal{N}(p,r)$ we always implicitly assume p, q, $r \in \mathcal{H}$ with $[p] = [q]$, $[p] = [r]$, $\mu(p,q) = 1$ and $\mu(p,r) = 2$.

Consider the universal cover $\tau : \tilde{M} \to M$ with induced orientation as topological manifold. For all $\tilde{z} \in \tilde{M}$ the orientation induces an isomorphism $H_2(\tilde{M}, \tilde{M}\backslash\{\tilde{z}\}) \simeq \mathbb{Z}$ and the contractibility of $\tilde{M} \simeq \mathbb{R}^2$ implies $H_1(\tilde{M}\backslash\{\tilde{z}\}) \simeq H_2(\tilde{M}, \tilde{M}\backslash\{\tilde{z}\})$. Denote by $[S^1]$ the fundamental class of S^1.
Now consider a continuous path $\tilde{\gamma} : S^1 \to \tilde{M}$ and $\tilde{z} \in \tilde{M}\backslash \operatorname{Im}(\tilde{\gamma})$. We define the **winding number of** $\tilde{\gamma}$ **w.r.t.** $\tilde{z}$ by $\operatorname{Ind}_{\tilde{\gamma}}(\tilde{z}) := \tilde{\gamma}_*([S^1]) \in H_1(M\backslash\{\tilde{z}\}) \simeq \mathbb{Z}$.
Identifying $\tilde{M}$ with $\mathbb{R}^2$ by an orientation preserving diffeomorphism the winding number also can be seen as mapping degree of $S^1 \to S^1$, $t \mapsto \frac{\tilde{\gamma}(t)-\tilde{z}}{|\tilde{\gamma}(t)-\tilde{z}|}$.

For the concrete computation of the winding number the following observation is very helpful.

Remark 3.6. *Let $\tilde{\gamma} : S^1 \to \tilde{M}$, $\tilde{z} \in \tilde{M}\backslash \operatorname{Im}(\tilde{\gamma})$ and $\rho : \mathbb{R}_0^+ \to \tilde{M} \simeq \mathbb{R}^2$ be smooth and regular with $\rho(0) = \tilde{z}$ and such that ρ escapes to infinity for $t \to \infty$ and is transverse to $\tilde{\gamma}$. Then*

$$\operatorname{Ind}_{\tilde{\gamma}}(\tilde{z}) = \sum_{\substack{t\in S^1 \\ \tilde{\gamma}(t)\in \operatorname{Im}(\rho)}} \operatorname{sign}(\det(\dot{\rho}(s_t), \dot{\tilde{\gamma}}(t)))$$

where $\rho(s_t) = \tilde{\gamma}(t)$.

Since the di-gon D and the hearts D_b and D_c are contractible a continuous map u from the di-gon or a heart to M can be lifted to $\tilde{M}$. If p is a vertex and $\tilde{p} \in \tau^{-1}(p)$ then there is a unique lift $\tilde{u}$ with $\tilde{u}(-1, 0) = \tilde{p}$.

Definition 3.7. *Let A stand for D, D_b or D_c and consider $u : A \to M$ and a lift $\tilde{u} : A \to \tilde{M}$ of u.*
The **winding number** $\operatorname{Ind}_{\tilde{u}}(\tilde{z})$ *of $\tilde{u}$ w.r.t. $\tilde{z} \in \tilde{M}\backslash\tilde{u}(\partial A)$ is defined as the winding number of the path $\tilde{u}|_{\partial A}$ w.r.t. $\tilde{z}$ with ∂A parametrized counterclockwise.*
The **winding number** *of u w.r.t. $z \in M\backslash u(\partial A)$ is defined as*

$$\operatorname{Ind}_u(z) := \sum_{\tilde{z}\in\tau^{-1}(z)} \operatorname{Ind}_{\tilde{u}}(\tilde{z})$$

Ind_u does not depend on the choice of the lift $\tilde{u}$ and the sum is finite since $\operatorname{Ind}_{\tilde{u}}$ vanishes for all $\tilde{z}$ lying in the unbounded component of $\tilde{M}\backslash\tilde{u}(\partial A)$.

Apart from Remark 3.6 there is another way to compute the winding number of a continuous map $\tilde{u} : A \to \tilde{M}$ where A again stands for D, D_b or D_c. For $\tilde{z} \in \tilde{M}\backslash\tilde{u}(\partial A)$ let $\tilde{B}$ be a small ball around $\tilde{z}$ and similarly consider small balls B_i around the $z_i \in \tau^{-1}(\tilde{z})$. Identify $\partial B \simeq S^1 \simeq \partial B_i$ and set $\hat{A} := A\backslash(\bigcup_{z_i\in\tau^{-1}(\tilde{z})} B_i)$.

Then using some kind of 'local degree' (see Bredon [**Br**]) we obtain

$$\begin{aligned}\mathrm{Ind}_{\tilde{u}}(\tilde{z}) &= \deg(\partial A \to \partial\tilde{B}) \\ &= \deg(\partial\hat{A} \to \partial\tilde{B}) + \deg\Big(\bigcup_{z_i\in\tau^{-1}(\tilde{z})} \partial B_i \to \partial\tilde{B}\Big).\end{aligned}$$

Now recall the following fact from topology (see for example Milnor [**Mi**]): If N^n and P^n are compact orientable manifolds of dimension n without boundary and if a smooth $\alpha : N^n \to P^n$ can be extended smoothly to some $(n+1)$-dimensional manifold Q^{n+1} with $\partial Q^{n+1} = N^n$ then $\deg(\alpha) = 0$.
Recognizing $\hat{A}$ as Q^{n+1} and $(\bigcup_{z_i\in\tau^{-1}(\tilde{z})} \partial B_i) \cup \partial A$ as N^n we deduce $\deg(\partial\hat{A} \to \partial\tilde{B}) = 0$ whereas the term $\deg(\bigcup_{z_i\in\tau^{-1}(\tilde{z})} \partial B_i \to \partial\tilde{B})$ yields for orientation preserving immersions the following:

Remark 3.8. *For $u \in \mathcal{M}(p,q)$ and $z \in M\backslash u(\partial D)$ we have*

$$\mathrm{Ind}_u(z) = \#u^{-1}(z)$$

and therefore in particular $\mathrm{Ind}_u \geq 0$. *The analogous result is true for $v \in \mathcal{N}(p,r)$.*

Remark 3.8 yields a simple criterion if a given u might be an immersion or not:

Corollary 3.9 (Criterion). *Let A stand for D, D_b or D_c and consider a smooth $u : A \to M$. If there is a component of $M\backslash u(\partial A)$ with* $\mathrm{Ind}_u < 0$ *then u is no immersion.*

This criterion easily yields the following result:

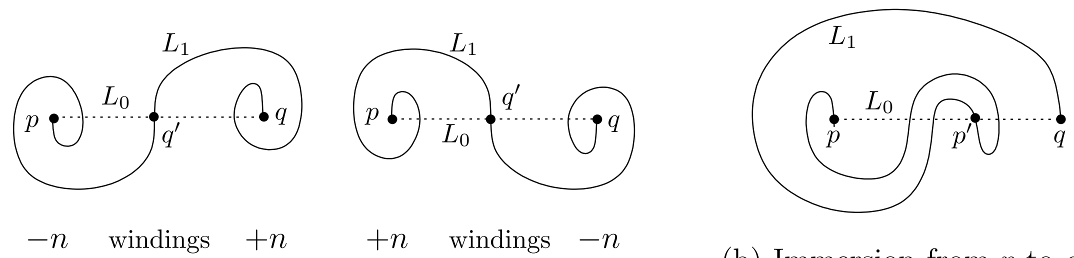
Remark 3.10. *Since in dimension two the Maslov index is twice the winding number of the tangent vector the shapes of figure 3.5 (a) are the only schematic sketches of relative index 0. From Corollary 3.9 follows that there cannot be immersions between points of relative index 0 since there are components with* $\mathrm{Ind} < 0$. *Note that for immersions from p to q with $\mu(p,q) = 1$ there might exist $p' \in \,]p,q[_0 \pitchfork \,]p,q[_1$ with $\mu(p,p') = 0$, see figure 3.5 (b).*

Now we want to express in terms of the winding number if a points lies in the image of the immersion or not.

Definition 3.11. *Let $u \in \mathcal{M}(p,q)$. The union of those components of $M\backslash u(\partial D)$ with* $\mathrm{Ind}_u > 0$ *is called the* **interior** $\mathrm{Int}(u)$ *of u. The union of the others is called the* **exterior** $\mathrm{Ext}(u)$ *of u (their winding number vanishes). And similarly for $v \in \mathcal{N}(p,r)$.*

Now consider the universal covering $\tau : (\tilde{M},\tilde{\omega}) \to (M,\omega)$ with $\tau^*\omega = \tilde{\omega}$ and observe

Lemma 3.12. *Let p, $q \in \mathcal{H}$ and $[p] = [q]$ and denote by $[\tilde{p},\tilde{q}]_i$ the lift of $[p,q]_i$ starting in $\tilde{p} \in \tau^{-1}(p)$ to the universal cover $(\tilde{M},\tilde{\omega})$. Then $\mu(p,q) = \mu(\tilde{p},\tilde{q})$.*



(a) No immersions from p to q

(b) Immersion from p to q with $\mu(p,q) = 1$ and $\mu(p,p') = 0$

FIGURE 3.5. Shapes for relative index 0

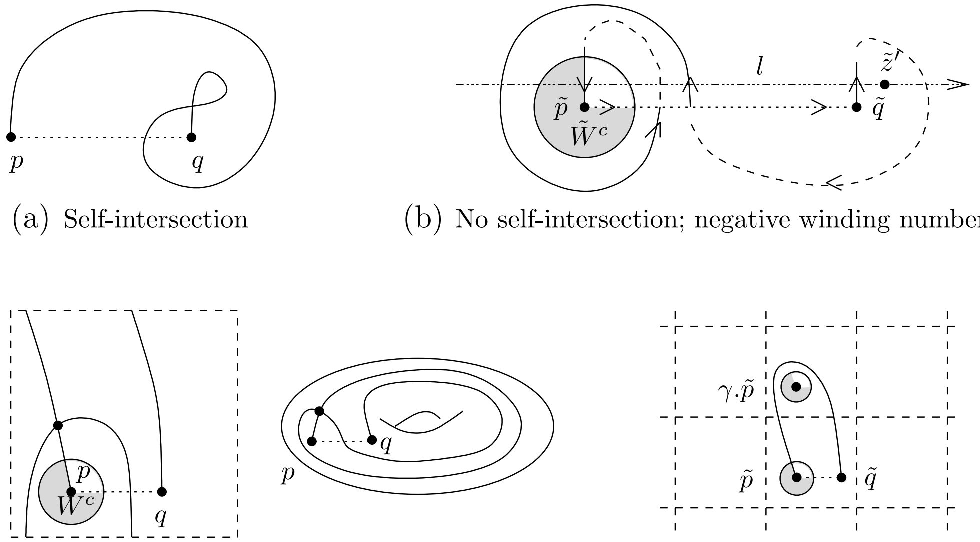
PROOF : Let u and the trivialization $\Phi : u^*TM \to [0,1]^2 \times \mathbb{R}^{2n}$ be as needed for Definition 2.10 and denote by $\tilde{u}$ the lift of u starting in $\tilde{p}$ and by $\tilde{q}$ the accordingly lifted q. Then $\tilde{\Phi} : \tilde{u}^*T\tilde{M} \to [0,1]^2 \times \mathbb{R}^{2n}$, $\tilde{\Phi}(z,v) := \Phi(z, D\tau(v)))$ is an analogous trivialization for $\tilde{u}$. Since $D\tau(T_{\tilde{u}}[\tilde{p},\tilde{q}]_i) = T_u L_i$ for $i \in \{0,1\}$ the loops Λ_u and $\Lambda_{\tilde{u}}$ coincide and thus $\mu(p,q) = \mu(\Lambda_u) = \mu(\Lambda_{\tilde{u}}) = \mu(\tilde{p},\tilde{q})$. ■

The next statement will be needed among others for the existence of the 'cutting points' in the cutting construction Theorem 3.16. There we will need to know that the vertices of an immersion are not multiply covered. Now choose a metric on M. Since the image of our immersions stays in a compact region the following does not depend on the choice of the metric.

PROPOSITION 3.13. *Let $u \in \mathcal{M}(p,q)$. Then there is $\varepsilon > 0$ such that $U_p := u^{-1}(B_\varepsilon(p))$ is a connected neighbourhood of $(-1,0) \in D$ with $u|_{U_p}$ injective.*
As a consequence $\mathrm{Ind}_u = 1$ on $B_\varepsilon(p) \cap u(D)$ and $u(U_p)$ is the wedge-shaped piece of $B_\varepsilon(p)$ bounded by $([p,q]_0 \cup [p,q]_1) \cap B_\varepsilon(p)$ with angle $< \pi$.
An analogous statement is true for q. For $v \in \mathcal{N}(p,r)$ with vertices p and r the only change is $> \pi$ for the concave vertex.

Here the lack of self-intersections of L_0 and L_1 plays an important role — otherwise the lemma is not true, see figure 3.6 (a).

PROOF : There is $\varepsilon > 0$ such that $B_\varepsilon(p) \cap ([p,q]_0 \cup [p,q]_1)$ consists of two segments $[p,p_0^\varepsilon]_0$ and $[p,p_1^\varepsilon]_1$. Denote the wedge-shaped region of $B_\varepsilon(p) \backslash ([p,p_0^\varepsilon]_0 \cup [p,p_1^\varepsilon]_1)$ with angle $< \pi$ by $W := W(p)$ and the other one by $W^c := W^c(p)$.
Consider a path ρ transverse to $[p,q]_0 \cup [p,q]_1$ starting in W and passing through W^c to the exterior of $u \in \mathcal{M}(p,q)$. Remark 3.6 implies $\mathrm{Ind}_u(W) = \mathrm{Ind}_u(W^c)+1$, i.e. points in W have one pre-image more than those in W^c according to Remark 3.8.



(a) Self-intersection

(b) No self-intersection; negative winding number

(c) Overlapping on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

(d) No overlapping on $\mathbb{R}^2$

FIGURE 3.6. Behaviour near vertices

We want to prove $\mathrm{Ind}_u(W^c) = 0$ via contradiction. Therefore assume (w.l.o.g.) $\mathrm{Ind}_u(W^c) = 1$.
Consider the lift $\tilde{u}$ of u starting in $\tilde{p} \in \tau^{-1}(p)$ and denote the lifts of $B_\varepsilon(p)$, W and W^c containing $\tilde{p}$ resp. having $\tilde{p}$ as vertex by $B_\varepsilon(\tilde{p})$, $\tilde{W}$ and $\tilde{W}^c$. Assume for simplicity that the lift $[\tilde{p}, \tilde{q}]_0$ of $[p, q]_0$ is a straight line segment and all intersections with $[\tilde{p}, \tilde{q}]_1$ are orthogonal. The Maslov index stays the same under lifting according to Lemma 3.12.
There are two cases: Either $\#\tilde{u}^{-1}(\tilde{z}) = 1$ for $\tilde{z} \in \tilde{W}^c$ as indicated in figure 3.6 (b) or $\#\tilde{u}^{-1}(\tilde{z}) = 0$ as sketched in figure 3.6 (b) and (c).
Case $\#\tilde{u}^{-1}(\tilde{z}) = 1$ *(figure 3.6 (b)):* Consider a parametrized line $l : \mathbb{R} \to \tilde{M}$ parallel to $[\tilde{p}, \tilde{q}]_0$ and passing sufficiently close. Let the tangent vector $\vec{l}$ and $\partial_s \tilde{u}(\cdot, 0)$ point in the same direction. Since $[\tilde{p}, \tilde{q}]$ is a straight line segment the Maslov index of $\tilde{u}$ equals twice the tangent winding number of $[\tilde{p}, \tilde{q}]_1$ which therefore equals $\frac{1}{2}$. This implies $\sum_{\tilde{u}(s,0) \in \mathrm{Im}(l)} \det(\vec{l}, \partial_s u(s, 1)) = 0$. For $\zeta \in [\tilde{p}, \tilde{q}]_0$ denote by l_ζ the point in $\mathrm{Im}(l)$ closest to ζ and set $l^{\geq \zeta} := l([l^{-1}(l_\zeta), \infty[)$. Then $\sum_{\tilde{u}(s,0) \in l^{\geq \tilde{p}}} \det(\vec{l}, \partial_s u(s, 1)) = 1$. Thus there is $\tilde{z}' \in l^{\geq \tilde{p}}$ such that $\sum_{\tilde{u}(s,0) \in l^{\geq \tilde{z}'}} \det(\vec{l}, \partial_s u(s, 1)) = -1$. But then $\mathrm{Ind}_{\tilde{u}}(\tilde{z}') = -1$ and $\tilde{u}$ is no immersion by Corollary 3.9.
Case $\#\tilde{u}^{-1}(\tilde{z}) = 0$ *(figure 3.6 (c), (d))*: we demonstrate the proof for $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. For surfaces with higher genus it is similar. Denote by $\Gamma \simeq \mathbb{Z}^2$ the group

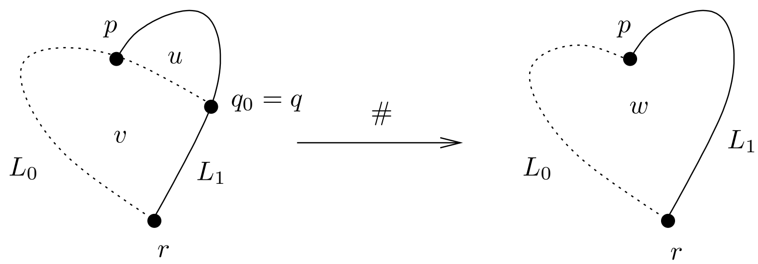


FIGURE 3.7. Gluing in 2 dimensions

of deck transformations of $\tau : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. Since $\#\tilde{u}^{-1}(\tilde{z}) = 0$ for $\tilde{z} \in \tilde{W}^c$, but $\#u^{-1}(\tau(\tilde{z})) = 1$ there is $\gamma \in \Gamma$ such that $\#\tilde{u}^{-1}(\gamma.\tilde{z}) = 1$ resp. $\mathrm{Ind}_{\tilde{u}}(\gamma.\tilde{z}) = 1$, see figure 3.6 (c) and (d). Therefore the segment $[\tilde{p}, \tilde{q}]_1$ has to pass $\gamma.\tilde{z}$ on the left and above before reaching $\tilde{q}$. Now consider this situation after applying τ: As sketched in figure 3.6 (c) we obtain a self-intersection of $[p, q]_1$, but L_1 is free of self-intersections — contradiction.
Therefore $\mathrm{Ind}_u(W^c) = 0$ and $\mathrm{Ind}_u(W) = 1$ and $U_p := u^{-1}(B_\varepsilon(p)) = u^{-1}(W)$ is the desired neighbourhood for $\varepsilon > 0$ small enough.
The proof for the other vertex q proceeds similarly and so does the proof for $v \in \mathcal{N}(p, r)$. ■

3. Gluing and cutting

In this section we will define and prove the gluing and cutting procedure which is the crucial technic in order to show that the square of the Floer differential vanishes.

Briefly, gluing of two immersed di-gons $u \in \widehat{\mathcal{M}}(p, q)$ and $v \in \widehat{\mathcal{M}}(q, r)$ with $\mu(p, q) = 1 = \mu(q, r)$ (and therefore $\mu(p, r) = 2$) is the construction which recognizes the tupel (u, v) as an element of $\widehat{\mathcal{N}}(p, r)$.
Cutting is the 'inverse' construction which starts with $w \in \widehat{\mathcal{N}}(p, r)$ and finds two significant points $q_0, q_1 \in \mathcal{H}$ such that w can be seen either as tupel $(u, v) \in \widehat{\mathcal{M}}(p, q_0) \times \widehat{\mathcal{M}}(q_0, r)$ or as tupel $(u', v') \in \widehat{\mathcal{M}}(p, q_1) \times \widehat{\mathcal{M}}(q_1, r)$.

First consider the gluing construction.

THEOREM 3.14 (Gluing). *Let $p, q, r \in \mathcal{H}$ with $[p] = [q] = [r]$ and $\mu(p, q) = 1 = \mu(q, r)$. Let $u \in \widehat{\mathcal{M}}(p, q)$ and $v \in \widehat{\mathcal{M}}(q, r)$. Then the* **gluing procedure** *$\#$ for u and v yields an immersed heart $w := v\#u \in \widehat{\mathcal{N}}(p, r)$.*

PROOF : Recall from Remark 3.5 and figure 3.4 the geometric positions of p, q and r forced by the index and the existence of u and v. Exemplarily consider the configuration of figure 3.4 (a) for $q = q_0$ which is resketched on the left of figure 3.7. The gluing construction glues $u \in \mathcal{M}(p, q_0)$ and $v \in \mathcal{M}(q_0, r)$ along the common boundary segment $[p, q_0]_0$. After adjusting the domain of definition this yields $w := v\#u \in \mathcal{N}(p, r)$ as sketched in figure 3.7. For technical details we refer to Chekanov [**Che**], de Silva [**dS**], Gautschi & Robbin & Salamon [**GauRS**] or Robbin [**R**]. ∎

Now we turn to the cutting procedure. The proof of the cutting construction differs considerably from the one for *compact* Lagrangian submanifolds which can be found in Chekanov [**Che**], de Silva [**dS**], Gautschi & Robbin & Salamon [**GauRS**] or Robbin [**R**].

For the existence of the 'cutting points' we need the *existence of certain homoclinic points*, more precisely we define

Definition 3.15. *We call L_0 and L_1* **strongly intersecting** *(w.r.t. x) if each branch of L_0 intersects each branch of L_1, i.e. $L_i^+ \cap L_j^+ \neq \emptyset \neq L_i^- \cap L_j^+$ for $i,\ j \in \{0, 1\}$ and $i \neq j$.*

We call a set **generic** (in the sense of Baire) if it is a countable intersection of open and dense sets. A property is called **generic** if it holds on a generic set.

Generically homoclinic points are transverse. If M is an orientable closed surface and if $\mathrm{Diff}_\omega(M)$ carries the C^1-topology then to be strongly intersecting w.r.t a fixed point is a generic property of a symplectomorphism (see Takens [**Ta**] for dimension two and Xia [**Xia1**] for higher dimensional compact manifolds). For C^r-topology with $1 \leq r \leq \infty$ there are results by Robinson, Pixton and Oliveira on S^2 and T^2 (cf. Xia [**Xia4**]). If the action of the symplectomorphism on the first homology group is irreducible then Oliveira [**Ol**] proved C^r-genericity for closed surfaces with genus $g \geq 2$. This hypothesis is not fulfilled by symplectomorphisms isotopic to the identity, for example Hamiltonian diffeomorphisms. For symplectomorphisms isotopic to the identity on closed surfaces Xia [**Xia3**] proved strongly intersecting to be C^r-generic.

Now we can formulate the cutting theorem which describes the 'inverse' procedure to the gluing construction.

Theorem 3.16 (Cutting). *Let L_0 and L_1 be strongly intersecting and transverse. Let $p,\ r \in \mathcal{H}$ with $[p] = [r]$ and $\mu(p, r) = 2$ and $w \in \mathcal{N}(p, r)$. Then there are unique $q_0,\ q_1 \in \mathcal{H}$ with $\mu(p, q_i) = 1 = \mu(q_i, r)$ and $u_i \in \mathcal{M}(p, q_i)$, $v_i \in \mathcal{M}(q_i, r)$ such that $v_i\#u_i = w$ for $i \in \{0, 1\}$.*

Before we turn to the proof of Theorem 3.16 we need to know in which way a symplectomorphism can act on its (un)stable manifolds. The following two classical theorems can be found for instance in Arrowsmith & Place [**ArP**]. First recall

Theorem 3.17 (Hartman-Grobman). *Let x be a hyperbolic fixed point of $\varphi \in \mathrm{Diff}(M)$. Then there is a neighbourhood $U \subset M$ of x and a neighbourhood $V \subset T_xM$ of $0_x \in T_xM$ such that $\varphi|_U$ is topologically conjugate to $D\varphi(x)|_V$.*

This is wrong for non-hyperbolic fixed points since in that case also the higher derivatives are important for the behaviour near x.

Theorem 3.17 relates the local behaviour of φ around x to the linearization $D\varphi(x)$. If we want to distinguish the behaviour of our systems near x we have to ask how many linear systems with 'different' behaviour exist.

Theorem 3.18. *Let A and B be two hyperbolic real matrices (i.e. no eigenvalues of modulus 1) and see them as linear diffeomorphisms of $\mathbb{R}^n$ and denote by E^i_A and E^i_B for $i \in \{u, s\}$ their (un)stable eigenspaces. Then A and B are topologically conjugate if and only if:*

(1) $\dim E^s_A = \dim E^s_B$ *(or equivalently* $\dim E^u_A = \dim E^u_B$*).*
(2) For $i \in \{s, u\}$ the restrictions $A|_{E^i_A}$ and $B|_{E^i_B}$ are either both orientation preserving or both orientation reversing.

If we count now the possible combinations for different behaviour we notice $4n$ topological types of hyperbolic linear diffeomorphisms on $\mathbb{R}^n$. Thus there are $4n$ types of local behaviour of a diffeomorphism on a n-dimensional manifold around a hyperbolic fixed point.

In this work we are mostly interested in symplectic diffeomorphisms which simplifies the situation considerably:

Corollary 3.19. *There are only 2 types of local behaviour of a* **symplectomorphism** *φ on an $2n$-dimensional manifold around a hyperbolic fixed point x: Either φ is orientation preserving on $W^s(x, \varphi)$ and $W^u(x, \varphi)$ or φ is orientation reversing on both.*

Proof : Let φ be a symplectomorphism with hyperbolic fixed point x. If λ is an eigenvalue of $D\varphi(x)$ so are $\bar{\lambda}$, $\frac{1}{\lambda}$ and $\frac{1}{\bar{\lambda}}$. Since $|\lambda| \neq 1$ we get w.l.o.g. $|\lambda| = |\bar{\lambda}| > 1$ and $|\frac{1}{\lambda}| = |\frac{1}{\bar{\lambda}}| < 1$ and therefore always $\dim W^s(x, \varphi) = \dim W^u(x, \varphi) = n$. From $\det D\varphi(x) = 1$ follows $D\varphi(x)$ orientation preserving and therefore either orientation preserving on the stable and unstable eigenspace or orientation reversing on both. Using Theorem 3.17 and Theorem 3.18 this leaves exactly two possibilities. ∎

In accordance to our convention $L_0 = W^u(x, \varphi)$ and $L_1 := W^s(x, \varphi)$ we define

Definition 3.20. *φ is called* L**-orientation preserving (reversing)** *if φ is orientation preserving (reversing) on L_0 and L_1.*

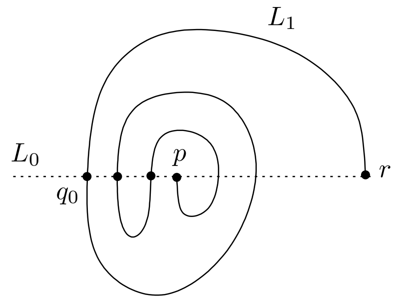


FIGURE 3.8. Motivation for the definition of q_0

The next statement will provide us with information of the oscillation behaviour of the (un)stable manifolds near the fixed point under the assumption of transverse intersection points.

Let N_1 and N_1 be two submanifolds of the manifold M and $i_1 : N_1 \to M$ and $i_2 : N_2 \to M$ their inclusions. Then N_2 is **ε-C^k-close to** N_1 if there is a diffeomorphism $\psi : N_1 \to N_2$ such that $\|i_1 - i_2 \circ \psi\|_{C^k} < \varepsilon$.

THEOREM 3.21 (λ-lemma [**Pa**]). *Let M be a compact m-dimensional manifold or $\mathbb{R}^m$ and $\varphi : M \to M$ a C^k-diffeomorphism with hyperbolic fixed point x. Let $\dim W^u(x, \varphi) = m_u$ and let $D^u \subset W^u(x, \varphi)$ be a small m_u-dimensional disk centered around x. Let $p \in W^s(x, \varphi)$ and let D be a m_u-dimensional disk around p intersecting $W^s(x, \varphi)$ transversely. Then $\bigcup_{n \geq 0} \varphi^n(D)$ contains an m_u-disk arbitrarily C^k-close to D^u.*

After these preparations we are able to turn to the proof of Theorem 3.16.

PROOF of Theorem 3.16: According to Corollary 3.19 φ is either L-orientation preserving or L-orientation reversing. Let us start with the *L-orientation preserving case* and assume p to be the concave vertex of $w \in \mathcal{N}(p, r)$. Thus the domain of definition of w is the standard heart D_b.
Fix parametrizations $\gamma_i : \mathbb{R} \to L_i$ for $i \in \{0, 1\}$ satisfying w.l.o.g. $\gamma_i^{-1}(r) < \gamma_i^{-1}(p)$ and thus inducing an ordering $<_i$ according to Notation 2.3 with $r <_i p$.
According to Theorem 3.17 (Hartman-Grobman) the stable and unstable manifolds look locally around the fixed point x like the transverse intersection of the according eigenspaces in the linearization.
Given any small disk neighbourhood $D(x)$ of x in L_0 there is $n \in \mathbb{N}$ large enough such that $\varphi^{-n}(p)$ and $\varphi^{-n}(r)$ lie in $D(x)$. If we can prove the existence of 'cutting points' q_0 and q_1 for $\varphi^{-n}(p)$, $\varphi^{-n}(r)$ and $\varphi^{-n} \circ w$ then $\varphi^n(q_0)$ and $\varphi^n(q_1)$ are cutting points for p, r and w.
Now choose $D(x)$ to be the 'convergence disk' $D^u \subset L_0$ of Theorem 3.21 (λ-lemma) and assume from now on w.l.o.g. p, $r \in D^u$.

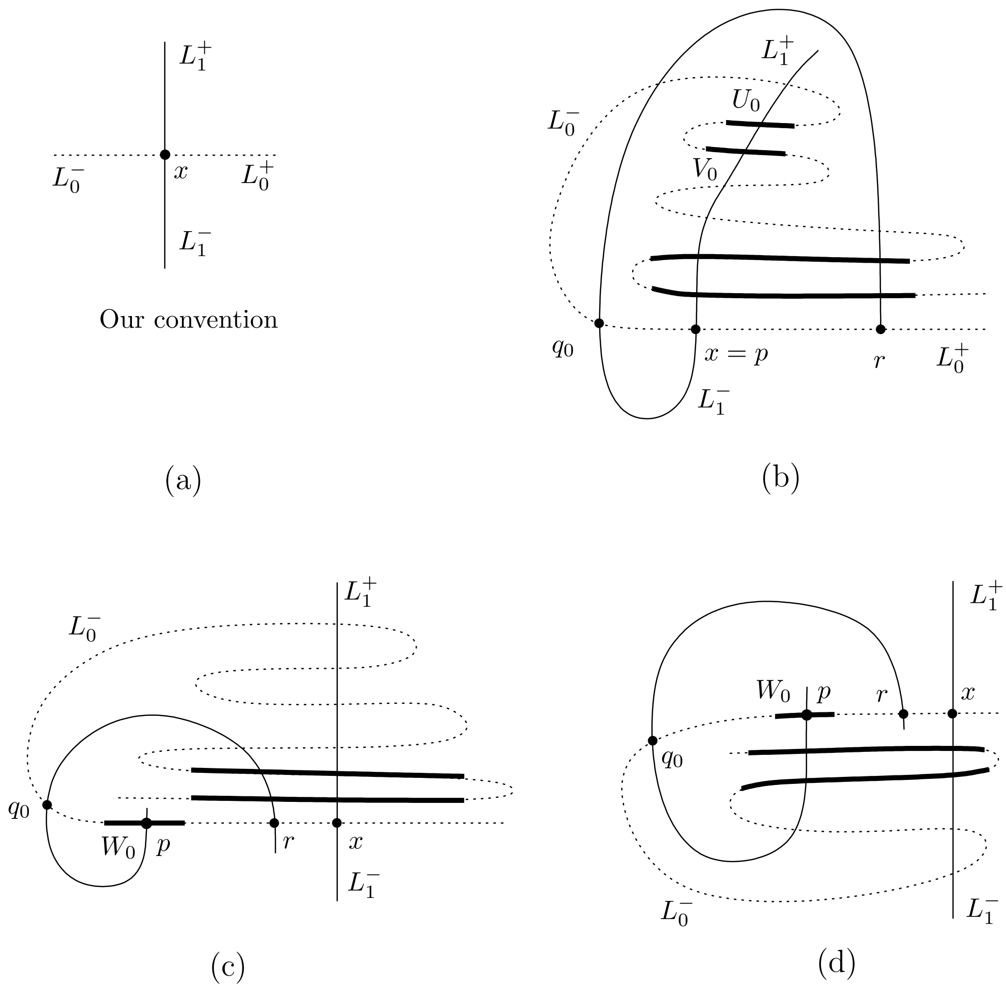


FIGURE 3.9. Constructions for q_0

The idea to find the cutting point q_0 is to follow the segment $[p, \infty[_0$. For a certain time after p it stays in the interior of w. We claim that at some point it passes $w(\partial D_b)$ to the exterior of w and the first such point will be our desired q_0. If w would be an embedding we could define

$$q_0 := \min\{q \in L_0 \mid p <_0 q,\ q \in]r, p[_1\},$$

but unfortunately for not globally injective immersions this might not yield the desired result, see figure 3.8. In order to avoid this difficulty we define formally

$$q_0 := \min\{q \in L_0 \mid p <_0 q,\ q \in]r, p[_1,\ [q, q + \varepsilon[_0 \cap w(D_b)^c \neq \emptyset \text{ for } \varepsilon > 0\}.$$

Now we prove that such a minimum always exists. We use the notation for the branches $L_0^\pm$ and $L_1^\pm$ as sketched in figure 3.9 (a).

Let us start with *case* $p = x \neq r$: For the relative positions of p and r see figure 3.9(b). As sketched in figure 3.9 (b) L_0^+ is the branch of L_0 containing r. L_1^+ is the branch of L_1 which starts in the local picture on the same side of L_0 as $[r, r+\varepsilon]_1$ for $\varepsilon > 0$ small. Since L_0 and L_1 are strongly intersecting and transverse $L_0^- \pitchfork L_1^+ \neq \emptyset$ and there is $q \in L_0^- \pitchfork L_1^+$ with $p <_0 q$. Let $t_i^q := \gamma_i^{-1}(q)$ and consider a small neighbourhood of q in L_0^-. If $\operatorname{sign}(\det(\dot\gamma_0(t_0^q), \dot\gamma_1(t_1^q)))$ is negative we denote the neighbourhood by U_0 and otherwise by V_0.
Since $L_0 \pitchfork L_1$ these neighbourhoods meet L_1 transversely in q. Now Theorem 3.21 (λ-lemma) implies the C^k-convergence of disks $D_n \subset \varphi^n(U_0)$ resp. $D_n \subset \varphi^n(V_0)$ to D^u for $n \to \infty$. Recall $r \in D^u$ and that $[r, p]_1$ intersects L_0 in r transversely. Thus for given $\varepsilon > 0$ there is n_0 large enough such that D_n and $[r, r+\varepsilon]_1$ intersect for $n \geq n_0$, see the extra bold long segments in figure 3.9 (b). Proposition 3.13 states that for $\varepsilon > 0$ small enough the ball $B_\varepsilon(r)$ splits into two wedge-shaped regions $W_{int} \subset \operatorname{Int}(w)$ and $W_{ext} \subset \operatorname{Ext}(w)$ with common boundary $([r,p]_0 \cup [r,p]_1) \cap B_\varepsilon(r)$.
Thus for n large enough D_n meets W_{ext} before or after passing $]r, r+\varepsilon[_1$ depending on if D_n lies in an iterate of U_0 or V_0. Therefore the segment $[p, \infty[_0$ leaves $\operatorname{Int}(w)$ and meets $Ext(w)$ such that points as claimed in the definition of q_0 exist and so does the minimum q_0.

Now consider the *case* $p \neq x$. Here we do not need the (un)stable manifolds to be strongly intersecting as it was necessary in case $p = x$. The sketches in figure 3.9 are schematical and it is unimportant if x lies in the exterior of w or not. Again we use the conventions for the branches from figure 3.9 (a) and assume $p \in L_0^-$ as sketched in figure 3.9 (c), (d). We have to distinguish *two subcases*, namely if $p \in L_0^- \pitchfork L_1^+$ as in (c) or if $p \in L_0^- \pitchfork L_1^-$ as in (d).
We start with $p \in L_0^- \pitchfork L_1^+$ and consider a small neighbourhood around p in L_0^-. If $\operatorname{sign}(\det(\dot\gamma_0(t_0^p), \dot\gamma_1(t_1^p)))$ is negative we denote as above the small neighbourhood by U_0 and otherwise by V_0. Since $p \in L_1^+$ the disks $D_n \subset \varphi^n(U_0)$ resp. $D_n \subset \varphi^n(V_0)$ from Theorem 3.21 (λ-lemma) approach the 'convergence disk' D^u centered at x from the L_1^+-side for $n \to \infty$, see the extra bold long segments in figure 3.9 (c). As in the proof of *case* $p = x$ we consider the special neighbourhood sectors $W_{int} \subset \operatorname{Int}(w)$ and $W_{ext} \subset \operatorname{Ext}(w)$ of r provided by Proposition 3.13 and conclude that $]p, \infty[_0$ passes somewhere through W_{ext} and therefore meets the exterior of w. Thus points as claimed in the definition of q_0 exist and so does the minimum q_0.
The *case* $p \in L_0^- \pitchfork L_1^-$ as sketched in figure 3.9 (d) proceeds analogously to *case* $p \in L_0^- \pitchfork L_1^+$ except from the following fact: Now the disks $D_n \subset \varphi^n(U_0)$ resp. $D_n \subset \varphi^n(V_0)$ from Theorem 3.21 (λ-lemma) approach the 'convergence disk' D^u from the L_1^--side for $n \to \infty$, see the extra bold long segments in figure 3.9 (d).

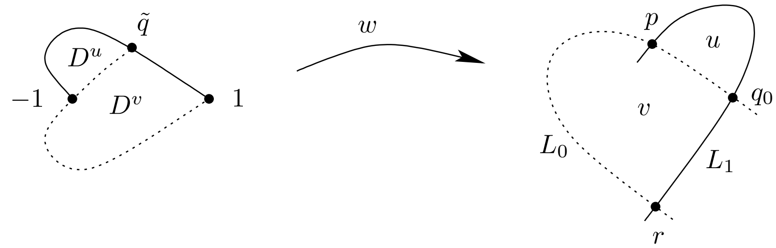


FIGURE 3.10. Cutting

Therefore we have to use the sectors $W_{int} \subset \text{Int}(w)$ and $W_{ext} \subset \text{Ext}(w)$ of p instead of those of r and then proceed as above.

Since we only need the oscillation behaviour predicted by Theorem 3.21 (λ-lemma) and those special neighbourhood sectors around the vertices the proof carries over to all possible relative positions of x, p and r within L_0 and L_1 in case $p \neq x$.
Under exchanging the roles of L_0 and L_1 the constructions for q_1 are similar to those for q_0. If r is the concave vertex the proof proceeds similarily.

Now we will describe the cutting procedure from p to q_0. The cut from p to q_1 is performed analogously.
Recall from Proposition 3.13 that w is injective on a small neighbourhood of p. If we consider $w^{-1}([p, q_0]_0)$ then there is a unique segment in D_b denoted by I whose start point is $w^{-1}(p) = -1$. By definition of q_0 the segment $[p, q_0 + \varepsilon]_0$ leaves $w(D_b)$ through q_0 for $\varepsilon > 0$. Thus there is $\tilde{q} \in w^{-1}(q_0)$ which has to be the endpoint of I. In fact, since q_0 lies per definitionem on a boundary segment parting the interior from the exterior of w we deduce from Remark 3.6 that w is injective in a neighbourhood of q_0 such that $\{\tilde{q}\} = w^{-1}(q_0)$ is even unique.
We now cut D_b along I into D_b^u and D_b^v like in figure 3.10. The boundary conditions of D_b^u are $B_0^u = I$ and B_1^u is the segment from -1 to $\tilde{q}$ in B_1. And for D_b^v we have $B_0^v = I \cup B_0$ and B_1^v is the segment from $\tilde{q}$ to 1 in B_1. Identify D_b^u and D_b^v with the di-gon D via $h^u : D_b^u \to D$ with $h^u(B_i^u) = B_i^D$ and $h^v : D_b^v \to D$ with $h^v(b_i^v) = B_i^D$ for $i \in \{0, 1\}$ and define

$$\begin{aligned} u : D \to M, \quad & u(z) := w((h^u)^{-1}(z)), \\ v : D \to M, \quad & v(z) := w((h^v)^{-1}(z)). \end{aligned}$$

Since our techniques considered the branches of the (un)stable manifolds separately the L-orientation reversing case is reduced to the L-orientation preserving case by considering the L-orientation preserving φ^2 instead of φ. ∎

4. Coherent orientations of immersed di-gons and hearts

The gluing and cutting construction impplies that with $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$-coefficients the square of the Floer boundary operator vanishes.
But we are in fact able to assign a gluing compatible sign to each immersion which makes $\mathbb{Z}$-coefficients possible for the desired homology if the symplectomorphism φ is *L-orientation preserving.* If φ is L-orientation reversing we still can define and state in the following Definition 3.22, Definition 3.23 and Lemma 3.24. But we will not be able to divide by the $\mathbb{Z}$-action as explained more detailed when defining the Floer homology.

In classical Floer theory this procedure is know as 'coherent orientations' and they arise in the Fredholm set-up from the determinant bundle. Fortunately in the two dimensional situation here we can give a brief and purely geometrical definition of this phenomenon.

If we want to assign a sign to each immersion in $\mathcal{M}(p,q)$ we need something to which we can 'compare' the immersions. This will be done by comparing the parameter direction on the L_0-boundary of the immersion with a fixed orientation on L_0.

Thus first of all, we **fix an orientation on** L_0. We will discuss later if our definitions depend on this choice.

Definition 3.22. *Let $p,\ q \in \mathcal{H}$ with $\mu(p,q) = 1$ and $u \in \mathcal{M}(p,q)$. Provide $u(B_0) = [p,q]_0$ with the orientation induced by the parametrization from p to q and define*

$$m(p,q,u) := \begin{cases} +1 & \text{if the orientation induced on } L_0 \text{ by } u(B_0) \\ & \text{and the fixed one coincide,} \\ -1 & \text{otherwise.} \end{cases}$$

The image of an immersion between two homoclinic points is determined by the unique connecting stable and unstable segments between them. Thus we can neglect the immersion in the definition and assign a 'relative' sign to the two homoclinic points in question depending on if there are immersions between them or not.

Definition 3.23. *Since $\#\widehat{\mathcal{M}}(p,q) \in \{0,1\}$ for $p,\ q \in \mathcal{H}$ with $\mu(p,q) = 1$ we can set*

$$m(p,q) := \begin{cases} m(p,q,u) & \text{if } u \in \mathcal{M}(p,q) \neq \emptyset, \\ 0 & \text{if } \mathcal{M}(p,q) = \emptyset. \end{cases}$$

And setting $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ we define $m_2(p,q) \in \mathbb{Z}_2$ via $m_2(p,q) := m(p,q) \bmod 2$.

Now we show the skew-symmetry for the individual signs of the endpoints of u_i and v_i involved in the cutting and gluing construction.

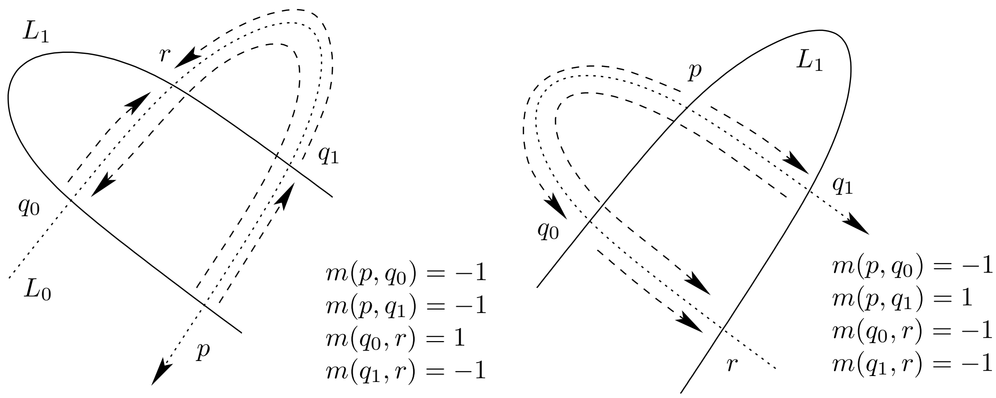


FIGURE 3.11. Coherent orientations

LEMMA 3.24. *Let p, $r \in \mathcal{H}$ with $\mu(p, r) = 2$ and $w \in \mathcal{N}(p, r)$. For $i \in \{0, 1\}$ consider $q_i \in \mathcal{H}$ with $\mu(p, q_i) = 1 = \mu(q_i, r)$ and $u_i \in \mathcal{M}(p, q_i)$ and $v_i \in \mathcal{M}(q_i, r)$ such that $v_i \# u_i = w$. Then*

$$m(p, q_0) \cdot m(q_0, r) = -m(p, q_1) \cdot m(q_1, r)$$

and this relation also is true for m_2.

PROOF : In figure 3.11 the two possible cuts of $w \in \mathcal{N}(p, r)$ and the signs w.r.t. a chosen orientation on L_0 are sketched. We calculate

$$\begin{aligned} m(p, q_0) \cdot m(q_0, r) &= (-1) \cdot 1 = -1 = -(-1) \cdot (-1) = -m(p, q_1) \cdot m(q_1, r), \\ m(p, q_0) \cdot m(q_0, r) &= (-1) \cdot (-1) = 1 = -(1 \cdot (-1)) = -m(p, q_1) \cdot m(q_1, r). \end{aligned}$$

If in the figure the other orientation on L_0 is chosen all signs swap and the relation remains true. ■

The coefficients of the Floer differential will be defined using the signs defined in Definition 3.23. The vanishing of the square of the Floer differential will be due to Lemma 3.24.

Now we discuss if Definition 3.22, Definition 3.23 and Lemma 3.24 depend on the choice of the orientation. If we choose in the beginning the other orientation of L_0 all signs in $m(p, q)$ etc. swap and therefore Lemma 3.24 remains true. If we take L_1 as reference instead of L_0 the definition of the signs proceeds analogously and Lemma 3.24 is valid.
Now consider L_i for $i \in \{0, 1\}$ simultanously and recall $\mathcal{H}^n_{[x]} = \{p \in \mathcal{H} \mid \mu(p, x) = n, [p] = [x]\}$. Choose parametrizations $\gamma_i : \mathbb{R} \to L_i$ with $\gamma_i(0) = x$ and provide L_i with the orientation induced by $\dot{\gamma}_i$. Let $\sigma_{01} := \operatorname{sign}(\det(\dot{\gamma}_0(0), \dot{\gamma}_1(0)))$ and denote

the signs defined by means of the orientation on L_i by $m(p,q,L_i)$. Then

$$
\begin{aligned}
m(p,q,L_0) &= \sigma_{01} m(p,q,L_1) && \text{for } p \in \mathcal{H}^{2n}_{[x]}, \\
m(p,q,L_0) &= -\sigma_{01} m(p,q,L_1) && \text{for } p \in \mathcal{H}^{2n+1}_{[x]}
\end{aligned}
\tag{3.25}
$$

for all $q \in \mathcal{H}_{[x]}$ and $n \in \mathbb{Z}$.

CHAPTER 4

Primary homoclinic Floer homology

In this chapter we give the definition of primary homoclinic points and discuss some properties of immersions of relative index one and two between them. We show that the gluing and cutting procedure is compatible with the restriction to primary homoclinic points. This enables us to define a Floer homology generated by the primary homoclinic points, called *primary homoclinic Floer homology*.

Within this chapter the symplectic manifold (M,ω) is either a closed surface with genus $g \geq 1$ or $(\mathbb{R}^2,\omega)$. $\varphi : M \to M$ is a symplectomorphism with hyperbolic fixed point x with (un)stable manifolds $L_0 := W^u(x,\varphi)$, $L_1 := W^s(x,\varphi)$ and $\mathcal{H} := L_0 \cap L_1$. The connected components of $L_i \backslash \{x\}$ are denoted by L_i^+ and L_i^- for $i \in \{0,1\}$ and are called the branches of the (un)stable manifolds.
There is a $\mathbb{Z}$-action of φ on $\mathcal{H}$ given by $\mathbb{Z} \times \mathcal{H} \to \mathcal{H}$, $(n,p) \mapsto \varphi^n(p)$ which is free on $\mathcal{H}\backslash\{x\}$. Usually we abbreviate $p^n := \varphi^n(p)$ etc. for homoclinic points p and $n \in \mathbb{Z}$. If not stated otherwise we assume all appearing homoclinic points to be contractible. Our convention for the Maslov index is $\mu(p) := \mu(p,x)$ and therefore

$$\mu(p,q) = \mu(p,x) + \mu(x,q) = \mu(p,x) - \mu(q,x) = \mu(p) - \mu(q).$$

1. Primary homoclinic points

Poincaré [**Po1, Po2**] discovered the existence of transverse homoclinic points. He pointed out the complicated structure of the (un)stable manifolds and the existence of many, many further homoclinic points due to the oscillation and resulting overlap of the stable and unstable manifold. Nevertheless there is a subset which will turn out to be finite modulo $\mathbb{Z}$-action and admit the definition of Lagrangian Floer theory. Recall $\mathcal{H}_{[x]} := \{p \in \mathcal{H} \mid [p] = [x]\}$.

Definition 4.1. *$p \in \mathcal{H}\backslash\{x\}$ is called* **semi-primary** *if* $]x,p[_0 \cap\,]x,p[_1 = \emptyset$. $p \in \mathcal{H}_{[x]}\backslash\{x\}$ *is* **primary** *if* $]x,p[_0 \cap\,]x,p[_1 \cap \mathcal{H}_{[x]} = \emptyset$. *Nonprimary points are called* **secondary**.

In figure 5.2 and 5.3 the extra bold intersection points of L_0 and L_1 different from the fixed point x are primary.

Semi-primary points play a crucial role in the literature around the Melnikov method (see Appendix A and for example Rom-Kedar [**RK1, RK2**]).

Clearly iterates of a (semi-)primary point are again (semi-)primary. We require $[p] = [x]$ in the definition of primary points, since this condition was already necessary for the invariance of the Maslov index and the homotopy classes of homoclinic points under the $\mathbb{Z}$-action of φ. The condition '$\ldots \cap \mathcal{H}_{[x]}$' will be necessary in the invariance discussion.

For $L_0 \cap L_1 \neq \emptyset$ semi-primary points always exist, but they are not necessarily contractible, compare the example beginning on page 76.

LEMMA 4.2. *(1) Let φ be L-orientation preserving, consider $p \in \mathcal{H}$ semi-primary and denote the branches containing p by L_0^p and L_1^p. Then for every semi-primary $q \in (L_0^p \cap L_1^p) \backslash \{p^n \mid n \in \mathbb{Z}\}$ there is a unique $n \in \mathbb{Z}$ such that $q^n \in \,]p, p^1[_0 \cap \,]p, p^1[_1$.*

If φ is L-orientation reversing then p^1 has to be replaced by p^2 and n by $2n$.

(2) For primary points the analogon of the above item also is true.

(3) Let p be semi-primary and q primary within the same pair of branches. If $q \in \,]p, p^1[_0$ then $q \notin \,]x, p^1[_1$ and if $q \in \,]p, p^1[_1$ then $q \notin \,]x, p[_0$.

PROOF : *First item:* Let φ be L-orientation preserving and consider a semi-primary $q \in L_0^p \cap L_1^p$ with $q \notin \{p^n \mid n \in \mathbb{Z}\}$. There is a unique $n \in \mathbb{Z}$ such that $q^n \in \,]p, p^1[_1$ and w.l.o.g. $n = 0$. We want to know where q lies in L_0^p.
Keep in mind that $]p, x[_1 = \,]p^1, x[_1 \cup \{p^1\} \cup \,]p, p^1[_1$ and that p semi-primary means $]p, x[_0 \cap \,]p, x[_1 = \emptyset$ and therefore $q \notin \,]p, x[_0$. Now assume $q \in L_0^p \backslash \,]p^1, x[_0$. This implies $p^1 \in \,]q, x[_0$ and since also p^1 in $]q, x[_1$ the point q cannot be semi-primary – contradiction. This leaves $q \in \,]p, p^1[_0$ as only possibility.
If φ is L-orientation reversing then φ^2 is L-orientation preserving and the claim follows from the L-orientation preserving case.
The proofs of the *second* and *third item* proceed analogously. ■

REMARK 4.3. *Let p be semi-primary and q primary within the same pair of intersecting branches. Then there is $k \in \mathbb{N}_0$, $n \in \mathbb{Z}$ such that $q \in \,]p^n, p^{n+1}[_0 \cap \,]p^{n+k}, p^{n+k+1}[_1$. There are schematic tangles with $k > 0$.*

Now consider the universal covering $\tau : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ with $\tilde{\omega} = \tau^*\omega$. For $\tilde{x} \in \tau^{-1}(x)$ denote by $\tilde{L}_i(\tilde{x})$ the lift of L_i passing through $\tilde{x}$ for $i \in \{0, 1\}$ and let $\tilde{x}_0$, $\tilde{x}_1 \in \tau^{-1}(x)$. $\tilde{p} \in \tilde{L}_0(\tilde{x}_0) \cap \tilde{L}_1(\tilde{x}_1)$ is called **homoclinic** if $\tilde{x}_0 = \tilde{x}_1$ and otherwise **heteroclinic**.
The lift of the segment $[p, q]_i$ starting in $\tilde{p} \in \tau^{-1}(p)$ and ending in $\tilde{q} \in \tau^{-1}(q)$ we denote by $[\tilde{p}, \tilde{q}]_i$.

Let $p \in \mathcal{H}$ and $\tilde{p} \in \tau^{-1}(p)$. If $[p] = [x]$ then the lifts of $[p, x]_0$ and $[p, x]]_1$ starting at $\tilde{p}$ end at the same point $\tilde{x} \in \tau^{-1}(x)$. Thus $\tilde{p} \in \tilde{L}_0(\tilde{x}) \cap \tilde{L}_1(\tilde{x})$, i.e. contractible homoclinic points lift to homoclinic points.

If $[p] \neq [x]$ then the lifts of $[p,x]_0$ and $[p,x]_1$ starting at $\tilde{p}$ end at two different points $\tilde{x}_0$, $\tilde{x}_1 \in \tau^{-1}(x)$. Thus $\tilde{p} \in \tilde{L}_0(\tilde{x}_0) \cap \tilde{L}_1(\tilde{x}_1)$, i.e. noncontractible homoclinic points lift to heteroclinic points.

Definition 4.4. *$\tilde{p} \in \tilde{L}_0(\tilde{x}) \cap \tilde{L}_1(\tilde{x})$ is* **primary** *if $]\tilde{p},\tilde{x}[_0 \cap]\tilde{p},\tilde{x}[_1 = \emptyset$.*

Note that there are no noncontractible points in $\tilde{L}_0(\tilde{x}) \cap \tilde{L}_1(\tilde{x})$.

Notation 4.5. *Now fix some $\tilde{x} \in \tau^{-1}(x)$ and consider the tangle generated by L_0 and L_1.* **Lifting the tangle (to $\tilde{x}$)** *means that we consider the tangle generated by $\tilde{L}_i := \tilde{L}_i(\tilde{x})$ for $i \in \{0,1\}$ on $\tilde{M}$. To contractible $p \in L_0 \cap L_1$ we associate $\tilde{p} \in \tau^{-1}(p)$ such that the lift of $[p,x]_i$ starting in $\tilde{p}$ ends in $\tilde{x}$. To noncontractible p we associate $\tilde{p}$ such that the lift of $[p,x]_0$ starting in $\tilde{p}$ ends in $\tilde{x} = \tilde{x}_0$.*

Let $\tilde{p}$, $\tilde{q} \in \tilde{L}_0 \cap \tilde{L}_1$. If $\mu(\tilde{p},\tilde{q}) = 1$ we define analogously $\mathcal{M}(\tilde{p},\tilde{q})$ and $\widehat{\mathcal{M}}(\tilde{p},\tilde{q})$ and if $\mu(\tilde{p},\tilde{q}) = 2$ then $\mathcal{N}(\tilde{p},\tilde{q})$ and $\widehat{\mathcal{N}}(\tilde{p},\tilde{q})$.

For the following constructions consider the tangle lifted to $\tilde{x} \in \tau^{-1}(x)$. The results are independent of the chosen reference point $\tilde{x}$.

Since primary points in $L_0 \cap L_1$ are contractible they lift to homoclinic points. Moreover we notice

Remark 4.6. *$p \in L_0 \cap L_1$ is primary if and only if $\tilde{p} \in \tilde{L}_0 \cap \tilde{L}_1$ is primary. Moreover Lemma 4.2 holds also for the primary points in $\tilde{L}_0 \cap \tilde{L}_1$.*

The property ‘primary’ has the following geometric implications:

Lemma 4.7. *Let $\tilde{p} \in \tilde{L}_0 \cap \tilde{L}_1$ be primary. Then $\mu(\tilde{p}) := \mu(\tilde{p},\tilde{x}) \in \{\pm1,\pm2,\pm3\}$. There is either an embedded digon or an embedded heart or an embedded 2-gons with two concave vertices from $\tilde{p}$ to $\tilde{x}$ (resp. from $\tilde{x}$ to $\tilde{p}$ depending on the sign of the index). For the primary $p := \tau(\tilde{p})$ follows $\mu(p) := \mu(p,x) \in \{\pm1,\pm2,\pm3\}$.*

Proof : Since $[\tilde{p}] = [\tilde{x}]$ the two points can be connected by a path in $\mathcal{P}(L_0,L_1)$. Since $]\tilde{p},x[_0 \cap]\tilde{p},x[_1 = \emptyset$ the region enclosed by $[\tilde{p},\tilde{x}]_0$ and $[\tilde{p},\tilde{x}]_1$ is an embedded polygon with two vertices. Assume the intersections in $\tilde{p}$ and $\tilde{x}$ to be orthogonal and parametrize the segment $[\tilde{p},\tilde{x}]_0$ from $\tilde{p}$ to $\tilde{x}$ and $[\tilde{p},\tilde{x}]_1$ from $\tilde{x}$ to $\tilde{p}$. Then the Maslov index is twice the winding number of the tangent vector of the segments. Thus only $\mu(\tilde{p},\tilde{x}) \in \{\pm1,\pm2,\pm3\}$ can be realized without violating the boundary condition $]\tilde{p},\tilde{x}[_0 \cap]\tilde{p},\tilde{x}[_1 = \emptyset$. Due to Lemma 3.12 we have $\mu(p) \in \{\pm1,\pm2,\pm3\}$ also for $p = \tau(\tilde{p})$. ■

Since primary $p \in L_0 \cap L_1$ might have noncontractible points in $]x,p[_0 \cap]x,p[_1$ the immersion between p and x needs not to be globally injective.

Now we investigate the geometric positions of primary points on $\tilde{M}$ w.r.t. each other.

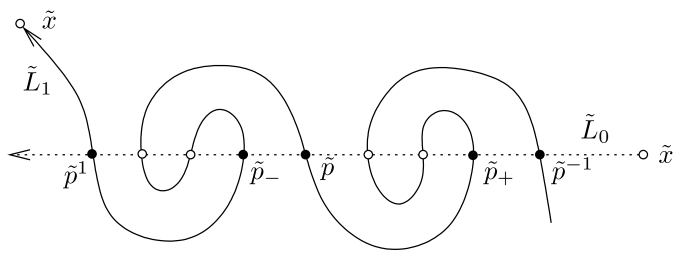


FIGURE 4.1. Adjacent primary points

LEMMA 4.8. *$\tilde{p} \in \tilde{L}_0 \cap \tilde{L}_1$ be primary and $p := \tau(\tilde{p})$. For $i \in \{0,1\}$ let $\gamma_i : \mathbb{R} \to L_i$ be a parametrization with $\gamma_i^{-1}(p) < \gamma_i^{-1}(\varphi(p))$ (for L-orientation reversing φ use $\gamma_i^{-1}(p) < \gamma_i^{-1}(\varphi^2(p))$. Parametrize $\tilde{\gamma}_i : \mathbb{R} \to \tilde{L}_i$ such that $\tau \circ \tilde{\gamma}_i = \gamma_i$ and obtain the ordering $<_i$ on $\tilde{L}_i$. Then*

$$\tilde{p}_+ := \max\{\tilde{q} \in \tilde{L}_1 \mid \tilde{q} <_1 \tilde{p},\ \tilde{q} \in]\tilde{x}, \tilde{p}[_0\},$$
$$\tilde{p}_- := \min\{\tilde{q} \in \tilde{L}_0 \mid \tilde{p} <_0 \tilde{q},\ \tilde{q} \in]\tilde{x}, \tilde{p}[_1\}$$

are primary and $\tilde{p}_\pm$ is called **adjacent** *to $\tilde{p}$.*

PROOF : Primary means $]\tilde{x}, \tilde{p}[_0 \cap]\tilde{p}, \tilde{x}[_1 = \emptyset$ and by definition of $\tilde{p}_+$ we have $]\tilde{p}_+, \tilde{x}[_1 =]\tilde{p}_+, \tilde{p}[_1 \cup [\tilde{p}, \tilde{x}[_1$ and $]\tilde{x}, \tilde{p}[_0 =]\tilde{x}, \tilde{p}_+[_0 \cup [\tilde{p}_+, \tilde{p}[_0$. Thus $]\tilde{p}_+, \tilde{x}[_1 \cap]\tilde{x}, \tilde{p}_+[_0 =]\tilde{p}_+, \tilde{p}[_1 \cap]\tilde{x}, \tilde{p}_+[_0 = \emptyset$ due to the maximality of $\tilde{p}_+$. The proof for $\tilde{p}_-$ is similar. ■

An example is sketched in figure 4.1 where we 'splitted' the fixed point into two copies in favour of a smaller sketch. Primary points are printed extra bold and $\{\tilde{p}^n\} = \tilde{L}_0 \cap \tilde{L}_1 \cap \tau^{-1}(p^n)$. We deduce

COROLLARY 4.9.
(1) Let $\tilde{p}$ be primary and $\tilde{q} = \tilde{p}_\pm$. Then $]\tilde{p}, \tilde{q}[_0 \cap]\tilde{p}, \tilde{q}[_1 = \emptyset$. If moreover $\tilde{p}$ and $\tilde{q}$ are transverse then $\mu(\tilde{p}, \tilde{q}) \in \{1, -1\}$ and there is an embedded di-gon between them.
(2) Let $\tilde{p}$ be primary and order the primary points in $[\tilde{p}, \tilde{p}^{-1}]_0 \cap [\tilde{p}, \tilde{p}^{-1}]_1$ via $\tilde{p}$, $\tilde{p}_+$, $(\tilde{p}_+)_+, \dots, \tilde{p}^{-1}$ and assume them transverse. Then their relative Maslov index alternates between $+1$ and -1.
(3) Let all primary points $p \in L_0 \cap L_1$ be transverse. Then there are modulo $\mathbb{Z}$-action only finitely many primary points. The same is true for the primary points in $\tilde{L}_0 \cap \tilde{L}_1$.

PROOF : *First item:* $]\tilde{p}, \tilde{q}[_0 \cap]\tilde{p}, \tilde{q}[_1 = \emptyset$ is clear. Now assume $\tilde{p}$ and $\tilde{q}$ transverse and consider the embedded 2-gon between $\tilde{p}$ and $\tilde{x}$. Only the vertex type at $\tilde{p}$ (convex or concave) is important for the relative index between $\tilde{p}$ and $\tilde{p}_+$ and it can only be ± 1. Since $]\tilde{p}_+, \tilde{p}[_0 \cap]\tilde{p}_+, \tilde{p}[_1 = \emptyset$ there is an embedded di-gon between them.

Second item: Follows from the first item and Lemma 4.8.
Third item: Let $p \neq q$ be primary. Every orbit $(q^n)_{n\in\mathbb{Z}}$ has exactly one representant in $[p,p^1]_0 \cap [p,p^1]_1$ according to Lemma 4.2. Since all primary points are transverse the second item and the compactness of $[p,p^1]_0$ and $[p,p^1]_1$ imply the claim. ∎

2. Primary homoclinic Floer homology

In this section we will define the primary homoclinic Floer homology: We will take the primary homoclinic points graded by $\mu(\cdot) := \mu(\cdot, x)$ as generators of the Floer complex. The differential will count immersions only to other primary homoclinic points. Then we will divide by the $\mathbb{Z}$-action in order to obtain finite rank over $\mathbb{Z}$ resp. $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. The proofs are postponed to the following sections in order to gain better readability.

From now on we assume all primary points to be transverse.

If not stated otherwise all appearing homoclinic points (except from x itself) are primary within this section.

Definition 4.10. *We define*

$$\begin{aligned}
&\mathcal{H}_{pr} := \{p \in \mathcal{H} \mid p \text{ primary}\},\\
&p \sim q \text{ for } p,\ q \in \mathcal{H}_{pr} \quad \text{if and only if} \quad \exists\, n \in \mathbb{Z} : q^n = p,\\
&\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/_{\sim}
\end{aligned}$$

an denote by $\langle p \rangle$ the equivalence class of p w.r.t. the equivalence relation $\sim$.

$\tilde{\mathcal{H}}_{pr}$ is finite as was shown in Lemma 4.9.

Using Lemma 2.7 and Proposition 2.11 we can establish a well-defined homotopy class and a Maslov index for the equivalence class.

Definition 4.11. *Setting*

$$[\langle p \rangle] := [p], \quad \mu(\langle p \rangle, \langle q \rangle) := \mu(p,q) \quad \text{and} \quad \mu(\langle p \rangle) := \mu(p,x)$$

is well-defined.

Before we define the chain complex we have to think about the coefficients. As already mentioned before defining and stating Definition 3.22, Definition 3.23 and Lemma 3.24 we have to distinguish if φ is L-orientation preserving or reversing. Now fix an orientation on L_0.

First assume φ to be *L-orientation preserving* and recall the signs defined in Definition 3.23.

Definition 4.12. *Let φ be L-orientation preserving. We define*

$$\mathfrak{C}_m := \mathfrak{C}_m(x, \varphi; \mathbb{Z}) := \bigoplus_{\substack{p \in \mathcal{H}_{pr} \\ \mu(p)=m}} \mathbb{Z}p,$$

$$\mathfrak{d}_m : \mathfrak{C}_m \to \mathfrak{C}_{m-1}, \qquad \mathfrak{d}(p) = \sum_{\substack{q \in \mathcal{H}_{pr} \\ \mu(q)=\mu(p)-1}} m(p,q)q$$

on a generator p and extend $\mathfrak{d}$ by linearity. φ induces $\varphi_* : \mathfrak{C}_* \to \mathfrak{C}_*$ satisfying

$$\varphi_* \circ \mathfrak{d} = \mathfrak{d} \circ \varphi_*.$$

The sum is finite since $\tilde{\mathcal{H}}_{pr}$ is finite due to Lemma 4.9 and $\#\{n \in \mathbb{Z} \mid \mathcal{M}(p, q^n) \neq \emptyset\} < \infty$ due to Proposition 4.27 as we will see later on.

Unfortunately $\mu(p) = \mu(p^n)$ for $n \in \mathbb{Z}$ implies that the chain groups have infinite rank over $\mathbb{Z}$. But since $\mu(p) := \mu(p, x) \in \{\pm 1, \pm 2, \pm 3\}$ for $p \in \mathcal{H}_{pr}$ due to Lemma 4.7 there are at most six nonvanishing chain groups.

The next theorem enables us to pass to the homology since

Theorem 4.13. *Let φ be L-orientation preserving. Then $\mathfrak{d} \circ \mathfrak{d} = 0$, i.e. $(\mathfrak{C}_*, \mathfrak{d}_*)$ is a chain complex.*

The proof of Theorem 4.13 is postponed to the following sections. Now we define the homology of $(\mathfrak{C}_*, \mathfrak{d})$ via

Definition 4.14. *Let φ be L-orientation preserving and define*

$$\mathfrak{H}_m := \mathfrak{H}_m(x, \varphi; \mathbb{Z}) := \frac{\ker \mathfrak{d}_m}{\operatorname{Im} \mathfrak{d}_{m+1}}.$$

$\mathfrak{H}_*$ does not depend on the choice of the orientation for the definition of the signs in $m(p,q)$ from Definition 3.23.

Proof : We recall from the discussion after Lemma 3.24 that changing the orientation of L_0 changes the sign of the $m(p,q)$. This means that $\mathfrak{d}$ transforms into $-\mathfrak{d}$ which has by linearity the same kernel and image as $\mathfrak{d}$. (3.25) implies that the differential obtained by using an orientation on L_1 instead of L_0 equals for fixed Maslov index ± 1 times the L_0-induced differential. Thus $\ker \mathfrak{d}_k^{L_0} = \ker \mathfrak{d}_k^{L_1}$ and $\operatorname{Im} \mathfrak{d}_k^{L_0} = \operatorname{Im} \mathfrak{d}_k^{L_1}$ for all k such that the homologies coincide. ∎

Since the chain groups have infinite rank over $\mathbb{Z}$ this might also happen for the homology groups. In order to get rid of this unconvenience we will now divide by the $\mathbb{Z}$-action of φ.

Definition 4.15. *Let φ be L-orientation preserving. For $\langle p \rangle$, $\langle q \rangle \in \tilde{\mathcal{H}}_{pr}$ set*

$$m(\langle p \rangle, \langle q \rangle) := \sum_{n \in \mathbb{Z}} m(p, q^n)$$

and define

$$C_m := C_m(x,\varphi;\mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle)=m}} \mathbb{Z}\langle p\rangle,$$

$$\partial_m : C_m \to C_{m-1}, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity. The compatibility with the Maslov index and the homotopy classes implies that ∂ is well-defined.

Since $\tilde{\mathcal{H}}_{pr}$ is finite so is the rank of C_m over $\mathbb{Z}$, more precisely $\mathrm{rk}_{\mathbb{Z}}(C_m) = \#\{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \mid \mu(\langle p\rangle) = m\}$. Furthermore due to Lemma 4.7 at most $C_{\pm 1}$, $C_{\pm 2}$ and $C_{\pm 3}$ are nonzero. And Lemma 4.9 implies $\mathrm{rk}_{\mathbb{Z}} C_{\pm 2} = \mathrm{rk}_{\mathbb{Z}} C_{\pm 1} + \mathrm{rk}_{\mathbb{Z}} C_{\pm 3}$.
If we generalize the notion of equivalence classes to finite sums via $\langle p+q\rangle = \langle p\rangle + \langle q\rangle$ the differential can also be written as

$$\partial\langle p\rangle = \langle \mathfrak{d}p\rangle = \sum_{\substack{q\in\mathcal{H}_{pr} \\ \mu(q)=\mu(p)-1}} m(p,q)\langle q\rangle.$$

Therefore $\mathfrak{d}^2 = 0$ implies immediately

Theorem 4.16. *Let φ be L-orientation preserving. Then*

$$\partial \circ \partial = 0,$$

i.e. (C_*, ∂_*) is a chain complex.

And we can proceed to the homology groups:

Definition 4.17. *Let φ be L-orientation preserving. We define the* **primary homoclinic Floer homology of φ in** *x as*

$$H_m := H_m(x,\varphi;\mathbb{Z}) := \frac{\ker \partial_m}{\mathrm{Im}\, \partial_{m+1}}.$$

H_m does not depend on the choice of the orientation for the same reasons as $\mathfrak{H}_m$.

Since already the C_m have finite rank over $\mathbb{Z}$ so has H_m and at most the homology groups $H_{\pm 1}$, $H_{\pm 2}$ and $H_{\pm 3}$ are nonzero.

Now we consider symplectomorphisms φ which are *L-orientation reversing.*

In order to define the signs $m(p,q)$ in Definition 3.23 we compare the orientation of $u(B_0)$ induced on L_0 to the fixed one of L_0. Since φ is orientation reversing on L_0 the positions of p and q relative to each other are exchanged under φ. Therefore the orientation induced by the parametrization of

$\varphi(u(B_0))$ on L_0 does not coincide with the one induced by $u(B_0)$. Since we keep the orientation which was fixed in the beginning on L_0 unchanged we obtain $m(p,q) = -m(\varphi(p),\varphi(q))$. Considering $\mathfrak{d}p = \sum_{\mu(q)=\mu(p)-1} \mu(p,q)q$ and $\mathfrak{d}\varphi(p) = \sum_{\mu(q)=\mu(p)-1} \mu(\varphi(p),\varphi(q))\varphi(q)$ such that we cannot pass to the equivalence classes as in the orientation preserving case. However, using the $\mathbb{Z}_2$-signs m_2 from Definition 3.23 we can proceed as in the L-orientation preserving case.

DEFINITION 4.18. *If φ is L-orientation reversing we define*

$$\mathfrak{C}_m := \mathfrak{C}_m(x,\varphi;\mathbb{Z}_2) := \bigoplus_{\substack{p\in\mathcal{H}_{pr}\\ \mu(p)=m}} \mathbb{Z}_2 p,$$

$$\mathfrak{d}_m : \mathfrak{C}_m \to \mathfrak{C}_{m-1}, \qquad \mathfrak{d}(p) = \sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=\mu(p)-1}} m_2(p,q)q$$

on a generator p and extend $\mathfrak{d}$ by linearity. φ induces $\varphi_* : \mathfrak{C}_* \to \mathfrak{C}_*$ satisfying

$$\varphi_* \circ \mathfrak{d} = \mathfrak{d} \circ \varphi_*.$$

The well-definedness carries over from the L-orientation preserving case. Analogously to Theorem 4.13 follows

THEOREM 4.19. *Let φ be L-orientation reversing. Then $\mathfrak{d}\circ\mathfrak{d} = 0$, i.e. $(\mathfrak{C}_*, \mathfrak{d}_*)$ is a chain complex.*

Now we define the homology of $(\mathfrak{C}_*, \mathfrak{d})$ via

DEFINITION 4.20. *Let φ be L-orientation reversing. We define*

$$\mathfrak{H}_m := \mathfrak{H}_m(x,\varphi,\mathbb{Z}_2) := \frac{\ker \mathfrak{d}_m}{\operatorname{Im} \mathfrak{d}_{m+1}}.$$

As discussed above the $\mathbb{Z}_2$-coefficients allow us to divide by the $\mathbb{Z}$-action:

DEFINITION 4.21. *If φ is L-orientation reversing we set for $\langle p\rangle$, $\langle q\rangle \in \tilde{\mathcal{H}}_{pr}$*

$$m_2(\langle p\rangle,\langle q\rangle) := \sum_{\substack{q\in\langle q\rangle\\ n\in\mathbb{Z}}} m_2(p,q^n) \ mod\ 2$$

and define

$$C_m := C_m(x,\varphi;\mathbb{Z}_2) := \bigoplus_{\substack{\langle p\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle p\rangle)=m}} \mathbb{Z}_2\langle p\rangle,$$

$$\partial_m : C_m \to C_{m-1}, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m_2(\langle p\rangle,\langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity. The compatibility with the Maslov index and the homotopy classes implies that ∂ is well-defined.

We deduce as in the L-orientation preserving case

THEOREM 4.22. *Let φ be L-orientation reversing. Then*

$$\partial \circ \partial = 0.$$

And we can proceed to the homology groups:

DEFINITION 4.23. *Let φ be L-orientation reversing. We define the* **primary homoclinic Floer homology of φ in** x *as*

$$H_m := H_m(x, \varphi, \mathbb{Z}_2) := \frac{\ker \partial_m}{\operatorname{Im} \partial_{m+1}}.$$

When working with primary homoclinic Floer homology groups H_* we always mean $H_*(x, \varphi, \mathbb{Z})$ for L-orientation preserving φ and $H_*(x, \varphi, \mathbb{Z}_2)$ in the L-orientation reversing case.

One important point is that the primary homoclinic Floer homology is already determined by the intersection behaviour of two fixed large *compact segments* of L_0 and L_1: We have defined ∂ on the equivalence classes via representatives and their differential $\mathfrak{d}$. There are only finitely many equivalence classes and $\mathfrak{d}$ yields a finite sum. Thus we can choose compact segments in L_0 and L_1 large enough to contain a representative system and the points appearing when applying $\mathfrak{d}$ to the representatives. Therefore the primary homoclinic Floer homology is already encoded in a compact subset of the tangle.

3. Primary homoclinic Floer cohomology

After defining $H_*(x, \varphi)$ it is natural to ask if or how it might be related to $H_*(x, \varphi^{-1})$. In order to answer this question we consider for *L-orientation preserving φ*

$$C^m(x, \varphi; \mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle)=m}} \mathbb{Z}\langle p\rangle$$

with differential $\delta : C^m(x, \varphi; \mathbb{Z}) \to C^{m+1}(x, \varphi; \mathbb{Z})$ defined on the generators by

$$\delta(\langle p\rangle) := \sum_{\substack{q \in \mathcal{H}_{pr} \\ \mu(q)=m+1}} m(q, p)\langle q\rangle.$$

Then $\delta \circ \delta = 0$ is proven analogously to $\partial \circ \partial = 0$ and

$$H^*(x, \varphi; \mathbb{Z}) := \frac{\ker \delta}{\operatorname{Im} \delta}$$

is called **primary homoclinic Floer cohomology of** φ **in** x. For L-*orientation reversing* φ we set

$$C^m(x,\varphi;\mathbb{Z}_2) := \bigoplus_{\substack{\langle p\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle p\rangle)=m}} \mathbb{Z}_2\langle p\rangle$$

with differential $\delta : C^m(x,\varphi;\mathbb{Z}_2) \to C^{m+1}(x,\varphi;\mathbb{Z}_2)$ defined on the generators by

$$\delta(\langle p\rangle) := \sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=m+1}} m_2(q,p)\langle q\rangle.$$

$\delta\circ\delta = 0$ in the L-orientation reversing case follows as in the L-orientation preserving case and

$$H^*(x,\varphi;\mathbb{Z}_2) := \frac{\ker\delta}{\operatorname{Im}\delta}$$

is called **primary homoclinic Floer cohomology of** φ **in** x. If there is no need to distinguish between L-orientation preserving and L-orientation reversing φ we simply write $H^*(x,\varphi)$ or even shorter H^*.

Changing from φ to φ^{-1} transforms L_0 into L_1 and vice versa, but apart from this leaves the homoclinic tangle untouched. Therefore the sign of the Maslov index of a homoclinic point $p = p_\varphi$ in the tangle generated by φ changes when considered as homoclinic point $p = p_{\varphi^{-1}}$ in the tangle corresponding to φ^{-1}, i.e. $\mu(p_\varphi) = -\mu(p_{\varphi^{-1}})$. This implies

Theorem 4.24. *$H^*(x,\varphi) = H_{-*}(x,\varphi^{-1})$.*

4. Immersions between primary points

Recall the lifting procedure of a homoclinic tangle from Notation 4.5 and fix some $\tilde{x}\in\tau^{-1}(x)$. The Maslov index stays invariant under the lifting procedure due to Lemma 3.12. Given primary p, $q\in L_0\cap L_1$ with associated primary $\tilde{p}$, $\tilde{q}\in\tilde{L}_0\cap\tilde{L}_1$ the immersions in $\mathcal{M}(p,q)$ resp. $\mathcal{N}(p,q)$ lift exactly to the immersions in $\mathcal{M}(\tilde{p},\tilde{q})$ resp. $\mathcal{N}(\tilde{p},\tilde{q})$.
Therefore the combinatorial data needed for primary homoclinic Floer (co)homology stay untouched when lifting the tangle. Thus primary homoclinic Floer (co)homology is well-defined for (φ,x) on M if and only if it is well-defined for the lifted homoclinic tangle generated by $\tilde{L}_0$ and $\tilde{L}_1$ on $\tilde{M}$.
Thus it is enough to prove the primary cutting and gluing procedure for the lifted tangle $\tilde{L}_0\cap\tilde{L}_1$ on $\tilde{M}$.
This simplifies the proofs considerably since immersions between primary homoclinic points of $\tilde{L}_0\cap\tilde{L}_1$ turn out to be in fact *embeddings. Thus most of the proofs in the following sections are worked out for $\tilde{L}_0\cap\tilde{L}_1$ on $\tilde{M}\simeq\mathbb{R}^2$.*

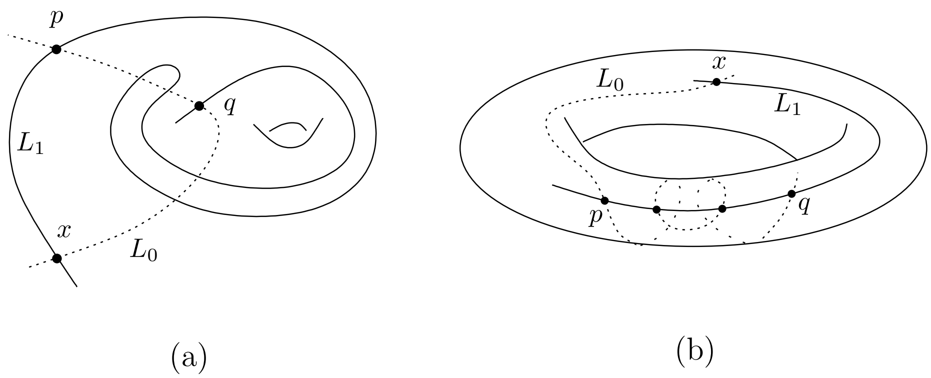


FIGURE 4.2. Immersion, but no embedding

In this section we will prove that for $p,\, q \in \mathcal{H}_{pr}$ the set $\{n \in \mathbb{Z} \mid \mathcal{M}(p, q^n) \neq \emptyset\}$ is finite. This is together with the finiteness of $\tilde{\mathcal{H}}_{pr}$ (see Lemma 4.9) crucial for the well-definedness of the differentials in Definition 4.12 and Definition 4.15.

The first step will be to generalize Lemma 4.7 and recognize immersions between primary homoclinic points on $\tilde{M}$ as embeddings.

LEMMA 4.25 (Classification for index difference 1). *Let $p,\, q \in \mathcal{H}$ be primary with $\mu(p, q) = 1$ and let $\tilde{p}$ and $\tilde{q}$ the associated primary points in $\tilde{L}_0 \cap \tilde{L}_1$. Then either $\mathcal{M}(\tilde{p}, \tilde{q}) = \emptyset$ or $u \in \mathcal{M}(\tilde{p}, \tilde{q})$ is in fact an embedding.*

The elements of $\mathcal{M}(p, q)$ do not need to be embeddings, see figure 4.2 (a). Nor is it true for noncontractible semi-primary points, see figure 4.2 (b). The immersion overlaps after wrapping once around resp. through the hole of the torus.

PROOF : In the following we work with the lifted tangle on $\tilde{M}$. For sake of better readability we drop the tilde associated to symbols on $\tilde{M}$. Thus identify $p = \tilde{p}$ and $q = \tilde{q}$ etc.
The proof is tedious, but simple. $[p] = [q] = [x]$ allows us to write $1 = \mu(p, q) = \mu(p, x) + \mu(x, q)$ and Lemma 4.7 provides the four cases $(\mu(p, x), \mu(x, q)) \in \{(-2, 3), (-1, 2), (2, -1), (3, -2)\}$. Since there are always two possibilities to place the concave vertex of a standard heart the number of cases multiplies by two. Moreover we will distinguish $]x, p[_i \cap]x, q[_i = \emptyset$ or $\neq \emptyset$ for $i \in \{0, 1\}$. Since L_i is self-intersection free and one-dimensional we conclude in case $]x, p[_i \cap]x, q[_i \neq \emptyset$ either $[x, p]_i \subset [x, q]_i$ or $[x, q]_i \subset [x, p]_i$. This yields a lot of cases, but fortunately some of them are symmetric.
We recall from Lemma 4.7 that there is modulo parametrization exactly one embedding between p and x and q and x. Since embeddings do not overlap themselves there is — together with the boundary conditions — almost no degree

of freedom in sketching them and we are able to give the complete list in the relevant cases in figure 4.3 and 4.4.
There are different reasons why there are no immersions in some cases. We will explain figure 4.3 and figure 4.4 by analysing representative sketches and classifying which other sketches belong to the same type. References like (a).(-2,3).(i) refer to *item (a) with* $(\mu(p,x),\mu(x,q)) = (-2,3)$, *subcase (i)* in figure 4.3 or figure 4.4. Dashed lines in sketches mean that this sketch cannot occur within a homoclinic tangle or that it contradicts the conditions of the (sub)case in question.

Case I: Consider (a).(-2,3).(i) in figure 4.3. The boundary conditions are fine, but there is no immersion since there are *nonremovable* components of $M\backslash([p,q]_0 \cup [p,q]_1)$ with Ind < 0 which contradicts Corollary 3.9. Nonremovable means that their existence is forced by the geometric position of $[p,x]_i$ and $[x,q]_i$ caused by the index prescription for $\mu(p,x)$ and $\mu(x,q)$ and that there is no way to get rid of them while fulfilling the demands of the (sub)case.
For the same reason there are no immersions in (a).(-1,2).(i), (a).(1,-2).(ii), (a).(3,-2).(ii), (b).(-2,3).(ii), (b).(2,-1).(i) and (c).(-2,3).(i).
Case II: Consider (a).(-2,3).(ii) in figure 4.3. The index together with the choice x as concave vertex for the immersion in $\widehat{\mathcal{N}}(q,x)$ forces the branches of the (un)stable manifolds to emanate from x in an unnatural way, i.e. in contradiction to the behaviour predicted by Theorem 3.17 (Hartman-Grobman). So this case does not occur.
For analogous reasons the cases (a).(-1,2).(ii), (a).(2,-1).(i), (a).(3,-2).(i), (c).(-2,3).(ii), (c).(2,-1).(ii), (c).(-1,2).(iii) and (c).(3.-2).(iii) do not occur.
Case III: Consider (b).(-2,3).(i) in figure 4.3. Here the index together with the choice of the concave vertex for the immersion in $\widehat{\mathcal{N}}(x,p)$ forces the branches of the (un)stable manifolds to contradict the conditions of the (sub)case. So this case actually does not appear.
The same holds for (b).(2,-1).(ii), (b).(-1,2).(iii), (b).(3,-2).(iii), (c).(2,-1).(i), (d).(-2,3).(iii), (d).(-1,2).(iii), (d).(2,-1).(iii) and (d).(3,-2).(iii).
Case IV: Consider (b).(-1,2).(i) and (ii) in figure 4.3. There is an immersion $u \in \widehat{\mathcal{M}}(p,q)$ in (b).(-1,2).(i) which is in fact an embedding since $]p,q[_0 \pitchfork\,]p,q[_1 = \emptyset$. But there is a certain degree of freedom in sketching $[p,q]_1$ such that a different behaviour of $[p,q]_1$ as sketched in (b).(-1,2).(ii) can prevent an immersion by producing components of $M\backslash([p,q]_0 \cup [p,q]_1)$ with $\mathrm{Ind}_u < 0$, compare Corollary 3.9.
The same phenomenon is found in (b).(3,-2).(i) and (ii), (c).(-1,2).(i) and (ii), (c).(3,-2).(i) and (ii), (d).(-2,3).(i) and (ii), (d).(-1,2).(i) and (ii), (d).(2.-1).(i) and (ii) and (d).(3,-2).(i) and (ii).

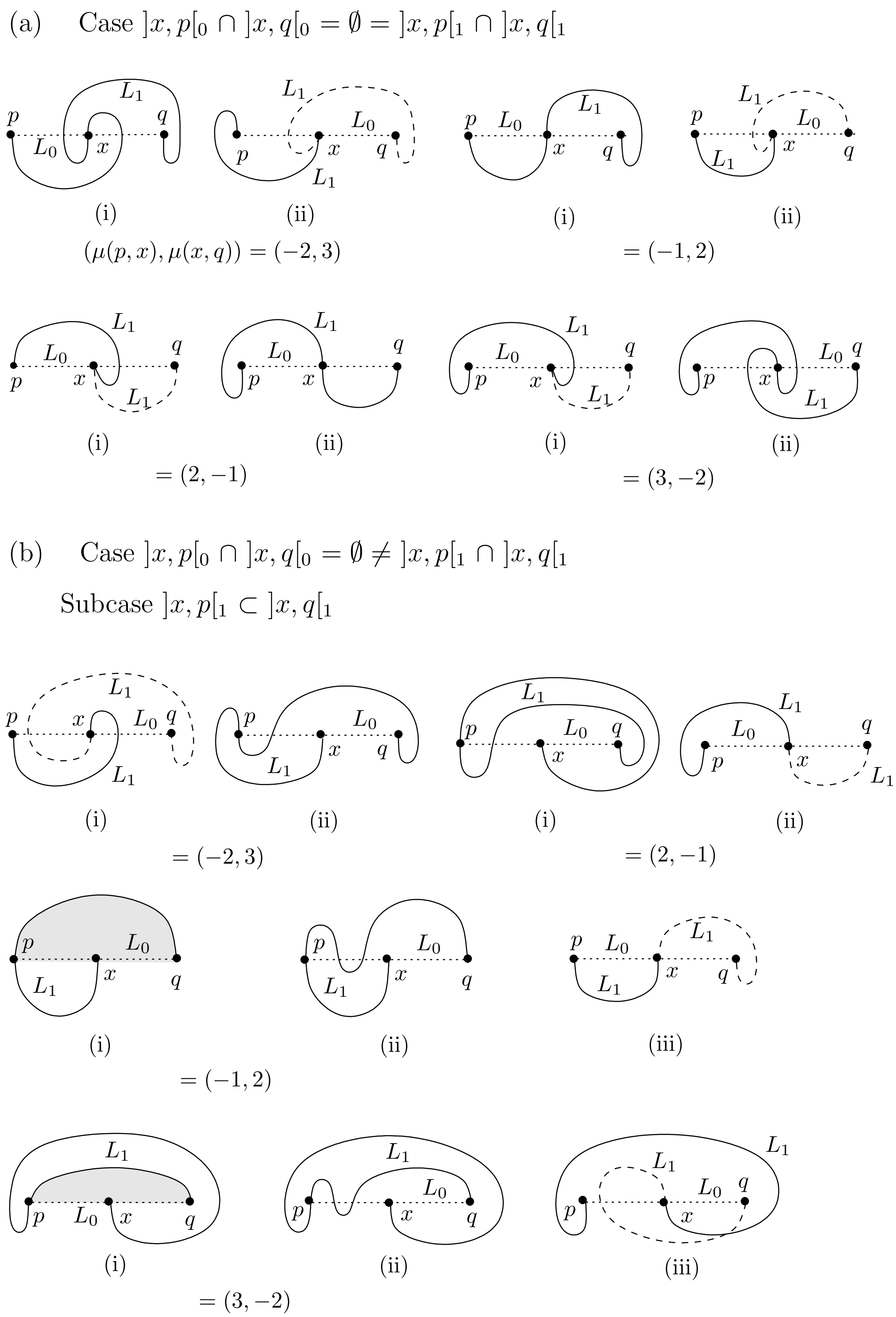
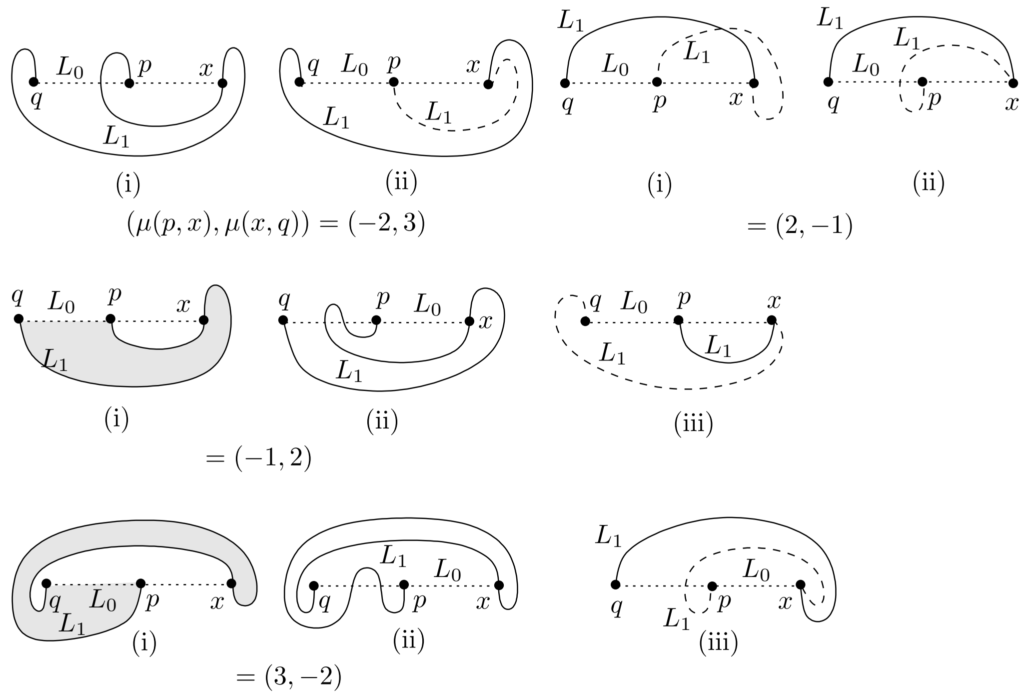


FIGURE 4.3. Geometric realization of primary points of index 1

(c) Case $]x,p[_0 \cap]x,q[_0 \neq \emptyset =]x,p[_1 \cap]x,q[_1$, Subcase $]x,p[_0 \subset]x,q[_0$



(d) Case $]x,p[_0 \cap]x,q[_0 \neq \emptyset \neq]x,p[_1 \cap]x,q[_1$, $]x,p[_0 \subset]x,q[_0$, $]x,q[_1 \subset]x,p[_1$

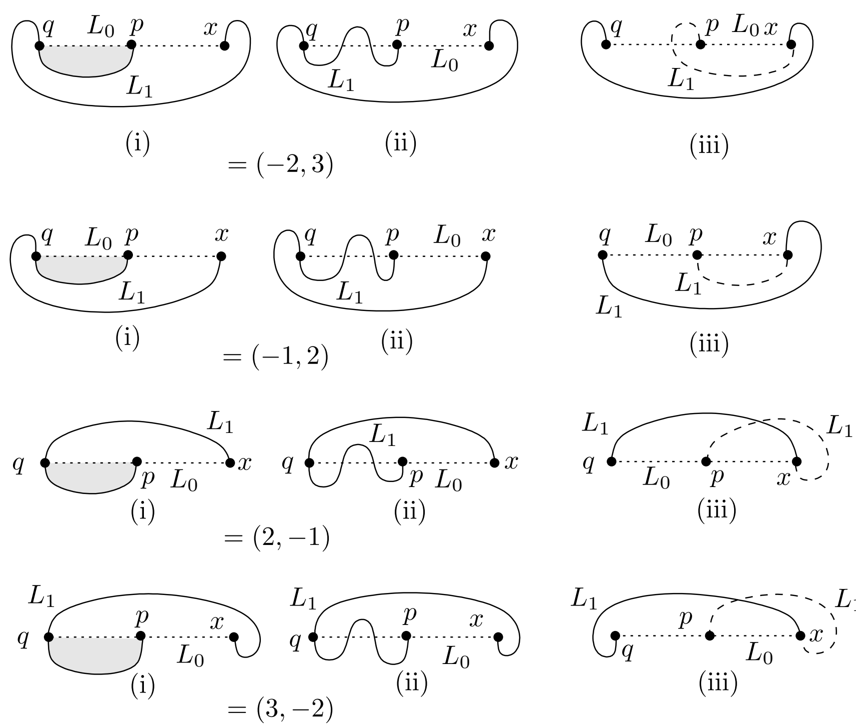


Figure 4.4. Continuation of figure 4.3

Considering the case $]x,p[_0 \cap]x,q[_0 = \emptyset \neq]x,p[_1 \cap]x,q[_1$ we have not treated the subcase $]x,q[_1 \subset]x,p[_1$ in figure 4.3 (b). But the geometric positions of $[x,p]_i$ and $[x,q]_i$ correspond to the treated subcase if we change the sign of the tupel $(\mu(p,x),\mu(x,q))$, exchange p and q and reflect everything on the L_0-segment lying in the horizontal axis.
The same holds for the not sketched subcase $]x,q[_0 \subset]x,p[_0$ and $]x,p[_1 \subset]x,q[_1$ of case $]x,p[_0 \cap]x,q[_0 \neq \emptyset \neq]x,p[_1 \cap]x,q[_1$ w.r.t. the subcase $]x,p[_0 \subset]x,q[_0$ and $]x,q[_1 \subset]x,p[_1$ sketched in figure 4.4 (d).
The not sketched subcase $]x,q[_0 \subset]x,p[_0$ of case $]x,p[_0 \cap]x,q[_0 \neq \emptyset =]x,p[_1 \cap]x,q[_1$ goes over to the subcase $]x,p[_0 \subset]x,q[_0$ sketched in figure 4.4 (c) if we change the sign of the tupel $(\mu(p,x),\mu(x,q))$, exchange p and q and reflect everything on the (imagined) vertical axis. ■

Since according to Lemma 4.25 immersions between primary homoclinic points $\tilde{p}$ and $\tilde{q}$ of $\tilde{L}_0 \cap \tilde{L}_1$ are in fact embeddings it is enough to show $]\tilde{p},\tilde{q}[_0 \cap]\tilde{p},\tilde{q}[_1 \neq \emptyset$ to prevent their existence.

Lemma 4.26. *Let p, $q \in \mathcal{H}_{[x]}\backslash\{x\}$ and $p^n := \varphi^n(p)$ etc. for $n \in \mathbb{Z}$. Let $\tilde{p}$, $\tilde{q}$ and $\tilde{p}^n$ etc. be the associated points in $\tilde{L}_0 \cap \tilde{L}_1$. Then there is $N \in \mathbb{N}_0$ such that for $n \in \mathbb{Z}$ with $|n| \geq N$ we have $]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1 \neq \emptyset$.*

Proof : Let $\tilde{p}$ etc. be the point associated to p in the lifted tangle on $\tilde{M}$. Let φ be *L-orientation preserving*
Consider the case $x \notin]p,q[_0$ and $x \notin]p,q[_1$. Then there is $N \in \mathbb{N}_0$ such that $\tilde{p}^1 \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all $n \geq N$ and $\tilde{p}^{-1} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all $n \leq -N$.
If $x \in]p,q[_0 \cap]p,q[_1$ then $\tilde{x} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all $n \in \mathbb{Z}$.
Consider the case $x \in]p,q[_0$ and $x \notin]p,q[_1$. Then there is $N \in \mathbb{N}_0$ such that $\tilde{q}^{N-1} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all $n \geq N$ and $\tilde{p}^{-1} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all $n \leq -N$.
In the case $x \notin]p,q[_0$ and $x \in]p,q[_1$ conclude analogously.
Now consider *L-orientation reversing* φ. Here we have to distinguish between even and odd $n \in \mathbb{Z}$. Since φ^2 is orientation preserving the above proof carries over for even n if we replace p^1 by ^{2} etc. Thus we only have to prove the claim for *odd* n.
If $x \notin]p,q[_0$, $]p,q[_1$ then $\tilde{x} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all odd n.
If $x \in]p,q[_0 \cap]p,q[_1$ there is $N \in \mathbb{N}_0$ such that $\tilde{p}^2 \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all odd $n \geq N$ and $\tilde{p}^{-2} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for all odd $n \leq -N$.
If $x \in]p,q[_0$ and $x \notin]p,q[_1$ then there is an odd $N \in \mathbb{N}_0$ such that $\tilde{p}^2 \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for odd $n \geq N$ and $\tilde{q}^{N+2} \in]\tilde{p},\tilde{q}^n[_0 \cap]\tilde{p},\tilde{q}^n[_1$ for odd $n \leq -N$.
If $x \notin]p,q[_0$ and $x \in]p,q[_1$ conclude analogously. ■

Now we are able to prove that for $p \in \mathcal{H}_{pr}$ the differential

$$\mathfrak{d}p = \sum_{\substack{q \in \mathcal{H}_{pr} \\ \mu(q)=\mu(p)-1}} m(p,q)q$$

from Definition 4.12 does not contain infinitely many iterates $m(p, q^n)q^n$ for some q. This is crucial for ∂ in Definition 4.15 in order to pass from p to $\langle p\rangle$.

PROPOSITION 4.27. *Let p, $q \in \mathcal{H}_{pr}$ and $\mathcal{M}(p,q) \neq \emptyset$ and set $q^n := \varphi^n(q)$ for $n \in \mathbb{Z}$. Then*

$$\#\{n \in \mathbb{Z} \mid \mathcal{M}(p, q^n) \neq \emptyset\} < \infty.$$

PROOF : Denote by $\tilde{p}$, $\tilde{q}$, $\tilde{q}^n$ etc. the associated points in $\tilde{L}_0 \cap \tilde{L}_1$ and recall that $u \in \mathcal{M}(p,q)$ exists if and only its lift $\tilde{u} \in \mathcal{M}(\tilde{p}, \tilde{q})$ exists.
Lemma 4.26 yields the existence of some $N > 0$ such that $]\tilde{p}, \tilde{q}^n[_0 \cap]\tilde{p}, \tilde{q}^n[_1 \neq \emptyset$ for all $n \in \mathbb{Z}$ with $|n| \geq N$.
Assume $\tilde{u}_n \in \mathcal{M}(\tilde{p}, \tilde{q}^n) \neq \emptyset$ for some n with $|n| \geq N$. Since $[\tilde{p}, \tilde{q}^n]_0 = \tilde{u}_n(B_0)$ and $[\tilde{p}, \tilde{q}^n]_1 = \tilde{u}_n(B_1)$ there is $z_0 \in B_0$ and $z_1 \in B_1$ such that $\tilde{u}_n(z_0) = \tilde{u}_n(z_1)$. Since $\tilde{L}_0$ and $\tilde{L}_1$ don't have self-intersections it follows z_0, $z_1 \notin \{(-1,0),(1,0)\}$. Therefore $\tilde{u}_n$ is not globally injective and thus no embedding. The claim now follows from Lemma 4.25. ∎

Now the well-definedness of Definition 4.12 and Definition 4.15 is proven.

5. Gluing and cutting for primary homoclinic points

In this section we will show that the restriction to primary homoclinic points is gluing and cutting compatible. As in the previous section we will mostly work on the lifted tangle generated by $\tilde{L}_0$ and $\tilde{L}_1$ on $\tilde{M}$. First we consider the gluing construction.

THEOREM 4.28 (Gluing for primary points). *Let p, q, $r \in \mathcal{H}_{pr}$ with $\mu(p,q) = 1 = \mu(q,r)$ and $u \in \widehat{\mathcal{M}}(p,q)$ and $v \in \widehat{\mathcal{M}}(q,r)$. Then gluing of u and v yields an immersion $v\#u \in \widehat{\mathcal{N}}(p,r)$.*

PROOF : This is clearly a special case of the general gluing construction Theorem 3.14. ∎

The lift of the outcoming immersion $v\#u$ is in fact an embedding as we will see in the discussion of the cutting procedure.

Starting with p and r primary with $\mu(p,r) = 2$ and $w \in \mathcal{N}(p,r)$ we have to ask if the 'cutting points' q_0 and q_1 delivered by the cutting procedure Theorem 3.16 are again primary. Therefore we investigate what kind of immersed hearts exist between primary homoclinic points. We will see that the lifts of those immersed hearts relevant for the cutting procedure are in fact embedded.

LEMMA 4.29 (Classification for index difference 2). *Let p, $r \in \mathcal{H}_{pr}$ with $\mu(p,r) = 2$ and $\tilde{p}$ and $\tilde{r}$ the associated points in $\tilde{L}_0 \cap \tilde{L}_1$. The possible immersed hearts $w \in \widehat{\mathcal{N}}(\tilde{p}, \tilde{r})$ appear shadowed in figures 4.5 (b) and 4.6 (c). In both figures the shadowed w is an embedding except from case (ii) of $(\mu(\tilde{p}, \tilde{x}), \mu(\tilde{x}, \tilde{r})) = (1,1)$ where it is not globally injective.*

PROOF : In the following we work with the lifted tangle on $\tilde{M}$. For sake of better readability we drop the tilde associated to symbols on $\tilde{M}$. Thus we identify $p = \tilde{p}$ and $r = \tilde{r}$ etc.

Since $[p] = [r] = [x]$ we can write $\mu(p, r) = \mu(p, x) + \mu(x, r) = 2$. Now we proceed as in the proof of Lemma 4.25 and check the possible combinations for $(\mu(p, x), \mu(x, r))$. Lemma 4.7 restricts the possibilities to

$$(\mu(p, x), \mu(x, r)) \in \{(3, -1), (1, 1), (-1, 3)\}$$

and we recall from 4.7 that the immersions of index $\mu(p, x)$ and $\mu(x, r)$ between p and x and x and r are embeddings. As before we will consider the cases $]x, p[_i \cap]x, r[_i = \emptyset$ or $\neq \emptyset$. If $]x, p[_i \cap]x, r[_i \neq \emptyset$ this implies $[x, p]_i \subset [x, r]_i$ or $[x, r]_i \subset [x, p]_i$ since L_i is free of self-intersections and $\dim L_i = 1$. These considerations yield the sketches of figure 4.5 and 4.6 which we will now discuss in detail. Dashed segments in a sketch mean that this sketch cannot occur within a homoclinic tangle or that it violates the conditions of the (sub)case in question.

(a) $]x, p[_0 \cap]x, r[_0 = \emptyset =]x, p[_1 \cap]x, r[_1$**:** There are no possible immersions since the indices prescribed by the combinations $(\mu(p, x), \mu(x, r)) \in \{(3, -1), (1, 1), (-1, 3)\}$ force the two branches of L_1 to emanate from x in an unnatural way, compare figure 4.5 (a). So this case cannot happen.

(b) $]x, p[_0 \cap]x, r[_0 \neq \emptyset =]x, p[_1 \cap]x, r[_1$**:** For $]x, p[_0 \subset]x, r[_0$ compare the first part of figure 4.5 (b): The case (1,1) yields an immersion (see (ii)) which might be an embedding (see (i)). In the other two cases (3,-1) and (-1,3) there might be an embedding as sketched in (i), but a different behaviour of the segment $[x, p]_1$ can destroy it as showed in (ii). There components with Ind < 0 appear in contradiction to Corollary 3.9.

The subcase $]x, r[_0 \subset]x, p[_0$ is sketched in the second part of figure 4.5 (b) and behaves similar.

(c) $]x, p[_0 \cap]x, r[_0 = \emptyset \neq]x, p[_1 \cap]x, r[_1$**:** This is sketched in figure 4.6 (c). As in case (b) the case (1,1) always yields an immersion (see (ii)) which might be an embedding (see (i)). In the other cases there might — depending on the intersection behaviour of $[p, r]_1$ — be an embedding (see (i)) or not (see (ii)) due to the same reason as in (b).

(d) $]x, p[_0 \cap]x, r[_0 \neq \emptyset \neq]x, p[_1 \cap]x, r[_1$**:** There are four cases:

$]x, p[_0 \subset]x, r[_0$ and $]x, p[_1 \subset]x, r[_1$: This implies $p \in]x, r[_0 \pitchfork]x, r[_1$, but r is primary, so this is impossible.

$]x, p[_0 \subset]x, r[_0$ and $]x, r[_1 \subset]x, p[_1$: Consider figure 4.6 (d) and notice that the indices prescribed by $(\mu(p, x), \mu(x, r))$ force p and r to lie on different branches of L_1 in contradiction to the assumption $]x, p[_0 \subset]x, r[_0$ and $]x, r[_1 \subset]x, p[_1$. So this case is not possible.

$]x, r[_0 \subset]x, p[_0$ and $]x, p[_1 \subset]x, r[_1$: Similarily to the case before a look at the second part of figure 4.6 (d) shows that due to the prescribed indices

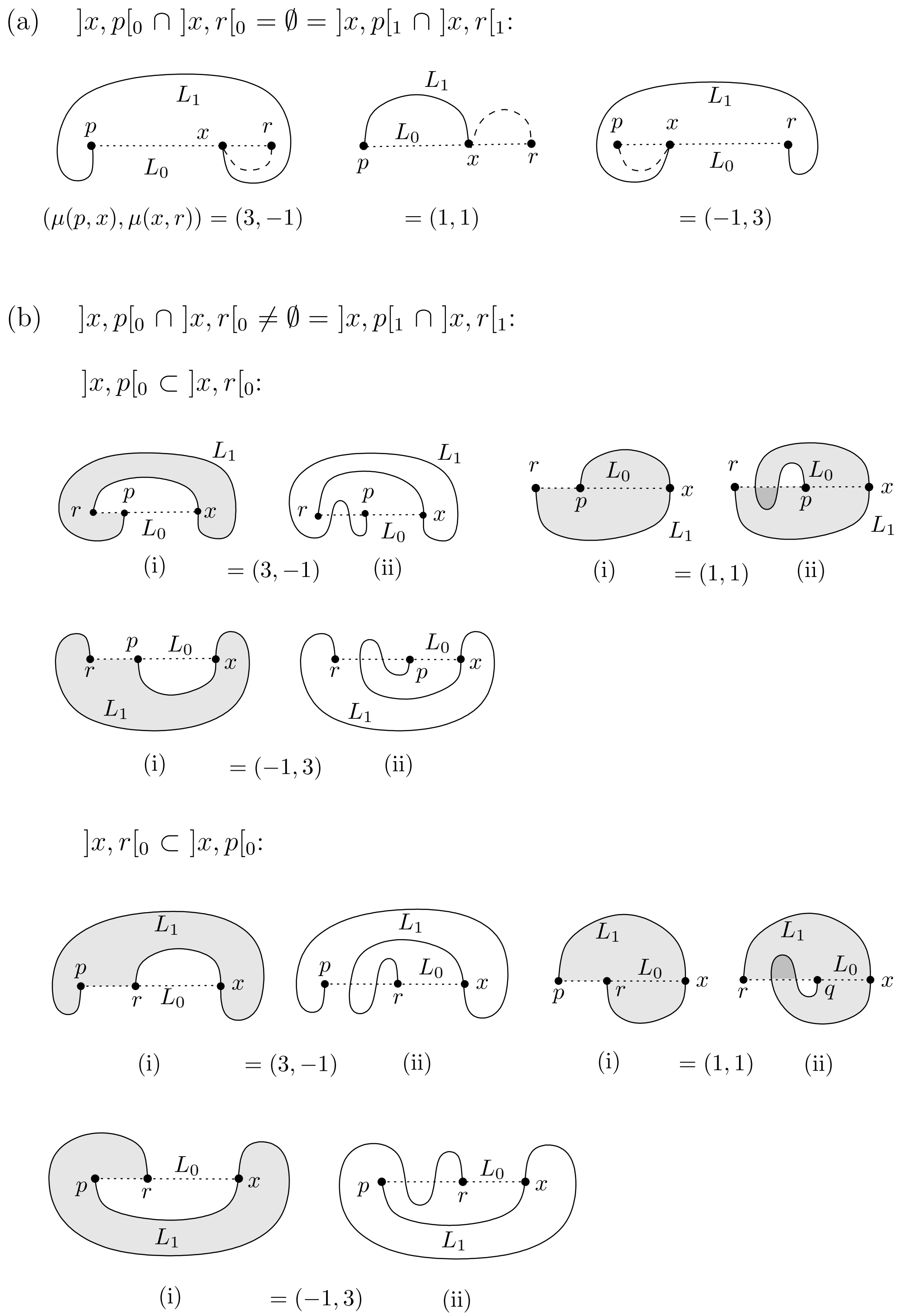
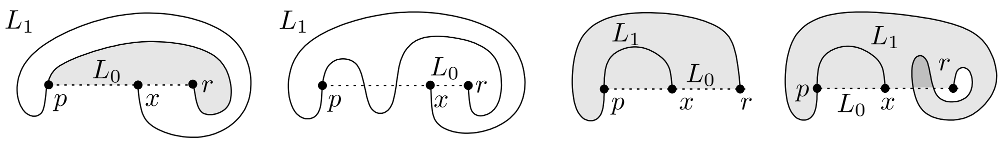


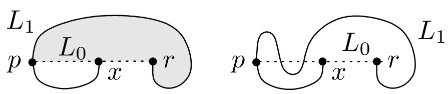
FIGURE 4.5. Geometric realization of primary points of relative index 2

(c) $]x,p[_0 \cap]x,r[_0 = \emptyset \neq]x,p[_1 \cap]x,r[_1$:

$]x,p[_1 \subset]x,r[_1$:

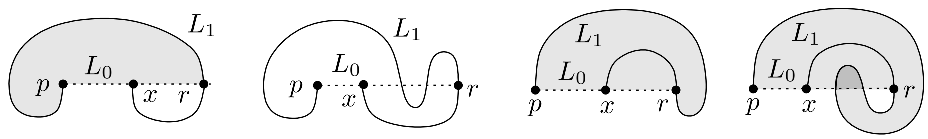


(i) $(\mu(p,x),\mu(x,r)) = (3,-1)$ (ii)   (i) $= (1,1)$ (ii)

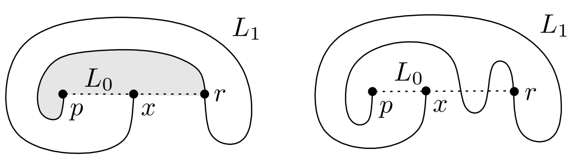


(i) $= (-1,3)$ (ii)

$]x,r[_1 \subset]x,p[_1$:



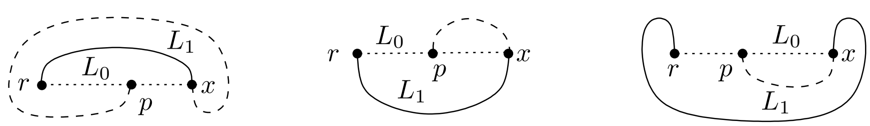
(i) $= (3,-1)$ (ii)   (i) $= (1,1)$ (ii)



(i) $= (-1,3)$ (ii)

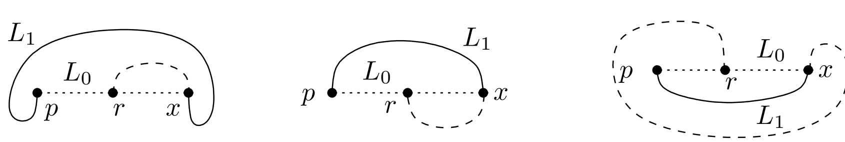
(d) $]x,p[_0 \cap]x,r[_0 \neq \emptyset \neq]x,p[_1 \cap]x,r[_1$:

$]x,p[_0 \subset]x,r[_0$ and $]x,r[_1 \subset]x,p[_1$:



$= (3,-1)$ $= (1,1)$ $= (-1,3)$

$]x,r[_0 \subset]x,p[_0$ and $]x,p[_1 \subset]x,r[_1$:



$= (3,-1)$ $= (1,1)$ $= (-1,3)$

FIGURE 4.6. Continuation of figure 4.5

$(\mu(p,x),\mu(x,r))$ the points p and r must lie on different branches of L_1 in contradiction to the subcase itself $]x,r[_0 \subset]x,p[_0$ and $]x,p[_1 \subset]x,r[_1$.
$]x,r[_0 \subset]x,p[_0$ and $]x,r[_1 \subset]x,p[_1$: This implies $r \in]x,p[_0 \pitchfork]x,p[_1$, but p is primary, so this is impossible. ■

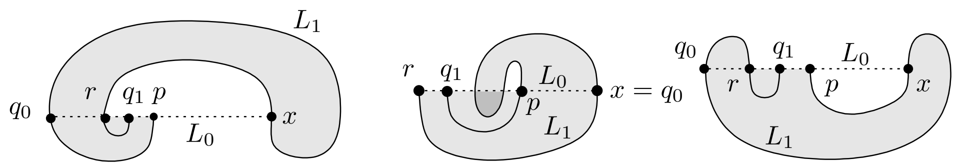
Now we are ready to approach the cutting construction for primary points. Recall the cutting points q_0 and q_1 from the general cutting construction Theorem 3.16. It will turn out that for p and r *primary* with $\mu(p,r)=2$ either q_0 *and* q_1 are primary or none of both where the latter case corresponds to the 'bad case' (ii) of $(\mu(p,x),\mu(x,r))=(1,1)$ in figure 4.5 (b) and figure 4.6 (c).
A look at the proof of Theorem 3.16 tells us that *strongly intersecting* L_0 and L_1 were only needed if the concave vertex of the heart was the fixed point. Since $x \notin \mathcal{H}_{pr}$ we can drop this assumption on L_0 and L_1 in the following theorem. In Theorem 3.16 the λ-lemma Theorem 3.21 was only applied to the intersection at the concave vertex of the immersion in question. Thus it is enough for the *well-definedness* of primary homoclinic Floer homology to require only the primary points to be transverse.

THEOREM 4.30 (Cutting for primary points). *Let p, $r \in \mathcal{H}_{pr}$ with $\mu(p,r)=2$ and $w \in \mathcal{N}(p,r)$. Then there are unique points q_0 and q_1 such that either both q_i are primary admitting $u_i \in \mathcal{M}(p,q_i)$ and $v_i \in \mathcal{M}(q_i,r)$ with $v_i \# u_i = w$ for $i \in \{0,1\}$ or none of them is primary.*

PROOF : It is sufficient to show the claim for the lifted tangle generated by $\tilde{L}_0$ and $\tilde{L}_1$ on $\tilde{M}$ where we will work in the following. For sake of better readability drop the tilde associated to symbols on $\tilde{M}$ and identify $\tilde{p}=p$ etc.
Let p and r be primary with $\mu(p,r)=2$. The general cutting procedure Theorem 3.16 requires all homoclinic points to be transverse in order to obtain transverse $q_i \in \mathcal{H}$ such that $u_i \in \mathcal{M}(p,q_i)$ and $v_i \in \mathcal{M}(q_i,r)$ for $i \in \{0,1\}$ with $w = v_i \# u_i$ are well-defined. If we require only the primary points to be transverse the proof of Theorem 3.16 nevertheless yields unique q_0 and q_1 since the vertices p and r still are transverse. But q_0 and q_1 might be nontransverse. We will prove that q_0 and q_1 are either both primary or both not primary. If both are primary then they are by assumption transverse and the claim follows from Theorem 3.16.
We proof this claim by checking all possible immersions between primary points of index difference 2. Those were investigated in Lemma 4.29 and listed in figures 4.5 (b) and 4.6 (c). We resketch them in figure 4.7 together with the cuts to the points q_0 and q_1.
We recall from the proof of the general cutting procedure Theorem 3.16 the definition of q_i. We choose a parametrization of L_i from the convex vertex to the concave one in order to obtain an ordering $<_i$ on L_i for $i \in \{0,1\}$. If p is the concave vertex then the formula for the point q_0 in the proof of the general

Case $\quad]x,p[_0 \cap\,]x,r[_0 \neq \emptyset =\,]x,p[_1 \cap\,]x,r[_1$

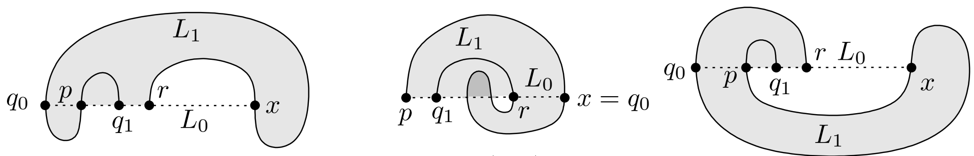
(i) $\quad]x,p[_0 \subset\,]x,r[_0$



$(\mu(p,x), \mu(x,r)) = (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_0 nor q_1 primary!

(ii) $\quad]x,r[_0 \subset\,]x,p[_0$

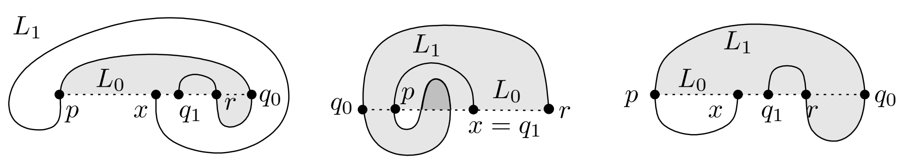


$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_0 nor q_1 primary!

Case $\quad]x,p[_0 \cap\,]x,r[_0 = \emptyset \neq\,]x,p[_1 \cap\,]x,r[_1$

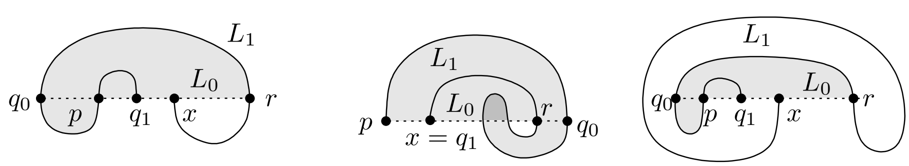
(i) $\quad]x,p[_1 \subset\,]x,r[_1$



$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_0 nor q_1 primary!

(ii) $\quad]x,r[_1 \subset\,]x,p[_1$



$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_0 nor q_1 primary!

FIGURE 4.7. Cutting for primary points

cutting construction Theorem 3.16 was

$$q_0 := \min\{q \in L_0 \mid p <_0 q,\ q \in [p,r]_1,\ [q, q+\varepsilon[_0 \cap w(D_b)^c \neq \emptyset \text{ for } \varepsilon > 0\}.$$

Now consider the cuts to q_0 and q_1 in figure 4.7 and check if q_0 and q_1 are primary. A priori the definition 'primary' does not care if the point is transverse or not. Thus for simplicity we sketched the q_i transverse.
Checking the shapes in figure 4.7 we find that for all cases $(\mu(p,x), \mu(x,r)) \in \{(3,-1), (-1,3)\}$ the immersion w is an embedding and that q_0 and q_1 are both primary. In the case $]x,p[_0 \cap]x,r[_0 \neq \emptyset =]x,p[_1 \cap]x,r[_1$ we only sketched the case $q_0 \in [x,p]_1$, but also $q_0 \in [x,r]_1$ would be primary. In the case $]x,p[_0 \cap]x,r[_0 = \emptyset \neq]x,p[_1 \cap]x,r[_1$ we have to distinguish $q_1 \in [x,p]_0$ or $q_1 \in [x,r]_0$, but in both cases q_1 is primary.
Now consider the case $(\mu(p,x), \mu(x,r)) = (1,1)$. First we note that w is not necessarily an embedding. Looking at figure 4.7 we realize that one of the cutting points is the fixed point itself which is per definitionem not primary. But fortunately always those segments which join the other cutting point to x overcross in p or r such that this cutting point also is not primary.
As a consequence either both q_0 and q_1 are primary or none of them. ■

6. The proof of Theorem 4.13

After all these preparations we finally are able to to prove Theorem 4.13. Again it is enough to prove the claim for the lifted tangle on $\tilde{M}$.

Proof of Theorem 4.13: In the following we work with the lifted tangle on $\tilde{M}$, but drop the tilde associated to symbols on $\tilde{M}$ and identify $p = \tilde{p}$ etc.
Due to the linearity of $\mathfrak{d}$ it is sufficient to prove the claim on the generators. We compute for $p \in \mathcal{H}_{pr}$

$$\begin{aligned}
\mathfrak{d}_{m-1}(\mathfrak{d}_m(p)) &= \mathfrak{d}_{m-1}\left(\sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=\mu(p)-1}} m(p,q)q\right) \\
&= \sum_{\substack{r\in\mathcal{H}_{pr}\\ \mu(r)=\mu(p)-2}} \sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=\mu(p)-1}} m(p,q)\cdot m(q,r)r \\
&= \sum_{\substack{r\in\mathcal{H}_{pr}\\ \mu(r)=\mu(p)-2}} \left(\sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=\mu(p)-1}} m(p,q)\cdot m(q,r)\right) r.
\end{aligned}$$

Thus it is enough to show for p and fixed r

$$\sum_{\substack{q\in\mathcal{H}_{pr}\\ \mu(q)=\mu(p)-1}} m(p,q)\cdot m(q,r)=0.$$

If all sign products vanish we are done. If $m(p,q)\cdot m(q,r)\neq 0$ both signs $m(p,q)$ and $m(q,r)$ must be nonzero. In that case $\widehat{\mathcal{M}}(p,q)$ and $\widehat{\mathcal{M}}(q,r)$ are not empty and by the gluing construction Theorem 4.28 we obtain $\widehat{\mathcal{N}}(p,r)$ nonempty. The cutting procedure Theorem 4.30 tells us that for fixed p and r there are either exactly two primary cutting points q_0 and q_1 or none. We are in the first case since our q in one of them. Since $m(p,q)\cdot m(q,r)=0$ for all $q\neq q_0,\ q_1$ the sum simplifies to

$$m(p,q_0)\cdot m(q_0,r)+m(p,q_1)\cdot m(q_1,r)$$

which vanishes since $m(p,q_0)\cdot m(q_0,r)=-m(p,q_1)\cdot m(q_1,r)$ by Lemma 3.24. ■

The following statement shows that the proof of Theorem 4.13 implies the proof of Theorem 4.19 where we use $\mathbb{Z}_2$-coefficients.

Remark 4.31. *$m(p,q_0)\cdot m(q_0,r)+m(p,q_1)\cdot m(q_1,r)=0$ over $\mathbb{Z}$ clearly implies vanishing of $m_2(p,q_0)\cdot m_2(q_0,r)+m_2(p,q_1)\cdot m_2(q_1,r)=0$ over $\mathbb{Z}_2$ for the $\mathbb{Z}_2$-signs m_2 from Definition 3.23.*

CHAPTER 5

Examples

In this chapter we discuss the aptitude and accessibility for explicit computations of primary homoclinic Floer homology. We compute the primary homoclinic Floer homology of three important examples and give a rough classification for tangles having exactly two primary equivalence classes w.r.t. each pair of intersecting branches.

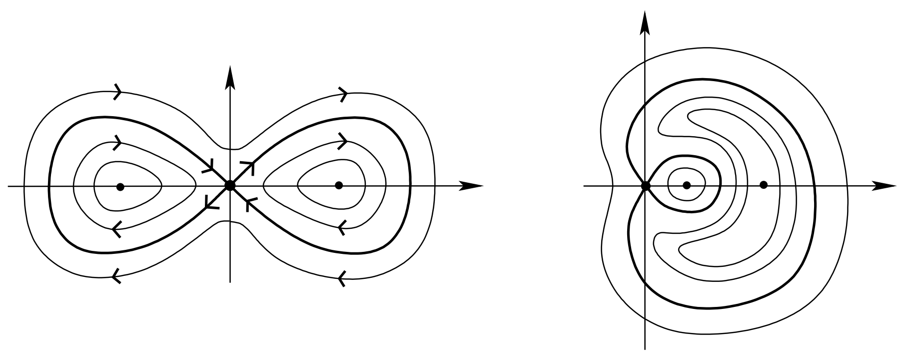
1. Intuition

If we want to compute the primary homoclinic Floer homology of an explicit tangle we have to locate the primary points within the tangle. For simplicity assume that there are only primary and no semi-primary points.

For a pair of intersecting branches we locate a primary point as follows: Start at x and follow simultanously both branches until they intersect for the first time. This intersection point p is primary. Lemma 4.2 now tells us that all other primary points arising from this pair of branches have exactly one representant in $]p,p^1[_0 \cap]p,p^1[_1$. Since all primary points are transverse there is only a finite number of primary equivalence classes and we locate their representants in $]p,p^1[_0 \cap]p,p^1[_1$ applying successively Lemma 4.8.
If we proceed in this way for all pairs of intersecting branches we obtain representatves for all primary equivalence classes.

To discover for a given p all q with $\mathcal{M}(p,q) \neq \emptyset$ is much more tricky. Unfortunately there is no general recipe, but only some strategies.

Denote by L_0^p and L_1^p the branches containing p. A glance at the figures 4.3 and 4.4 shows that q with $\mathcal{M}(p,q) \neq \emptyset$ lies at least on one of the branches L_0^p and L_1^p.
Embeddings from p to primary points in $L_0^p \cap L_1^p$ are easily found. Points q with $\mathcal{M}(p,q) \neq \emptyset$ have to lie in $]p^{-1},p^1[_0 \cap]p^{-1},p^1[_1$ since otherwise p^1 resp. $p^{-1} \in]p,q[_0 \cap]p,q[_1$ causing the lifted $\mathcal{M}(\tilde{p},\tilde{q})$ to be empty according to Lemma 4.25 and thus $\mathcal{M}(p,q^n) = \emptyset$.
Another strategy is to look out for embeddings of relative index two and try to find the cutting points. Due to Theorem 4.30 either both are primary and yield the desired embeddings or none of them is primary.



(a) (b)

FIGURE 5.1. 'Figure eight' and 'tilted figure eight' saddle connections

Or we glue (if possible) two already found embeddings to one of index difference two and cut it in order to locate the still unknown second cutting point.

As already discussed after Definition 4.23 primary homoclinic Floer homology is already determined by the intersection behaviour of two fixed *compact segments* of L_0 and L_1. They can be determined by choosing a representative system and adding the points which appear when applying $\mathfrak{d}$ to the representatives.

Moreover we can schematically iterate certain parts of the tangle without loosing information as long as we do not change the primary points and the embeddings between them. Since primary points are only a tiny part of the tangle this reduces the expenditure for iterations considerably.

So far primary homoclinic Floer homology is only defined for homoclinic tangles in two dimensions. We can also gain information of higher dimensional systems whenever we can reduce them somehow to dimension two.
Consider for example a four dimensional integrable Hamiltonian system and a 3-dimensional energy hypersurface therein. If it contains a hyperbolic periodic orbit we can consider the Poincaré map associated to a transverse 2-dimensional section of the periodic orbit. The periodic orbit yields a hyperbolic fixed point of the Poincaré map whose (un)stable manifolds lie in the intersecting surface. In this way we have reduced the 4-dimensional problem to two dimensions where we can apply primary homoclinic Floer homology.

2. 'Figure eight' example

Our first example is the schematically sketched homoclinic tangle of figure 5.2 to which we also refer as 'figure eight' tangle due to its shape.

For instance a tangle looking more or less like the schematic 'figure eight' tangle might appear when applying the Melnikov perturbation method (see Appendix A) to an integrable system as in figure 5.1 (a). There are also concrete examples:

- The generalized cubic standard map

  $$G_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (x_1, y_1), \quad y_1 := y + \varepsilon f(x), \quad x_1 := x + y_1$$

  for $\varepsilon > 0$ and $f(x) := x + rx^2 - x^3$ with $r \in \mathbb{R}$ as sketched in figure 8.2.
- The Hamiltonian

  $$H(x_1, x_2, y_1, y_2) = \tfrac{1}{2}(ax_1^2 + by_1^2 + x_2^2 + y_2^2) - \varepsilon x_1^2 y_1$$

  for certain a, b and ε: Reduce the two degree of freedom to one by considering a energy hypersurface and the Poincaré map as done in Contopoulos & Polymilis [**CP**]. Then we obtain a 'figure eight' tangle in a two dimensional setting.

Now we calculate the primary homoclinic Floer homology of figure 5.2. Let the tangle be associated to an L-orientation preserving symplectomorphism in $\mathbb{R}^2$. All homoclinic points are contractible due to $\pi_1(\mathbb{R}^2) = 0$. The fixed points x, y and $\tilde{y}$ and the *primary* homoclinic points are printed extra bold. Next to each primary point its Maslov index $\mu(\cdot) := \mu(\cdot, x)$ is given. Locate the primary point p in figure 5.2 and fix an orientation of L_0 by choosing a parametrization in direction from x to p.
There are eight equivalence classes $\langle p \rangle$, $\langle \tilde{p} \rangle$, $\langle q_1 \rangle$, $\langle q_2 \rangle$, $\langle \tilde{q}_1 \rangle$, $\langle \tilde{q}_2 \rangle$, $\langle r \rangle$ and $\langle \tilde{r} \rangle$ with

$$\begin{aligned}
&\mu(\langle p \rangle) = \mu(\langle \tilde{p} \rangle) = -1, \\
&\mu(\langle q_1 \rangle) = \mu(\langle q_2 \rangle) = \mu(\langle \tilde{q}_1 \rangle) = \mu(\langle \tilde{q}_2 \rangle) = -2, \\
&\mu(\langle r \rangle) = \mu(\langle \tilde{r} \rangle) = -3.
\end{aligned}$$

We obtain as chain groups

$$\begin{aligned}
C_{-1} &= \mathbb{Z}\langle p \rangle \oplus \mathbb{Z}\langle \tilde{p} \rangle, \\
C_{-2} &= \mathbb{Z}\langle q_1 \rangle \oplus \mathbb{Z}\langle q_2 \rangle \oplus \mathbb{Z}\langle \tilde{q}_1 \rangle \oplus \mathbb{Z}\langle \tilde{q}_2 \rangle, \\
C_{-3} &= \mathbb{Z}\langle r \rangle \oplus \mathbb{Z}\langle \tilde{r} \rangle, \\
C_n &= 0 \quad \text{for } n \in \mathbb{Z} \backslash \{-1, -2, -3\}
\end{aligned}$$

and want to determine their differentials

$$0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} C_{-3} \xrightarrow{\partial_{-3}} 0.$$

Recall $\langle p \rangle = \langle p^n \rangle$ etc. for all $n \in \mathbb{Z}$ and all homoclinic points. $-\langle \tilde{q}_2 \rangle$ appears in the differential of $\langle p^{-2} \rangle$ and thus $-\langle \tilde{q}_2^2 \rangle$ appears in the differential of $\langle p \rangle$. The

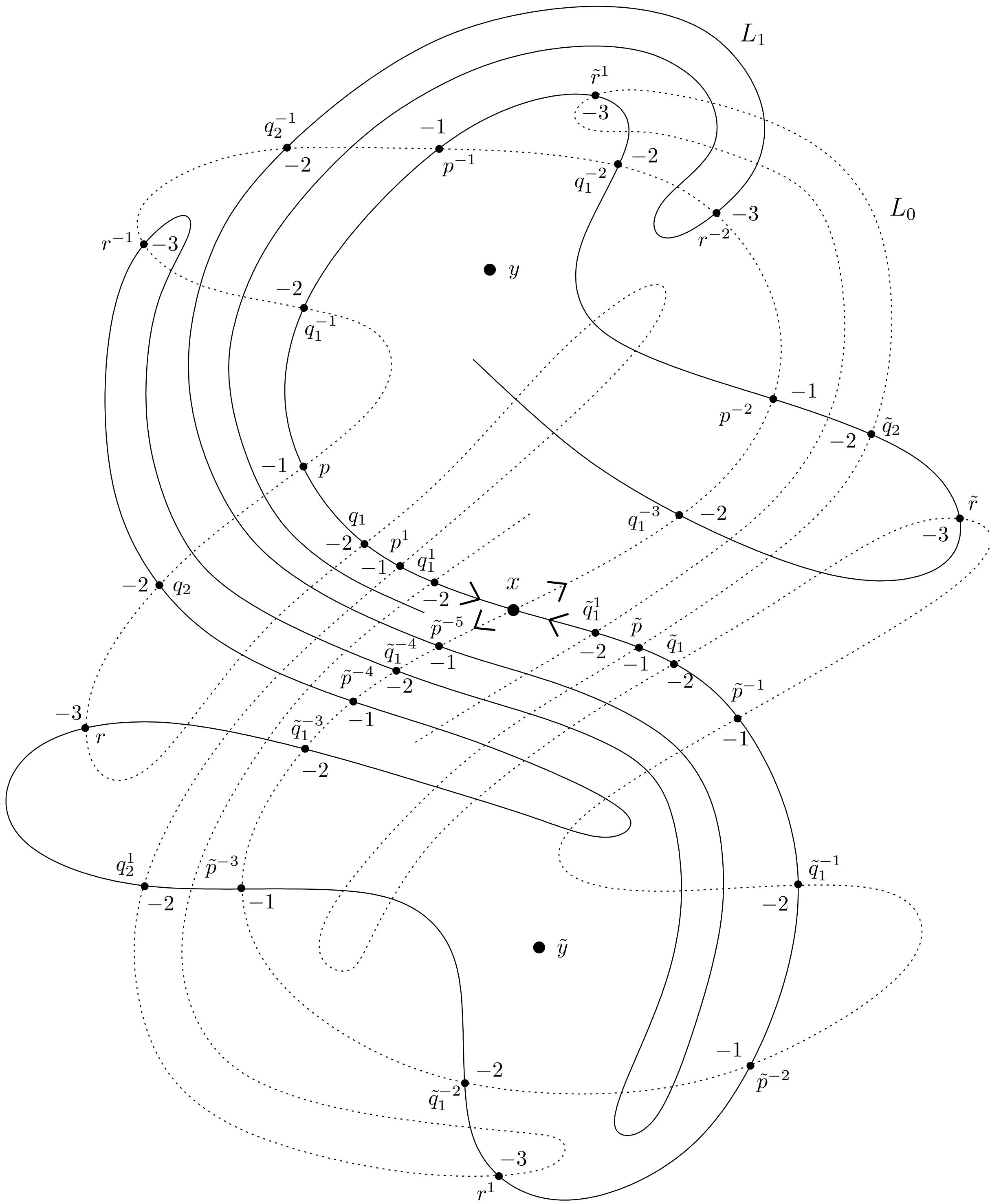


Figure 5.2. A 'figure eight' homoclinic tangle

analogous argument yields $\langle q_2^4 \rangle$ in the differential of $\langle \tilde{p} \rangle$ and altogether we find

$$\begin{aligned}
\partial\langle p\rangle &= \langle q_1\rangle - \langle q_1^{-1}\rangle + \langle q_2\rangle - \langle \tilde{q}_2^2\rangle = \langle q_2\rangle - \langle \tilde{q}_2\rangle,\\
\partial\langle \tilde{p}\rangle &= \langle \tilde{q}_1\rangle - \langle \tilde{q}_1^{-1}\rangle - \langle \tilde{q}_2\rangle + \langle q_2^4\rangle = \langle q_2\rangle - \langle \tilde{q}_2\rangle,\\
\partial\langle q_1\rangle &= -\langle r\rangle - \langle \tilde{r}^3\rangle = -\langle r\rangle - \langle \tilde{r}\rangle = -(\langle r\rangle + \langle \tilde{r}\rangle),\\
\partial\langle q_2\rangle &= \langle r\rangle - \langle r^{-1}\rangle = 0,\\
\partial\langle \tilde{q}_1\rangle &= \langle r^3\rangle + \langle \tilde{r}\rangle = \langle r\rangle + \langle \tilde{r}\rangle,\\
\partial\langle \tilde{q}_2\rangle &= \langle \tilde{r}\rangle - \langle \tilde{r}^1\rangle = 0,\\
\partial\langle r\rangle &= 0,\\
\partial\langle \tilde{r}\rangle &= 0.
\end{aligned}$$

Now we calculate

$$\begin{aligned}
&\ker\partial_{-1} = \mathbb{Z}(\langle p\rangle - \langle \tilde{p}\rangle), && \operatorname{Im}\partial_{-1} = \mathbb{Z}(\langle q_2\rangle - \langle \tilde{q}_2\rangle),\\
&\ker\partial_{-2} = \mathbb{Z}\langle q_2\rangle \oplus \mathbb{Z}\langle \tilde{q}_2\rangle \oplus \mathbb{Z}(\langle q_1\rangle + \langle \tilde{q}_1\rangle), && \operatorname{Im}\partial_{-2} = \mathbb{Z}(\langle r\rangle + \langle \tilde{r}\rangle),\\
&\ker\partial_{-3} = \mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle, && \operatorname{Im}\partial_{-3} = 0.
\end{aligned}$$

This yields as nonvanishing homology groups

$$\begin{aligned}
H_{-1} &= \frac{\ker\partial_{-1}}{\operatorname{Im}\partial_0} = \ker\partial_{-1} = \mathbb{Z}(\langle p\rangle - \langle \tilde{p}\rangle),\\
H_{-2} &= \frac{\ker\partial_{-2}}{\operatorname{Im}\partial_{-1}} = \frac{\mathbb{Z}\langle q_2\rangle \oplus \mathbb{Z}\langle \tilde{q}_2\rangle \oplus \mathbb{Z}(\langle q_1\rangle + \langle \tilde{q}_1\rangle)}{\mathbb{Z}(\langle q_2\rangle - \langle \tilde{q}_2\rangle)}\\
H_{-3} &= \frac{\ker\partial_{-3}}{\operatorname{Im}\partial_{-2}} = \frac{\mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle}{\mathbb{Z}(\langle r\rangle + \langle \tilde{r}\rangle)}.
\end{aligned}$$

3. ‘Tilted figure eight’ example

As second example we compute the primary homoclinic Floer homology of the tangle in figure 5.3 lying in $\mathbb{R}^2$ associated to an L-orientation preserving symplectomorphism. Heuristically it looks like shrinking the upper part of the tangle in figure 5.2 and expanding the lower one and tilting the latter over the first one, thus the name ‘tilted figure eight’. Compare also this ‘relation’ between figure 5.1 (a) and 5.1 (b).

This kind of homoclinic tangle might arise if we apply Melnikov’s perturbation method (see Appendix A) to the 2-dimensional integrable system sketched in figure 5.1 (b). An explicit example for figure 5.1 (b) is the averaged Hamiltonian

$$\varepsilon\bar{H}(x,y) := -\frac{\varepsilon}{4\omega}(\Omega(x^2+y^2) + \frac{3\alpha}{8}(x^2+y^2)^2 - 2\gamma x$$

associated to the Duffing equation for certain values of the constants $\varepsilon > 0$, ω, Ω, α and γ, see Guckenheimer & Holmes [**GH**].

As above primary homoclinic points are printed extra bold and their Maslov index $\mu(\cdot) := \mu(\cdot, x)$ is given. We locate the primary point p and fix an orientation of L_0 via a parametrization in direction from x to p.
There are the eight equivalence classes $\langle\tilde{p}\rangle$, $\langle\tilde{q}\rangle$, $\langle s\rangle$, $\langle\tilde{s}\rangle$, $\langle r\rangle$, $\langle\tilde{r}\rangle$, $\langle p\rangle$ and $\langle q\rangle$ with following Maslov index and nontrivial chain groups

$$\begin{aligned}
&\mu(\langle\tilde{p}\rangle) = 3, && C_3 = \mathbb{Z}\langle\tilde{p}\rangle,\\
&\mu(\langle\tilde{q}\rangle) = \mu(\langle s\rangle) = \mu(\langle\tilde{s}\rangle) = 2, && C_2 = \mathbb{Z}\langle\tilde{q}\rangle \oplus \mathbb{Z}\langle s\rangle \oplus \mathbb{Z}\langle\tilde{s}\rangle,\\
&\mu(\langle r\rangle) = \mu(\langle\tilde{r}\rangle) = 1, && C_1 = \mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle\tilde{r}\rangle,\\
&\mu(\langle p\rangle) = -1, && C_{-1} = \mathbb{Z}\langle p\rangle,\\
&\mu(\langle q\rangle) = -2, && C_{-2} = \mathbb{Z}\langle q\rangle
\end{aligned}$$

We want to determine their differentials

$$0 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} 0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} 0.$$

We recall $\langle p\rangle = \langle p^n\rangle$ for all homoclinic points and all $n \in \mathbb{Z}$. The appearance of $-\langle\tilde{s}\rangle$ in $\partial\langle\tilde{p}^3\rangle$ implies the appearance of $-\langle\tilde{s}^{-3}\rangle$ in $\partial\langle\tilde{p}\rangle$. Analogously follows $\langle\tilde{r}^{-3}\rangle$ in $\partial\langle\tilde{q}\rangle$.
Moreover $\langle\tilde{r}\rangle$ cannot appear in $\partial\langle s\rangle$ since they arise from the intersection of distinct pairs of branches of L_0 and L_1. Every $u \in \mathcal{M}(s, \tilde{r})$ would have $x \in]s, \tilde{r}[_0 \cap]s, \tilde{r}[_1$ such that its lift would not be an embedding in contradiction to Lemma 4.25. The same is true for $\langle r\rangle$ and $\partial\langle\tilde{s}\rangle$. Altogether we obtain

$$\begin{aligned}
\partial\langle\tilde{p}\rangle &= \langle\tilde{q}\rangle - \langle\tilde{q}^1\rangle + \langle s\rangle - \langle\tilde{s}^{-3}\rangle = \langle s\rangle - \langle\tilde{s}\rangle,\\
\partial\langle\tilde{q}\rangle &= \langle r^{-1}\rangle + \langle\tilde{r}^{-3}\rangle = \langle r\rangle + \langle\tilde{r}\rangle,\\
\partial\langle s\rangle &= \langle r\rangle - \langle r^{-1}\rangle = 0,\\
\partial\langle\tilde{s}\rangle &= \langle\tilde{r}\rangle - \langle\tilde{r}^1\rangle = 0,\\
\partial\langle r\rangle &= 0,\\
\partial\langle\tilde{r}\rangle &= 0,\\
\partial\langle p\rangle &= \langle q\rangle - \langle q^{-1}\rangle = 0,\\
\partial\langle q\rangle &= 0.
\end{aligned}$$

Now we calculate

$$\begin{aligned}
&\ker\partial_3 = 0, && \operatorname{Im}\partial_3 = \mathbb{Z}(\langle s\rangle - \langle\tilde{s}\rangle),\\
&\ker\partial_2 = \mathbb{Z}\langle s\rangle \oplus \mathbb{Z}\langle\tilde{s}\rangle, && \operatorname{Im}\partial_2 = \mathbb{Z}(\langle r\rangle + \langle\tilde{r}\rangle),\\
&\ker\partial_1 = \mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle\tilde{r}\rangle, && \operatorname{Im}\partial_1 = 0,\\
&\ker\partial_{-1} = \mathbb{Z}\langle p\rangle, && \operatorname{Im}\partial_{-1} = 0,\\
&\ker\partial_{-2} = \mathbb{Z}\langle q\rangle, && \operatorname{Im}\partial_{-2} = 0
\end{aligned}$$

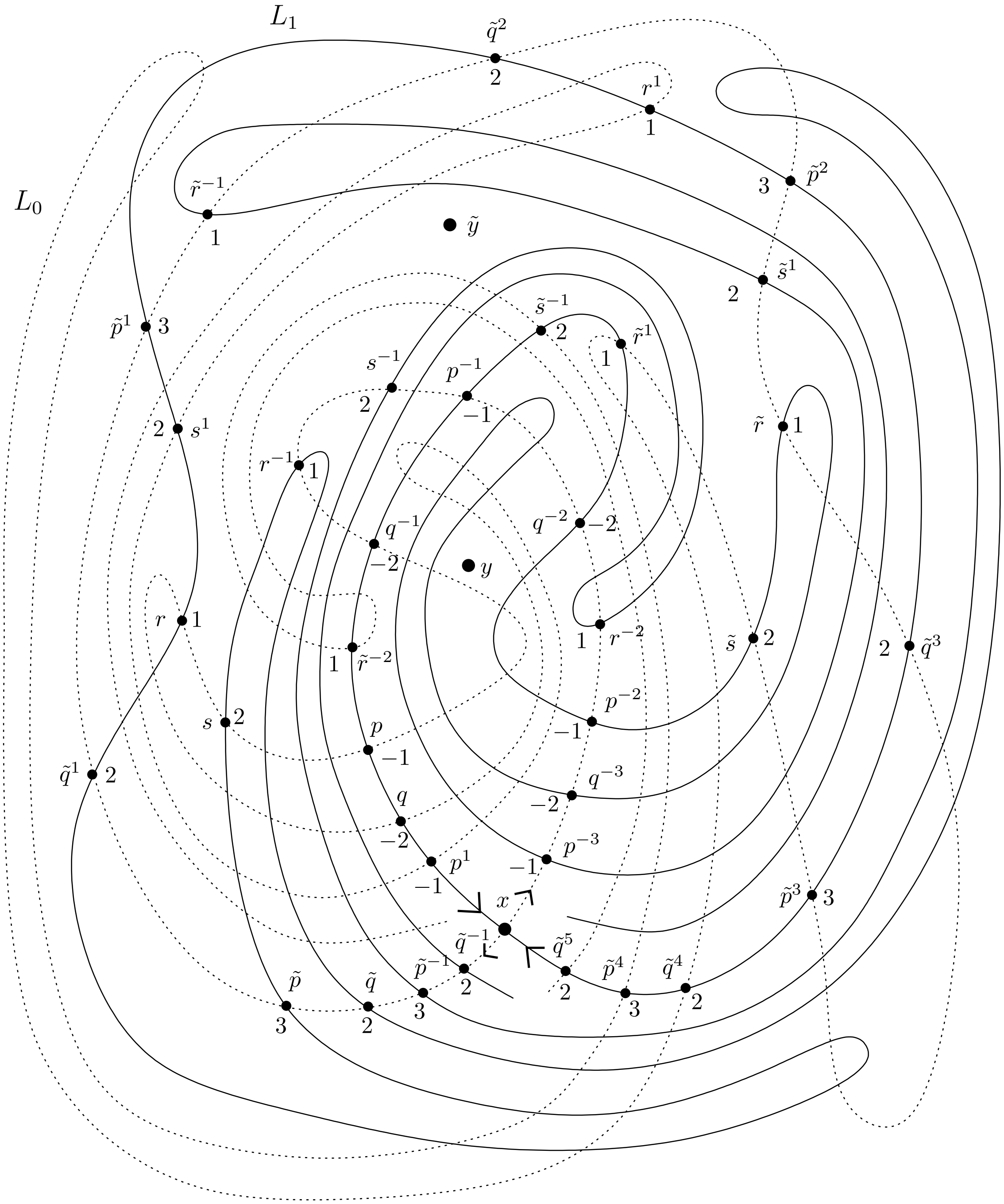


FIGURE 5.3. A 'tilted figure eight' homoclinic tangle

and obtain as homology

$$
\begin{aligned}
H_3 &= \frac{\ker \partial_3}{\operatorname{Im} \partial_4} = 0, \\
H_2 &= \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\mathbb{Z}\langle s\rangle \oplus \mathbb{Z}\langle \tilde{s}\rangle}{\mathbb{Z}(\langle s\rangle - \langle \tilde{s}\rangle)}, \\
H_1 &= \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle}{\mathbb{Z}(\langle r\rangle + \langle \tilde{r}\rangle)}, \\
H_{-1} &= \frac{\ker \partial_{-1}}{\operatorname{Im} \partial_0} = \mathbb{Z}\langle p\rangle, \\
H_{-2} &= \frac{\ker \partial_{-2}}{\operatorname{Im} \partial_{-1}} = \mathbb{Z}\langle q\rangle, \\
H_n &= 0 \quad \text{for } n \in \mathbb{Z}\backslash\{\pm 1, \pm 2, 3\}.
\end{aligned}
$$

4. Hyperbolic diffeomorphisms

Denote by $\tau : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 =: T^2$ the universal covering map. A diffeomorphism $\varphi : T^2 \to T^2$ is called **hyperbolic** if its covering map $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with $\det(\tilde{\varphi}) = 1$ and has no eigenvalue of modulus 1. In the literature they also appear under the name **Anosov automorphisms (with positive determinant)**. Hyperbolic diffeomorphisms are symplectic.

Since φ comes from a linear diffeomorphism $\tilde{\varphi}$ all possible fixed points of φ have the same eigen values and eigen vectors. If the eigen vectors are irrational directions the (un)stable manifolds of the considered fixed point of φ wrap densely around the torus. This is for instance the case for $\tilde{\varphi} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Especially *none* of their intersection points (different from the fixed point itself) is contractible to the fixed point!

Since primary homoclinic Floer homology is generated by contractible points the homology groups vanish completely for fixed points with irrational eigen directions of hyperbolic diffeomorphisms.

If we lift the tangle according to Notation 4.5 the lifted (un)stable manifolds can be identified with the linear (un)stable eigenspaces which only intersect in the origin.

5. Classification

We will give a rough classification of possible chain complexes under the assumption that each pair of intersecting branches gives rise to exactly two primary equivalence classes. Unfortunately this yields a priori not much information about the boundary operator.

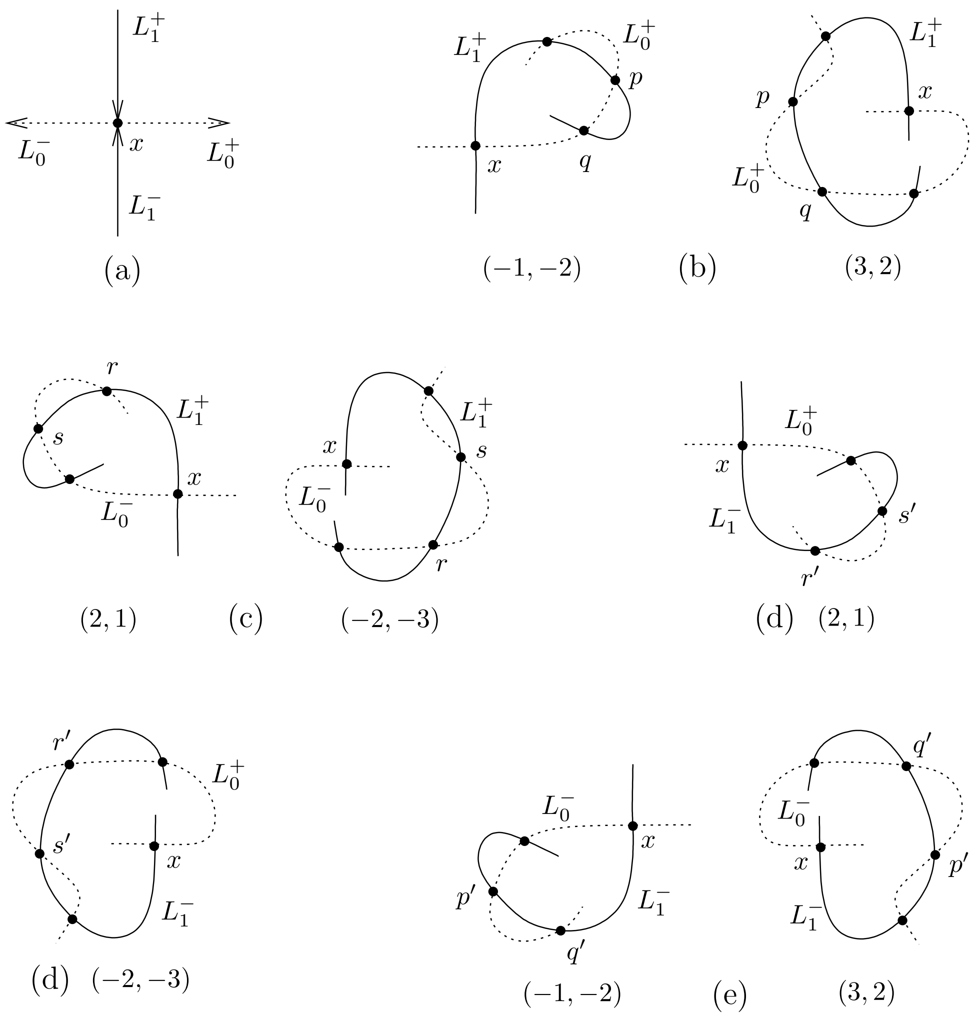


FIGURE 5.4. Intersection of the branches

There are four cases: No primary points at all or one, two, three or four pairs of intersecting branches and thus to 0, 2, 4, 6 or 8 primary homoclinic equivalence classes.

PROPOSITION 5.1. *Let L_0 and L_1 be strongly intersecting. Let each pair of intersecting branches give rise to exactly two primary equivalence classes. Then up to symmetry there are ten distinct tupels $(C_3, \ldots, C_{-3})$, compare table 5.2.*

PROOF : Let the four branches of the (un)stable manifolds be $L_0^\pm$ and $L_1^\pm$ emanating from x as sketched in figure 5.4 (a). The primary equivalence classes

$\mu(\cdot,\cdot) =$	(3,2)	(2,1)	(-1, -2)	(-2,-3)
(p,q)	y	n	y	n
(r,s)	n	y	n	y
(r',s')	n	y	n	y
(p',q')	y	n	y	n

TABLE 5.1

	C_3	C_2	C_1	C_{-1}	C_{-2}	C_{-3}	
(1)	p,p'	q,r,r',q'	s,s'				
(2)	p	q,r,r'	s,s'	p'	q'		$=(9')$
(3)	p,p'	q,r,q'	s		r'	s'	$=(5')$
(4)	p	q,r	s	p'	r',q'	s'	$=(13')$
(5)	p,p'	q,r',q'	s'		r	s	$=(3')$
(6)	p	q,r'	s'	p'	r,q'	s	$=(11')$
(7)	p,p'	q,q'			r,r'	s,s'	
(8)	p	q		p'	r,r',q'	s,s'	$=(15')$
(9)	p'	r,r',q'	s,s'	p	q		$=(2')$
(10)		r,r'	s,s'	p,p'	q,q'		
(11)	p'	r,q'	s	p	q,r'	s'	$=(6')$
(12)		r	s	p,p'	$q,r'q'$	s'	$=(14')$
(13)	p'	r',q'	s'	p	q,r	s	$=(4')$
(14)		r'	s'	p,p'	q,r,q'	s	$=(12')$
(15)	p'	q'		p	q,r,r'	s,s'	$=(8')$
(16)				p,p'	q,r,r',q'	s,s'	

TABLE 5.2

are

$$\langle p\rangle,\langle q\rangle \in L_0^+\cap L_1^+, \qquad \langle r\rangle,\langle s\rangle \in L_0^-\cap L_1^+,$$
$$\langle p'\rangle,\langle q'\rangle \in L_0^-\cap L_1^-, \qquad \langle r'\rangle,\langle s'\rangle \in L_0^+\cap L_1^-.$$

The relative index between two distinct classes within one pair of branches equals 1. Due to figure 5.4 (b) – (e) this leaves exactly two choices for the index of each pair of points: Table 5.1 gives an overview where 'y' stands for a possible index combination and 'n' if the combination is not possible.
Now table 5.2 lists all possible choices for the generators of C_m, $m = \pm1, \pm2, \pm3$ where we suppress the notion of equivalence class in favour of better readability. Table 5.2 shows that modulo symmetry $p \sim p'$ etc. there are ten cases. ∎

	H_3	H_2	H_1	H_{-1}	H_{-2}	H_{-3}
(1)	$*$	$*$	$*$	0	0	0
(2), (9)	$*$	$*$	$*$	C_{-1}	C_{-2}	0
(3), (5)	$*$	$*$	$*$	0	C_{-2}	C_{-3}
(4), (6), (11), (13)	$*$	$*$	$*$	$*$	$*$	$*$
(7)	$*$	$*$	0	0	$*$	$*$
(8), (15)	C_3	C_2	0	$*$	$*$	$*$
(10)	0	$*$	$*$	$*$	$*$	0
(12), (14)	0	C_2	C_1	$*$	$*$	$*$
(16)	0	0	0	$*$	$*$	$*$

TABLE 5.3

For example the homoclinic tangle of figure 5.2 corresponds to table 5.2 row 16 and the one of figure 5.3 corresponds to table 5.2 row 9.

Nevertheless table 5.2 also contains information about the associated homology.

COROLLARY 5.2. *Let (k) in table 5.3 denote the kth row of table 5.2. Then table 5.3 presents the homology information we can deduce from Proposition 5.1 and table 5.2.*

PROOF : The symbol '$*$' in table 5.3 stands for no additional information about the homology group in question. If there are no generators clearly the homology vanishes as for instance in the last three entries of the first row of table 5.3. Now consider exemplarily the second row: Since always $C_0 = 0$ the homology in positive (negative) degrees only depends on the positive (negative) chain groups. If $C_{-3} = 0$ and C_{-1} and C_{-2} have exactly one generator coming from a pair of adjacent points then $\partial \equiv 0$ in negative degrees and therefore $H_n = C_n$ for $n \in \{-1, -2, -3\}$. ∎

If there are less than four pairs of intersecting branches the associated behaviour can be deduced from Proposition 5.1 by neglecting the missing primary equivalence classes except if two pairs of intersecting branches imply the existence of further pairs of intersecting branches by means of the λ-lemma Theorem 3.21.

Consider $H_*(x, \varphi)$ obtained from $C_*(x, \varphi)$ and choose a pair of intersecting branches. Then the combinatorial technics used in Chapter 6 imply the existence of a chain complex $\mathcal{C}_*$ with $H_*(\mathcal{C}_*) = H_*(x, \varphi)$ such that the choosen pair of intersecting branches delivers only two generators for $\mathcal{C}$. If this can be done *simultanously* for all pairs of intersecting branches is an open problem.

CHAPTER 6

Invariance

We would very much like to imitate the modern approach to invariance of Floer homology using a homotopy argument as displayed for example in Schwarz [**Sch1, Sch2**]. But unfortunately exactly the abstractness of this argument makes it impossible for us since our methods need exact knowledge of the connecting immersions. Therefore our invariance proof here is inspired by Floer's original proof in [**Fl3**] who constructed explicit chain homotopies in order to show coinciding homologies.

In figures primary points are printed extra bold. In order to obtain smaller sketches we sometimes draw the hyperbolic fixed point x 'splitted' into two copies which have to be identified.

1. Main results

Let (M, ω) be a closed symplectic two-dimensional manifold with genus $g \geq 1$ or $(\mathbb{R}^2, dx \wedge dy)$. Denote by $\mathrm{Diff}(M)$ the group of smooth diffeomorphisms with the Whitney topology (which coincides on compact manifolds with the C^r-topology). Let $\mathrm{Diff}_\omega(M) \subset \mathrm{Diff}(M)$ be the group of symplectomorphisms. $\varphi \in \mathrm{Diff}_\omega(M)$ is **Hamiltonian** if it is the time-1 map of a time dependent Hamiltonian vector field. Recall the following perturbation result.

THEOREM 6.1 ([**PaT2**]). *Consider $\varphi \in \mathrm{Diff}^k(M)$ with $k \geq 1$ and $x \in \mathrm{Fix}(\varphi)$ and let $\psi \in \mathrm{Diff}^k(M)$ be sufficiently C^k-near to φ. Then ψ has a hyperbolic fixed point y near x and $W^i(y, \psi)$ is C^k-near $W^i(x, \varphi)$ for $i \in \{u, s\}$, at least if we restrict ourselves to compact neighbourhoods of y and x in $W^i(y, \psi)$ and $W^i(x, \varphi)$. y is called the* **continuation** *of x and the signs of the corresponding eigenvalues coincide.*

Now we give the definition of isotopies in our sense.

DEFINITION 6.2. *Let φ, $\psi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ both hyperbolic. An* **isotopy (between** (x, φ) **and** (y, ψ)**)** *is a smooth path $\Phi : [0, 1] \to \mathrm{Diff}_\omega(M)$, $\tau \mapsto \Phi(\tau) =: \Phi_\tau$ with $\Phi_0 = \varphi$, $\Phi_1 = \psi$, $x_0 = x$ and $x_1 = y$ and $x_\tau \in \mathrm{Fix}(\Phi_\tau)$ as continuation for all $\tau \in [0, 1]$ between x and y. Φ is called* **Hamiltonian** *if Φ_τ is Hamiltonian for all $\tau \in [0, 1]$.*

Attaching τ to a symbol associates it to (x_τ, Φ_τ), i.e. $\mathcal{H}^\tau_{pr}$ denotes the set of primary points of (x_τ, Φ_τ) etc.

Definition 6.3. *Let $\varphi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic. (x, φ) is called* **contractibly strongly intersecting (csi)** *if L_0 and L_1 are strongly intersecting and if each pair of branches has contractible homoclinic points. An isotopy Φ is csi if (x_τ, Φ_τ) is csi for all $\tau \in [0,1]$.*

Theorem 6.4 (Invariance). *Let (M, ω) be a closed symplectic two-dimensional manifold with genus $g \geq 1$. Let φ, $\psi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$. Let (x, φ) and (y, ψ) be csi and let all primary points of φ and ψ be transverse. Assume there is a csi isotopy Φ from (x, φ) to (y, ψ). Then*

$$H_*(x, \varphi) \simeq H_*(y, \psi).$$

For the genericity of 'strongly intersecting' compare the discussion before Theorem 3.16. We will prove Theorem 6.4 in the following sections. The proof carries over to compactly supported symplectomorphisms on $\mathbb{R}^2$:

Theorem 6.5 (Invariance). *Let φ, $\psi \in \mathrm{Diff}_{dx \wedge dy}(\mathbb{R}^2)$ be compactly supported with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$. Let (x, φ) and (y, ψ) be strongly intersecting and let all primary points of φ and ψ be transverse. Let Φ be a compactly supported strongly intersecting isotopy from (x, φ) to (y, ψ). Then*

$$H_*(x, \varphi) \simeq H_*(y, \psi).$$

'Csi' and 'compactly supported' are crucial since

Remark 6.6. *There are φ, $\psi \in \mathrm{Diff}_{dx \wedge dy}(\mathbb{R}^2)$ with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ and*

(1) different number of pairs of intersecting branches,
(2) $H_*(x, \varphi) \neq H_*(y, \psi)$

which can be joint by a symplectic isotopy.

Proof : Let $\varepsilon > 0$ be small and consider the path $(\Phi^\varepsilon_\tau)_{\tau \in [0,1]} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Phi^\varepsilon_\tau(x, y) := (x + y + \varepsilon f_\tau(x), y + \varepsilon f_\tau(x))$$

with $f_\tau(x) := -\tau x^3 - (1-\tau)x^2 + x$ for $\tau \in [0,1]$. $(\Phi^\varepsilon_\tau)^{-1}(\xi, \eta) = (\xi - \eta, \eta - \varepsilon f_\tau(\xi - \eta))$ is its inverse and $(\Phi^\varepsilon_\tau)^*(dx \wedge dy) = dx \wedge dy$, thus $\Phi^\varepsilon_\tau \in \mathrm{Diff}_{dx \wedge dy}(\mathbb{R}^2)$ for all $\tau \in [0,1]$. We have $\Phi^\varepsilon_\tau(0,0) = (0,0)$ with $D\Phi^\varepsilon_\tau(0,0) = \begin{pmatrix} 1+\varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix}$ as hyperbolic fixed point with positive eigenvalues for all τ. Now set $\varphi := \Phi^\varepsilon_0$ and $\psi := \Phi^\varepsilon_1$. φ is the *volume preserving Hénon map* and its homoclinic tangle is sketched in figure 8.2 (a): φ has one pair of intersecting branches. The tangle of ψ is a symmetric version of figure 8.2 (b) and admits four pairs of intersecting branches. Both are L-orientation preserving and we compute similarly to Chapter 5

$$H_2((0,0),\varphi) \simeq \mathbb{Z}, \quad H_1((0,0),\varphi) \simeq \mathbb{Z}, \quad H_n((0,0),\varphi) = 0 \quad \text{otherwise.}$$

But ψ has $H_3((0,0),\psi) \neq 0$, thus $H_*((0,0),\varphi) \neq H_*((0,0),\psi)$. ∎

Thus intersecting branches can be torn apart during an isotopy which satisfies no further conditions. This phenomenon prevents invariance of homology.

We note the following properties of primary homoclinic Floer homology.

REMARK 6.7. *(1) Whereas primary homoclinic Floer homology can be defined even for nonsymplectic diffeomorphisms (see Theorem 7.1) invariance only is natural within the class of symplectomorphisms. Thus primary homoclinic Floer homology is a symplectic invariant.*

(2) In Conjecture 9.1 we motivate a stronger invariance result for Hamiltonian diffeomorphisms.

(3) In contrast to classical Lagrangian Floer theory invariance of primary homoclinic Floer homology relies on the nontrivial result of (generical) existence of intersection points of the Lagrangians.

There are certain cases where the combinatorial results of the proof of Theorem 6.4 are valid, but the genericity discussion is difficult or impossible. Most generic properties only make sense for compact manifolds. Few is known about genericity of diffeomorphisms or paths of diffeomorphisms on noncompact manifolds apart from Robinson [**Ro**]. For these cases we formulate a combinatorial version of Theorem 6.4.

DEFINITION 6.8. *We call an isotopy Φ from (x,φ) to (y,ψ)* **good** *if $p \in L_0^\tau \cap L_1^\tau$ is either transverse or a* simple tangency *(defined before Remark 6.13) for $\tau \in [0,1]$ and if a pair of branches admits primary points either for all τ or for none. Moreover in the first case we require for all τ the existence of a transverse primary point.*

We deduce from the proof of Theorem 6.4

THEOREM 6.9. *Let (M,ω) be a closed symplectic two-dimensional manifold with genus $g \geq 1$ or $(\mathbb{R}^2, dx \wedge dy)$. Let φ, $\psi \in \mathrm{Diff}_\omega(\mathbb{R}^2)$ with $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ both hyperbolic. Assume all primary points of φ and ψ to be transverse. Let Φ be a good isotopy from (φ,x) to (ψ,y). Then*

$$H_*(x,\varphi) \simeq H_*(y,\psi).$$

As application we obtain the following existence and bifurcation criterion.

COROLLARY 6.10 (existence and bifurcation criterion). *Assume the conditions of Theorem 6.4 resp. Theorem 6.5 for (M,ω), (x,φ) and (y,ψ), but $H_*(x,\varphi) \neq$*

$H_*(y,\psi)$. Then (x,φ) and (y,ψ) cannot be joint by a csi (resp. compactly supported) isotopy.
Thus if there is a path $(\Phi_\tau)_{\tau\in[0,1]} \in \mathrm{Diff}_\omega(M)$ between φ and ψ then

(1) either Φ is no isotopy, i.e. there is $\tau_0 \in [0,1]$ where x_{τ_0} vanishes or undergoes a bifurcation,
(2) or if Φ is a (compactly supported) isotopy there has to be a pair of branches and some $\tau_0 \in [0,1]$ where all contractible homoclinic points vanish, i.e. there are homoclinic bifurcations.
(3) or Φ is no compactly supported *isotopy.*

In the combinatorial situation we deduce

Corollary 6.11. *Theorem 6.9 implies analogous conclusions to Corollary 6.10.*

2. Generic isotopies and their local picture

From now on let (M,ω) be compact and denote by $\tau : (\tilde{M},\tilde{\omega}) \to (M,\omega)$ with $\tau^*\omega = \tilde{\omega}$ its universal cover. A set is called **generic** if it is of second caregory of Baire.

The idea is the following: Recall that primary homoclinic Floer homology is already determined by compact segments centered around the fixed point.
Given (x,φ), (y,ψ) and Φ as in Theorem 6.4 we will perturb Φ slightly in order to obtain an isotopy whose affect on the chain complex can be modeled by a sequence of ‘moves’ as in knot theory.

First we have to discuss if the conditions on (x,φ) and (y,ψ) in Theorem 6.4 are compatible with this approach.
In Theorem 6.4 we impose the transversality condition only on the primary points of (x,φ) and (y,ψ). This is convenient for applications since it can be checked easily using Lemma 4.2 and Lemma 4.8. But our proof strategy requires perturbations of Φ. Thus we have to show that slight perturbations of the start and endpoint preserve their primary homoclinic Floer homologies.

Proposition 6.12. *Let (M,ω) be a closed two-dimensional symplectic manifold with genus $g \geq 1$. Let $\varphi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic. Let (x,φ) be csi and all primary points transverse. Then for all $\hat{\varphi} \in \mathrm{Diff}_\omega(M)$ sufficiently close to φ holds*

$$H_*(x,\varphi) = H_*(\hat{x},\hat{\varphi}).$$

where $\hat{x} \in \mathrm{Fix}(\hat{\varphi})$ is the continuation of x.

The proof is postponed to Section 7.

A fixed point of a symplectomorphism is called *elliptic* if the modulus of the eigen values equals one, but the eigenvalues are not $+1$ or -1. Generically the periodic

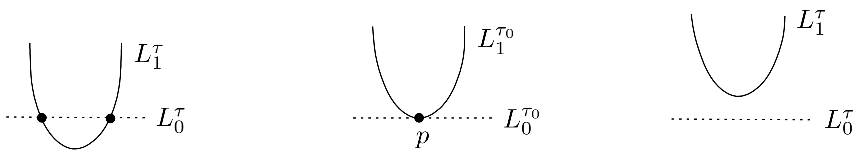


FIGURE 6.1. The local picture of a generic bifurcation

points of symplectomorphisms on 2-dimensional manifolds (also noncompact) are elliptic or hyperbolic with transversely intersecting (un)stable manifolds, see Robinson [**Ro**].

Now we ask for the generic properties of *isotopies*. A **homoclinic tangency** is a nontransverse intersection point of the stable and unstable manifold. A **simple tangency** is a homoclinic tangency where the (un)stable manifolds are tangent to each other, but do not have the same curvature. If an isotopy Φ has for $\tau_0 \in \,]0,1[$ a homoclinic tangency $p := p_{\tau_0} \in L_0^{\tau_0} \cap L_1^{\tau_0}$ we call passing from $\tau < \tau_0$ to $\tau > \tau_0$ the **unfolding** of the homoclinic tangency. A **bifurcation** is an unfolding of the homoclinic tangency p where p vanishes for $\tau \in \,]\tau_0 - \varepsilon, \tau_0[$ and splits into two points p_l^τ and p_r^τ for $\tau \in \,]\tau_0, \tau_0 + \varepsilon[$ for $\varepsilon > 0$ small or vice versa.

The results of Newhouse & Palis & Takens [**NePT**] § 2.6 and the birth-death discussion of critical points in Laudenbach [**Lau**] and Sullivan [**Su**] or the discussion of singularities of Lagrangian maps in Arnold & Gusein-Zade & Varchenko [**ArGZV**] state

REMARK 6.13. *Generically unfoldings are bifurcations associated to simple tangencies. In this case there exists for L_0^τ and L_1^τ having a bifurcation at p for $\tau = \tau_0$ the following* **local** *symplectic coordinate transformation around p for τ near τ_0 (see figure 6.1): p is mapped to the origin and a small unstable segment around p into $\{y = 0\}$ and a small stable segment around p into the graph of $f(x) \pm C(\tau - \tau_0)$ where f is homogeneous, quadratic and nondegenerate and $C > 0$.*

DEFINITION 6.14. *Let p be primary. $[p, p^1]_0 \cup [p, p^1]_1$ together with the positions of $[p, p^1]_0 \cap [p, p^1]_1$ and the immersions (embeddings on $\tilde{M}$) between adjacent points is called the* **frame induced by** *p.*

Let p be primary. Then every primary equivalence class different from p has according to Lemma 4.2 exactly one representative in the frame induced by p. As long as p persist as primary point under a perturbation its frame is ideal for observing the other primary points during this perturbation.

LEMMA 6.15. *Csi is an open property in $\mathrm{Diff}_\omega(M)$.*

Proof : Consider a csi $\varphi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed point x. We work now with the lifted tangle on the universal cover and drop the 'tilde' for sake of readability. Let p be primary. Since φ is csi there are no homoclinic loops, i.e. coinciding branches of the (un)stable manifolds. Since φ is symplectic the area enclosed by $[p,x]_0 \cup [p,x]_1$ equals the one enclosed by $[p^1,x]_0 \cup [p^1,x]_1$. Since the area swept in resp. out is nonzero $]p,p^1[_0 \cap]p,p^1[_1 \neq \emptyset$ and for small enough perturbations intersection points survive due to Theorem 6.1. ∎

Thus we can perturb (x,φ) and (y,ψ) joint by Φ to $(\hat{x},\hat{\varphi})$, $(\hat{y},\hat{\psi})$ and $\hat{\Phi}$ where $(\hat{x},\hat{\varphi})$, $(\hat{y},\hat{\psi})$ are perturbations in the sense of Proposition 6.12 and $\hat{\Phi}$ is a csi generic (in the sense of Remark 6.13) isotopy between them, i.e. all tangencies are simple.

Now consider for each pair of branches of the generic $\hat{\Phi}$ the open set

$$E := \{\tau \in [0,1] \mid \exists \text{ transverse primary points}\}$$

and its complement E^c. E^c is discrete and thus finite. For $\tau \in E^c$ perturb $\hat{\Phi}$ again slightly in order to obtain a transverse primary point within each pair of branches.

Now at all time $\tau \in [0,1]$ there exists a transverse primary point in each pair of branches. Since this point is transverse it persists for a small parameter interval and we can cover $[0,1]$ by a finite number of overlapping intervals associated to persistent primary points. Thus there is a finite number of frames within which we can observe the behaviour of the other primary points during the isotopy.
Since frames are compact and since primary points only can arise in certain distinguished parts of the frame (compare Lemma 4.8, later also Lemma 6.35) there are only finitely many $\tau \in [0,1]$ where primary points can arise or vanish as intersection points.
Since primary homoclinic Floer homology lives within compact segments centered around the fixed point we can model the relevant part of the isotopy by a sequence of moves as in knot theory.
A look at the proof of Proposition 6.12, more precisely Lemma 6.42 shows that as long as no primary point arises or vanishes the homology stays in fact untouched for purely combinatorial reasons.

3. Combinatorics of primary points

A **primary tangency** is a nontransverse primary point. Inspired by the second Reidemeister move in knot theory and the local picture of a bifurcation of Remark 6.13 we define

Definition 6.16. *Let the isotopy Φ have a bifurcation at $\tau = \tau_0$ in $p := p_{\tau_0}$. For small $\varepsilon > 0$ and $\tau \in]\tau_0 - \varepsilon, \tau_0 + \varepsilon[$ near p as sketched in figure 6.1 we call the* *local picture of an isotopy a* **move** *omitting the isotopy parameter τ. By abuse of notation we speak of an (r, s)-move if the arising points are called r and s. In fact we have a family of $(r^n, s^n)_{n\in\mathbb{Z}}$-moves.*

Due to the lack of self-intersections of the (un)stable manifolds the other Reidemeister moves do not have an equivalent in our framework.
Given an (r, s)-move there is always an embedded di-gon between r and s since $]r, s[_0 \cap\,]r, s[_1 = \emptyset$. If they are primary they are adjacent to each other. Moreover $x \notin [r, s]_0 \cup [r, s]_1$ and therefore r and s always lie on the same branches.

W.l.o.g. we will assume from now on that in case of a bifurcation in p at time τ_0 the tangency p unfolds into two points for $\tau > \tau_0$ and vanishes for $\tau < \tau_0$. This we briefly call **after** resp. **before** the bifurcation or move. We call a point **involved in a move** if it is either the homoclinic tangency at time τ_0 or one of the arising transverse homoclinic points. Persistent transverse primary points p and q are called **combinatorically affected by a move** if the value of $m(p, q)$ is changed by the move. By abuse of notation we call in this case also the elements of $\mathcal{M}(p, q)$ affected by the move.

Taking the discussion above into consideration generic isotopies are equivalent to a sequence of moves.

Nonprimary points are called **secondary**. Considering the definition of primary points there are different possibilities to generate (analogously destroy) a primary point p by a move:

(1) p arises as intersection point.
(2) p was secondary and becomes primary. This phenomenon we call a **primary-secondary flip**, briefly a **flip**.

Note that in the latter case the point needs not necessarily to be involved in the move itself, see figure 6.6. Primary points cannot switch to nontrivial homotopy classes or vice versa due to '$\cdots \cap \mathcal{H}_{[x]}$' in the definition of 'primary'.

Since there are always two points involved in a bifurcation the following types of moves are possible:

(1) If both arising points are primary the move is called **primary**.
(2) If one of the arising points is primary and the other one secondary the move is called **mixed**.
(3) If both arising points are secondary the move is called **secondary**.

We note

Lemma 6.17. *Let p be not involved itself in a given move, but let p undergo a primary-secondary flip. Then the move is a mixed one.*

Proof : Consider p primary before the move and the embedding between p and x as sketched in figure 6.2 (i) for $\mu(p, x) = -1$. In order to switch p secondary

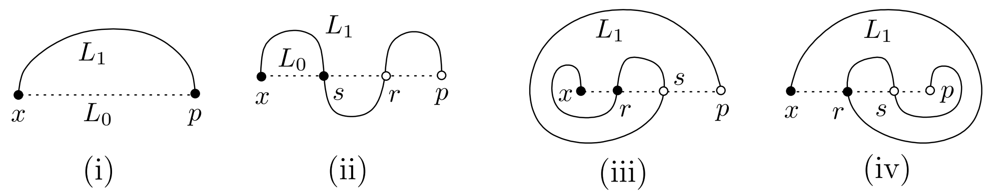


FIGURE 6.2. Causes for a primary-secondary flip

$]p,x[_0$ and $]p,x[_1$ have to intersect after the move. If we denote the intersection points produced by the move by r and s then $]r,s[_i \subset]x,p[_i$ for $i \in \{0,1\}$ which leaves up to (symplectic) diffeomorphism exactly the three possibilities of figure 6.2 (ii) – (iv). We see that in (ii) s is primary, but r secondary and that in (iii) and (iv) r is primary, but s secondary. We conclude that the primary-secondary flip of p only can be realized by a mixed (r,s)-move. ■

Having Lemma 6.17 and figure 6.2 in mind we conclude the following changes of the set of primary points under the different types of moves.

COROLLARY 6.18.
(1) A primary move generates two primary points and does not flip any.
(2) A mixed move generates one primary point, but flips a certain number of primary points secondary.
(3) A secondary move neither generates primary points nor can flip some of them secondary, i.e. the set of primary points stays untouched.

Thus the above specification characterizes how the different types of moves affect the generator set of the primary homoclinic Floer chain groups. We will inquire about the potential changes of the boundary operator in the next sections.

4. Invariance under secondary moves

In this section we show the invariance of primary homoclinic Floer homology under secondary moves. We work with the lifted homoclinic tangle on the universal cover. We already realized in Corollary 6.18 that the generator set of the chain complex stays unchanged under secondary moves and we will show now that this is also true for the boundary operator. Both implies the invariance of primary homoclinic Floer homology under secondary moves.

PROPOSITION 6.19. *Secondary moves do not affect embeddings between primary points.*

PROOF : We argue by contradiction: Let u be an embedding between primary points p and q with $\mu(p,q) = 1$. Consider an (r,s)-move such that $\{r,s\} =]p,q[_0 \pitchfork]p,q[_1$. We show: If r and s are secondary then the (r,s)-move already flipped either p or q secondary before r and s can arise.

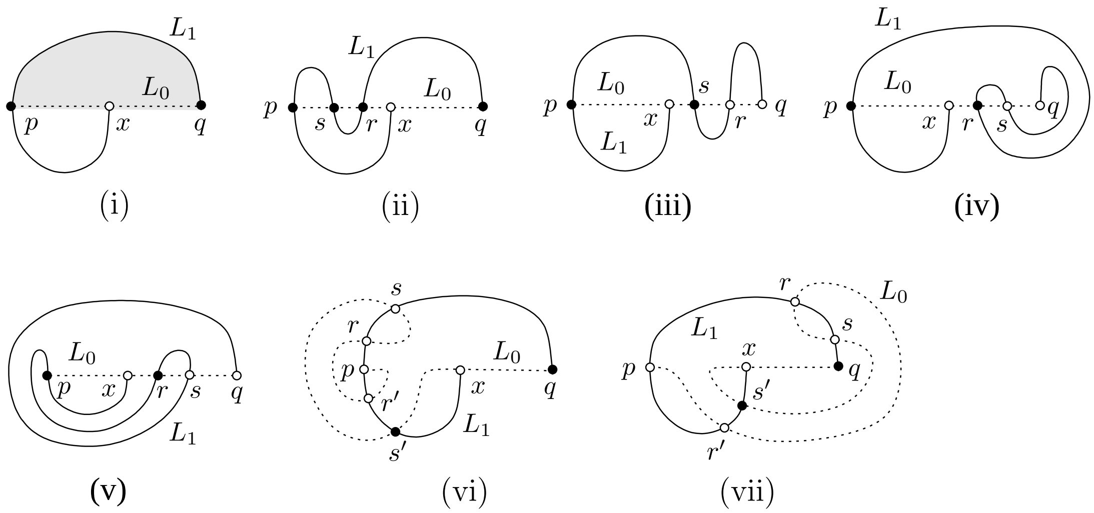


FIGURE 6.3. The effect of moves on an embedding between primary points p and q

The proof is tedious, but elementary. We just have to check for the embeddings between primary p and q of figure 4.3 and figure 4.4 all combinatorial possibilities of (r,s)-moves affecting the boundary $[p,q]_0 \cup [p,q]_1$ such that $\{r,s\} =]p,q[_0 \cap]p,q[_1$.
We only prove the assertion exemplarily in the case of figure 4.3 (b).(-1,2).(i) which is resketched in figure 6.3 (i). The strategy and result for the other cases in figure 4.3 and figure 4.4 is the same.
Consider figure 6.3 (i) and the boundary $[p,q]_0 \cup [p,q]_1$ of the embedding between p and q. $]p,q[_0 \backslash \{x\}$ consists of the two connected components $]p,x[_0$ and $]q,x[_0$. Since r and s always lie in the same branch we have to distinguish the cases r, $s \in]p,x[_0$ (see figure 6.3 (ii), (vi), (vii)) and r, $s \in]q,x[_0$ (see figure 6.3 (iii), (iv), (v)). Moreover we have to distinguish if p is connected within L_1 first to s (see figure 6.3 (ii), (iii)) or to r (see figure 6.3 (iv) – (vii)). The cases (iv) and (v) on the one hand and (vi) and (vii) on the other hand are basically the same. We deduce that (ii) is a primary move and that (iii), (iv) and (v) are mixed ones. In (vi) and (vii) the points r and s are both secondary. But before the move starting in the situation of sketch (i) generates the intersection points r and s in (vi) and (vii) it has to pass through $]p,x[_1$ generating the intersection points r' and s' which yields a mixed (r',s')-move flipping p secondary. ∎

We conclude

COROLLARY 6.20. *Proposition 6.19 and Corollary 6.18 imply the invariance of primary homoclinic Floer homology under secondary moves.*

We note

Remark 6.21. *According to the proof of Proposition 6.19 a mixed move affecting an embedding between two primary points always flips one of them secondary.*

5. Invariance under primary moves

In this section we prove the invariance of primary homoclinic Floer homology under primary moves. First we analyse where primary moves can take place and then we construct inspired by Floer's original proof in [**Fl3**] explicit chain maps and chain homotopies between the chain complexes before and after the move in order to obtain invariance of the homology. We work with the lifted homoclinic tangles on the universal cover.

Recall the notion of adjacent points from Lemma 4.8 and their properties in Lemma 4.9 and note that in a primary (r, s)-move the points r and s are adjacent to each other.
Now we have to inquire if embeddings between primary points can be affected by more than one member of the *family* of the primary move in question.

Lemma 6.22. *Let p and q be primary with $\mu(p, q) = 1$ and $u \in \mathcal{M}(p, q)$. Consider a primary (r, s)-move causing $[r^m, s^m]_i \subset\,]p, q[_i$ for $i \in \{0, 1\}$ after the move (compare figure 6.5) for some $m \in \mathbb{Z}$. Then there is no $n \in \mathbb{Z}^{\neq m}$ such that $[r^n, s^n]_i \subseteq\,]p, q[_i$ for $i \in \{0, 1\}$, i.e. an embedding of relative Maslov index 1 between primary points is combinatorically affected by at most* one *member of the primary move family.*

Proof : Let Φ be orientation preserving. Recall that r and s lie in the same branches. Since $x \notin\ [p, q]_0 \cap\ [p, q]_1$ at least one of the points p, q lies in the same branch as r and s and w.l.o.g. let it be p.
Now we argue by contradiction: Assume w.l.o.g. $m = 0$ and that there is $n \neq 0$ with $[r^n, s^n]_i \subset\,]p, q[_i$ for $i \in \{0, 1\}$. Then there is an iterate p^k with $p^k \in [r, r^n]_0 \cap\ [r, r^n]_1 \subset\,]p, q[_0 \cap\,]p, q[_1$. But then already $p^k \in\,]p, q[_0 \cap\,]p, q[_1$ before the primary (r, s)-move took place implying that u is no embedding.
In case of L-orientation reversing Φ consider Φ^2. ∎

Lemma 6.22 carries over to mixed moves as well.

We now investigate how and where primary moves can take place.

Lemma 6.23. *Let p and q be primary with $\mu(p, q) = 1$ and $\mathcal{M}(p, q) \neq \emptyset$. Consider a primary (r, s)-move such that after the move $]p, q[_0 \cap\,]p, q[_1 = \{r, s\}$. Then p and q remain primary and the geometric positions of p, q, r and s are as in figure 6.4.*

Proof : If $x \notin [p, q]_0 \cup [p, q]_1$ then p and q lie on the same branches and the claim follows from Lemma 4.8 and Lemma 4.9, compare figure 6.4 (i).

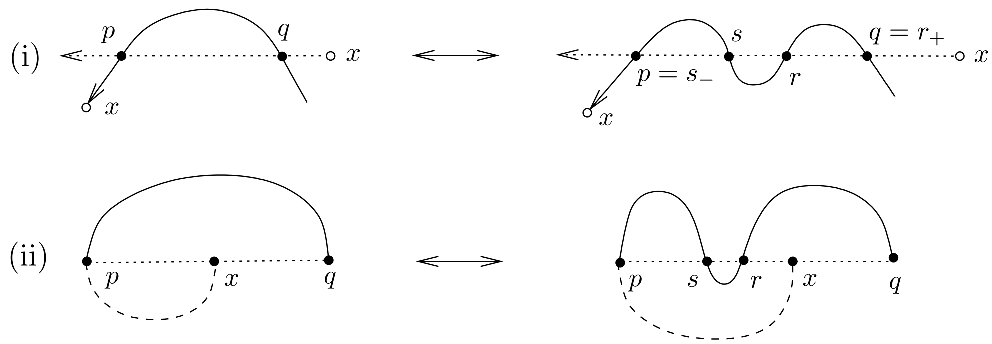


FIGURE 6.4. The primary (r, s)-move

If $x \in [p,q]_0 \cup [p,q]_1$ then x lies only in one of the segments. Now we have to check the embeddings from figure 4.3 and 4.4 as in the proof of Proposition 6.19. Figure 6.3 sketches all possible moves exemplarily for the case of figure 4.3 (b).(-1,2).(i) which is resketched in figure 6.4 (ii). The only sketch satisfying our hypothesis is figure 6.3 (ii). And analogously the other cases follow. ∎

We denote by

$$<\cdot,\cdot> \;:\; \mathcal{H}_{pr} \times \mathcal{H}_{pr} \to \{0,1\}, \quad <p,q> := \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{otherwise} \end{cases}$$

the **Kronecker symbol** and extend it by linearity to the chain complex.

We perform the following constructions on the chain complexes which still carry the $\mathbb{Z}$-action. We will divide by the $\mathbb{Z}$-action at the very end of this section. For an isotopy Φ which has a primary tangency at τ_0 and displays a primary (r,s)-move for $\tau \in [\tau_0 - \varepsilon, \tau_1 + \varepsilon]$ we abbreviate $\mathcal{H}_{pr} := \mathcal{H}_{pr}(\Phi_{\tau_0-\varepsilon}, x_{\tau_0-\varepsilon})$ and identify $\mathcal{H}'_{pr} := \mathcal{H}_{pr}(\Phi_{\tau_0+\varepsilon}, x_{\tau_0+\varepsilon}) = \mathcal{H}_{pr} \cup \{r^n, s^n \mid n \in \mathbb{Z}\}$. Moreover set

$$(\mathfrak{C}_*, \mathfrak{d}) := (\mathfrak{C}_*(x_{\tau_0-\varepsilon}, \Phi_{\tau_0-\varepsilon}), \mathfrak{d}_{x_{\tau_0-\varepsilon}, \Phi_{\tau_0-\varepsilon}}),$$
$$(\mathfrak{C}'_*, \mathfrak{d}') := (\mathfrak{C}_*(x_{\tau_0+\varepsilon}, \Phi_{\tau_0+\varepsilon}), \mathfrak{d}_{x_{\tau_0+\varepsilon}, \Phi_{\tau_0+\varepsilon}})$$

and signs after the move are marked by a prime as $m'(\cdot,\cdot)$. Given a primary (r,s)-move we define the **projection**

$$\pi : \mathfrak{C}'_* \to \mathfrak{C}_*, \quad \pi(p) = p - \sum_{n\in\mathbb{Z}} <p,r^n>r^n - <p,s^n>s^n$$

and the inclusion $\mathcal{H}_{pr} \hookrightarrow \mathcal{H}'_{pr}$ induces the homomorphism

$$i : \mathfrak{C}_* \to \mathfrak{C}'_*.$$

REMARK 6.24. *π and i commute with the $\mathbb{Z}$-action on the chain complexes.*

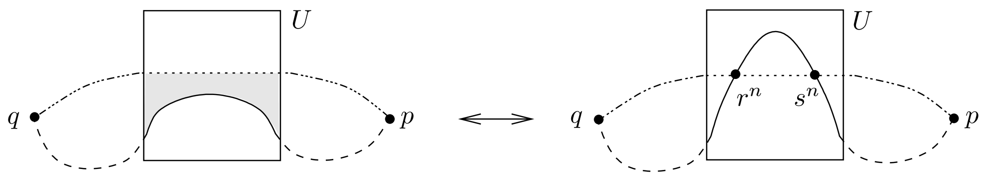


FIGURE 6.5. Behaviour of embeddings under primary (r, s)-moves

For $\mathbb{Z}_2$-coefficients and intersection points of *compact* Lagrangians L_0 and L_1 (satisfying certain additional conditions) an analogon of the following formula appears already in Floer [**Fl3**] and de Silva [**dS**]. We generalize it to our framework of noncompact Lagrangians whose intersection set carries a $\mathbb{Z}$-action and which admits $\mathbb{Z}$-coefficients in case of L-orientation preserving symplectomorphisms. The constructions in the proofs of the following statements for $\mathbb{Z}$-coefficients carry over to the case of $\mathbb{Z}_2$-coefficients in case of L-orientation reversing symplectomorphisms.

W.l.o.g. assume for the remaining section that for a primary (r, s)-move $\mu(r, s) = 1$ holds as sketched in figure 6.5.

THEOREM 6.25. *For all primary p, $q \in \mathcal{H}_{pr}$ and all primary (r, s)-moves holds*

$$m(p, q) = m'(i(p), i(q)) - \sum_{n \in \mathbb{Z}} m'(i(p), s^n) m'(r^n, s^n) m'(r^n, i(q))$$

where in fact at most one summand is nonzero.

PROOF : We know that the primary (r, s)-move changes $\mathcal{H}_{pr}$ to $\mathcal{H}'_{pr} = \mathcal{H}_{pr} \cup \{r^n, s^n \mid n \in \mathbb{Z}\}$. Lemma 6.23 yields the possible geometric positions of p, q, r, s. Lemma 6.22 ensures that for primary points p and q an embedding $u \in \mathcal{M}(p, q)$ is combinatorically affected by the primary (r, s)-move if and only if there is exactly one $n \in \mathbb{Z}$ such that $]p, q[_0 \pitchfork]p, q[_1 = \{r^n, s^n\}$ after the move as sketched in figure 6.5.
If the embedding is combinatorically affected by r^n and s^n then it corresponds under the move to three embeddings between r^n and q, r^n and s^n and p and s^n. Using some gluing construction within a small neighbourhood U containing the move as sketched in figure 6.5 we obtain $\widehat{\mathcal{M}}(p, q) \simeq \widehat{\mathcal{M}}(r^n, q) \times \widehat{\mathcal{M}}(r^n, s) \times \widehat{\mathcal{M}}(p, s^n)$. Counting with orientation we find $m(p, q) = m'(i(p), s^n) = m'(r^n, i(q)) = -m'(r^n, s^n)$ and thus $m(p, q) = -m'(i(p), s^n) m'(r^n, s^n) m'(r^n, i(q))$. For $k \in \mathbb{Z}^{\neq n}$ the embedding $u \in \mathcal{M}(p, q)$ stays unchanged and $m'(i(p), s^k) m'(r^k, s^k) m'(r^k, i(q)) = 0$.
If u is not combinatorically affected by the move then either $\widehat{\mathcal{M}}(p, s^l) = \emptyset$ or $\widehat{\mathcal{M}}(r^l, q) = \emptyset$ for all $l \in \mathbb{Z}$. In this case we have

$-m'(i(p),s^l)m'(r^l,s^l)m'(r^l,i(q)) = 0$ and $m(p,q) = m'(i(p),i(q))$, thus altogether $m(p,q) = m'(i(p),i(q)) - \sum_{n\in\mathbb{Z}} m'(i(p),s^n)m'(r^n,s^n)m'(r^n,i(q))$. ∎

In the remaining section we adjust Floer's [**Fl3**] original idea to our framework having $\mathbb{Z}$-coefficients and a $\mathbb{Z}$-action on the generator set performing a primary move. First we express the boundary operator $\mathfrak{d}$ in terms of $\mathfrak{d}'$.

Proposition 6.26.

$$\mathfrak{d}p = \pi(\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(i(p),s^n)m'(r^n,s^n)\mathfrak{d}'r^n) \quad \textit{for } \mu(i(p),r) = 0,$$

$$\mathfrak{d}p = \pi(\mathfrak{d}'i(p)) \quad \textit{otherwise.}$$

Proof : We compute formally

$$\mathfrak{d}'i(p) = \sum_{\substack{\mu(i(p),\tilde{q})=1 \\ \tilde{q}\notin\{r^n,s^n|n\in\mathbb{Z}\}}} m'(i(p),\tilde{q})\tilde{q} + \sum_{n\in\mathbb{Z}} m'(i(p),r^n)r^n + \sum_{n\in\mathbb{Z}} m'(i(p),s^n)s^n,$$

$$\mathfrak{d}'r^m = \sum_{\substack{\mu(r^m,\tilde{q})=1 \\ \tilde{q}\notin\{s^n|n\in\mathbb{Z}\}}} m'(i(p),\tilde{q})\tilde{q} + \sum_{n\in\mathbb{Z}} m'(r^m,s^n)s^n$$

and making use of the Kronecker symbol via $<\partial p,q> = m(p,q)$ etc. we rewrite Theorem 6.25 as

$$<\mathfrak{d}p,q> = <\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(i(p),s^n)m'(r^n,s^n)\mathfrak{d}'r^n, i(q)>. \tag{6.27}$$

Applying π to $\mathfrak{d}'i(p)$ and $\mathfrak{d}'r^m$ kills all r^n- and s^n-terms and we end up exactly with those terms which occur (maybe multiplied by $m'(i(p),s^n)m'(r^n,s^n)$) in (6.27). So we obtain $\mathfrak{d}p = \pi(\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(i(p),s^n)m'(r^n,s^n)\mathfrak{d}'r^n)$ for $\mu(i(p),r) = 0$ and $\mathfrak{d}p = \pi(\mathfrak{d}'i(p))$ otherwise. ∎

Now note the following technical statement:

Lemma 6.28. *Consider a primary (r,s)-move. Then for $k,\ l \in \mathbb{Z}$ holds $m'(r^k,s^l) = 0$ for $k \neq l$.*

Proof : For fixed m the points r^m and s^m are adjacent, but not r^m and s^{m-1} and s^m and r^{m+1} since otherwise $\langle r\rangle$ and $\langle l\rangle$ would be the only primary equivalence classes of their pair of intersecting branches implying nonintersecting branches before the move in contradiction the assumption on the isotopy. From Lemma 4.8 and Lemma 4.9 we deduce $]r^m,s^n[_0 \cap\,]r^m,s^n[_1 \neq \emptyset$ for $|m-n| \geq 1$ and thus $\mathcal{M}(r^m,s^n) = \emptyset$ and $m'(r^m,s^n) = 0$. ∎

For the following proofs keep in mind that

$$m(p,q)m(p,q) = \begin{cases} 1 & \text{if } m(p,q) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

On the following chain maps the chain equivalence will base.

LEMMA 6.29. *We define on the generators*

$$f:(\mathfrak{C}'_*,\mathfrak{d}')\to(\mathfrak{C}_*,\mathfrak{d}),\qquad f(p):=\pi(p-\sum_{n\in\mathbb{Z}}m'(r^n,s^n)\langle p,s^n\rangle\mathfrak{d}'r^n),$$
$$g:(\mathfrak{C}_*,\mathfrak{d})\to(\mathfrak{C}'_*,\mathfrak{d}'),\qquad g(p):=i(p)-\sum_{n\in\mathbb{Z}}m'(r^n,s^n)m'(i(p),s^n)r^n$$

and extend them by linearity. Then f and g are chain maps. Moreover they commute with the $\mathbb{Z}$-action on the chain complexes.

PROOF : For $m\in\mathbb{Z}$ we compute

$$f(r^m)=0,\quad f(s^m)=-m'(r^m,s^m)\pi\mathfrak{d}'r^m,\quad f(p)=\pi(p)\quad\text{for } p\neq r^m,\ s^m.$$

Recall $\mu(r^m,s^m)=1$ and $\widehat{\mathcal{M}}(r^m,s^m)\neq\emptyset$ such that $m'(r^m,s^m)=\pm1$ and keep the equations

$$\mathfrak{d}p=\pi(\mathfrak{d}'i(p)-\sum_{n\in\mathbb{Z}}m'(p,s^n)m'(r^n,s^n)\mathfrak{d}'r^n)\quad\text{for }\mu(i(p),r)=0,$$
$$\mathfrak{d}p=\pi(\mathfrak{d}'i(p))\quad\text{otherwise}$$

from Proposition 6.26 in mind. For f we obtain

$$\begin{aligned}
f(\mathfrak{d}'r^m)&=f(i\pi\mathfrak{d}'r^m+\sum_{n\in\mathbb{Z}}m'(r^m,s^n)s^n)\overset{6.28}{=}f(i\pi\mathfrak{d}'r^m+m'(r^m,s^m)s^m)\\
&=\pi i\pi\mathfrak{d}'r^m-0+0-m'(r^m,s^m)m'(r^m,s^m)\pi(\mathfrak{d}'r^m)\\
&=\pi\mathfrak{d}'r^m-\pi\mathfrak{d}'r^m=0=\mathfrak{d}0\\
&=\mathfrak{d}f(r^m),\\
\mathfrak{d}f(s^m)&=\mathfrak{d}(-m'(r^m,s^m)\pi\mathfrak{d}'r^m)\overset{6.26}{=}\pi\mathfrak{d}'(-m'(r^m,s^m)i\pi\mathfrak{d}'r^m)\\
&=-m'(r^m,s^m)\pi\mathfrak{d}'i\pi\mathfrak{d}'r^m\\
&=-m'(r^m,s^m)\pi\mathfrak{d}'(\mathfrak{d}'r^m-\sum_{n\in\mathbb{Z}}m'(r^m,s^n)s^n)\\
&\overset{6.28}{=}-m'(r^m,s^m)(\pi\mathfrak{d}'\mathfrak{d}'r^m-m'(r^m,s^m)\pi\mathfrak{d}'s^m)\\
&=\pi\mathfrak{d}'s^m\\
&=f(\mathfrak{d}'s^m).
\end{aligned}$$

For $p \neq r^m$, s^m for $m \in \mathbb{Z}$ we obtain

$$\begin{aligned}
f(\mathfrak{d}'p) &= f(i\pi\mathfrak{d}'p + \sum_{n\in\mathbb{Z}} m'(p,r^n)r^n + m'(p,s^n)s^n) \\
&= \pi i\pi\mathfrak{d}'p - \pi\left(\sum_{l\in\mathbb{Z}} m'(r^l,s^l)<i\pi\mathfrak{d}'p, s^l>\mathfrak{d}'r^l\right) \\
&\quad + \pi\left(\sum_{n\in\mathbb{Z}} m'(p,r^n)r^n\right) - \pi\left(\sum_{l\in\mathbb{Z}} m'(r^l,s^l)<\sum_{n\in\mathbb{Z}} m'(p,r^n)r^n, s^l>\mathfrak{d}'r^l\right) \\
&\quad + \pi\left(\sum_{n\in\mathbb{Z}} m'(p,s^n)s^n\right) - \pi\left(\sum_{l\in\mathbb{Z}} m'(r^l,s^l)<\sum_{n\in\mathbb{Z}} m'(p,s^n)s^n, s^l>\mathfrak{d}'r^l\right) \\
&= \pi i\pi\mathfrak{d}'p - 0 + 0 - 0 + 0 - \sum_{l\in\mathbb{Z}} m'(p,s^l)m'(r^l,s^l)\pi\mathfrak{d}'r^l \\
&= \pi\mathfrak{d}'p - \sum_{l\in\mathbb{Z}} m'(p,s^l)m'(r^l,s^l)\pi\mathfrak{d}'r^l \\
&= \pi\mathfrak{d}'i\pi p - \sum_{l\in\mathbb{Z}} m'(p,s^l)m'(r^l,s^l)\pi\mathfrak{d}'r^l \\
&\overset{6.26}{=} \mathfrak{d}\pi p \\
&= \mathfrak{d}f(p).
\end{aligned}$$

Now we extend the definition of $m'(p,q)$ etc. by linearity from primary points to elements of $\mathfrak{C}'_*$, i.e. $m'(\sum_j p_j, q) := \sum_j m'(p_j, q)$ and consider g:
Case $\mu(i(p), r) = 0$: We first show

$$i\pi\mathfrak{d}'g(p) = \mathfrak{d}'g(p) \tag{6.30}$$

which follows from $<\mathfrak{d}'g(p), r^m> = 0$ due to $\mu(i(p), r^m) = 0$ for $m \in \mathbb{Z}$ and

$$\begin{aligned}
<\mathfrak{d}'g(p), s^m> &= m'(g(p), s^m) = m'(i(p) - \sum_{n\in\mathbb{Z}} m'(r^n,s^n)m'(i(p),s^n)r^n, s^m) \\
&= m'(i(p),s^m) - \sum_{n\in\mathbb{Z}} m'(r^n,s^n)m'(i(p),s^n)m'(r^n,s^m) \\
&\overset{6.28}{=} m'(i(p),s^m) - m'(r^m,s^m)m'(i(p),s^m)m'(r^m,s^m) \\
&= m'(i(p),s^m) - m'(i(p),s^m) \\
&= 0.
\end{aligned}$$

Now we obtain

$$\begin{aligned} g(\mathfrak{d}p) &\overset{\mu(i(p),r)=0}{=} i(\mathfrak{d}p) \overset{6.26}{=} i\pi\mathfrak{d}'(i(p) - \sum_{n\in\mathbb{Z}} m'(i(p),s^n)m'(r^n,s^n)r^n) \\ &= i\pi\mathfrak{d}'g(p) \overset{6.30}{=} \mathfrak{d}'g(p). \end{aligned}$$

Case $\mu(i(p),r) \neq 0$: First note

$$m'(\mathfrak{d}'i(p),s^m) = <\mathfrak{d}'(\mathfrak{d}'i(p)),s^m> = 0 \tag{6.31}$$

and then compute

$$\begin{aligned} g(\mathfrak{d}p) &\overset{6.26}{=} g(\pi\mathfrak{d}'i(p)) \\ &= i\pi\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(r^n,s^n)m'(i\pi\mathfrak{d}'i(p),s^n)r^n \\ &= i\pi\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(r^n,s^n)m'(\mathfrak{d}'i(p) - \sum_{l\in\mathbb{Z}} m'(i(p),r^l)r^l - m'(i(p),s^l)s^l,s^n)r^n \\ &= i\pi\mathfrak{d}'i(p) - \sum_{n\in\mathbb{Z}} m'(r^n,s^n)(m'(\mathfrak{d}'i(p),s^n) - \sum_{l\in\mathbb{Z}} m'(i(p),r^l)m'(r^l,s^n) - 0)r^n \\ &\overset{(6.31)}{=} i\pi\mathfrak{d}'i(p) + \sum_{n\in\mathbb{Z}} m'(r^n,s^n)m'(i(p),r^n)m'(r^n,s^n)r^n \\ &= i\pi\mathfrak{d}'i(p) + \sum_{n\in\mathbb{Z}} m'(i(p),r^n)r^n \\ &\overset{\mu(i(p),r)\neq 0}{=} \mathfrak{d}'i(p) \\ &\overset{\mu(i(p),r)\neq 0}{=} \mathfrak{d}'(g(p)). \end{aligned}$$

That f and g commute with the $\mathbb{Z}$-action on the complexes relies on Remark 6.24. ∎

Now we show that f and g induce isomorphisms between the homologies of $(\mathfrak{C}'_*,\mathfrak{d}')$ and $(\mathfrak{C}_*,\mathfrak{d})$.

Theorem 6.32. *The homologies of* $(\mathfrak{C}'_*,\mathfrak{d}')$ *and* $(\mathfrak{C}_*,\mathfrak{d})$ *are isomorphic.*

Proof : For f and g from Lemma 6.29 we show that $f_* : H(\mathfrak{C}'_*,\mathfrak{d}') \to H(\mathfrak{C}_*,\mathfrak{d})$ and $g_* : H(\mathfrak{C}_*,\mathfrak{d}) \to H(\mathfrak{C}'_*,\mathfrak{d}')$ are inverse to each other. For that it is enough to show $f \circ g \simeq \mathrm{Id}_{\mathfrak{C}_*}$ and $g \circ f \simeq \mathrm{Id}_{\mathfrak{C}'_*}$ where $\simeq$ stands for homotopic by a chain homotopy. $f \circ g : (\mathfrak{C}_*, \mathfrak{d}) \to (\mathfrak{C}_*, \mathfrak{d})$ is even the identity:

$$\begin{aligned}
f(g(p)) &= f(i(p)) - \sum_{n\in\mathbb{Z}} m'(r^n, s^n) m'(i(p), s^n) f(r^n) = f(i(p)) \\
&= \pi i(p) - \pi \left(\sum_{n\in\mathbb{Z}} m'(r^n, s^n) {<} i(p), s^n {>} \mathfrak{d}' r^n \right) = \pi i(p) \\
&= \mathrm{Id}_{\mathfrak{C}_*}(p).
\end{aligned}$$

Unfortunately, this is not true for $g \circ f$. But we can find a chain homotopy $h : (\mathfrak{C}'_*, \mathfrak{d}') \to (\mathfrak{C}'_{*+1}, \mathfrak{d}')$ satisfying $g \circ f - \mathrm{Id}_{\mathfrak{C}'_*} = h \circ \mathfrak{d}' + \mathfrak{d}' \circ h$. Choose

$$h(p) := - \sum_{n\in\mathbb{Z}} {<} s^n, p {>} m'(r^n, s^n) r^n$$

and compute for $m \in \mathbb{Z}$

$$\begin{aligned}
(h \circ \mathfrak{d}' + \mathfrak{d}' \circ h)(r^m) &= - \sum_{n\in\mathbb{Z}} {<} s^n, \mathfrak{d}' r^m {>} m'(r^n, s^n) r^n - \mathfrak{d}' \left(\sum_{n\in\mathbb{Z}} {<} s^n, r^m {>} m'(r^n, s^n) r^n \right) \\
&= - \sum_{n\in\mathbb{Z}} m'(r^m, s^n) m'(r^n, s^n) r^n \\
&\overset{6.28}{=} -m'(r^m, s^m) m'(r^m, s^m) r^m \\
&= -r^m, \\
(h \circ \mathfrak{d}' + \mathfrak{d}' \circ h)(s^m) &= - \sum_{n\in\mathbb{Z}} {<} s^n, \mathfrak{d}' s^m {>} m'(r^n, s^n) r^n - \sum_{n\in\mathbb{Z}} {<} s^n, s^m {>} m'(r^n, s^n) \mathfrak{d}' r^n \\
&= -m'(r^m, s^m) \mathfrak{d}' r^m
\end{aligned}$$

and for $p \neq r^m$, s^m for $m \in \mathbb{Z}$

$$\begin{aligned}
(h \circ \mathfrak{d}' + \mathfrak{d}' \circ h)(p) &= - \sum_{n\in\mathbb{Z}} {<} s^n, \mathfrak{d}' p {>} m'(r^n, s^n) r^n - \mathfrak{d}' \left(\sum_{n\in\mathbb{Z}} {<} s^n, p {>} m'(r^n, s^n) r^n \right) \\
&= - \sum_{n\in\mathbb{Z}} m'(p, s^n) m'(r^n, s^n) r^n.
\end{aligned}$$

On the other hand we obtain

$$
\begin{aligned}
(g \circ f - \mathrm{Id}_{\mathfrak{C}'_*})(r^m) &= g(f(r^m)) - r^m = -r^m, \\
(g \circ f - \mathrm{Id}_{\mathfrak{C}'_*})(s^m) &= g(\pi(s^m) - \pi \left(\sum_{n\in\mathbb{Z}} m'(r^n, s^n) < s^m, s^n > \mathfrak{d}' r^n \right)) - s^m \\
&= g(-m'(r^m, s^m)\pi\mathfrak{d}' r^m) - s^m \\
&= -m'(r^m, s^m) g(\pi\mathfrak{d}' r^m) - s^m \\
&= -m'(r^m, s^m)(i\pi\mathfrak{d}' r^m - \sum_{n\in\mathbb{Z}} m'(r^n, s^n) m'(i\pi\mathfrak{d}' r^m, s^n) r^n) - s^m \\
&= -m'(r^m, s^m) i\pi\mathfrak{d}' r^m - s^m \\
&= -m'(r^m, s^m)(\mathfrak{d}' r^m - \sum_{n\in\mathbb{Z}} m'(r^m, s^n) s^n) - s^m \\
&\overset{6.28}{=} -m'(r^m, s^m)\mathfrak{d}' r^m + m'(r^m, s^m) m'(r^m, s^m) s^m - s^m \\
&= -m'(r^m, s^m)\mathfrak{d}' r^m
\end{aligned}
$$

and for $p \neq r^m$, s^m for $m \in \mathbb{Z}$

$$
\begin{aligned}
(g \circ f - \mathrm{Id}_{\mathfrak{C}'_*})(p) &= g(\pi(p) - \sum_{n\in\mathbb{Z}} m'(r^n, s^n) < p, s^n > \pi\mathfrak{d}' r^n) - p \\
&= g(\pi(p)) - p = i\pi(p) - \sum_{n\in\mathbb{Z}} m'(r^n, s^n) m'(i\pi(p), s^n) r^n - p \\
&= -\sum_{n\in\mathbb{Z}} m'(r^n, s^n) m'(p, s^n) r^n.
\end{aligned}
$$

Comparing the results yields $g \circ f - \mathrm{Id}_{\mathfrak{C}'_*} = h \circ \mathfrak{d}' + \mathfrak{d}' \circ h$ which proves the claim. ∎

Moreover note

Remark 6.33. *The chain homotopy h defined in the proof of Theorem 6.32 commutes with the $\mathbb{Z}$-action on the chain complexes.*

Now we get rid of the $\mathbb{Z}$-action. Define C_* and C'_* analogously to $\mathfrak{C}_*$ and $\mathfrak{C}'_*$. Since f, g and h commute with the $\mathbb{Z}$-action on the chain complexes they pass to C_* and C'_*. Thus we obtain

Theorem 6.34. *The homologies of (C_*, ∂) and (C'_*, ∂') are isomorphic, i.e. primary moves leave the primary homoclinic Floer homology invariant.*

6. Invariance under mixed moves

In this section we will show the invariance of primary homoclinic Floer homology under mixed moves. This will be done by recognizing them as concatenation of primary and secondary moves which leave the homology invariant as already

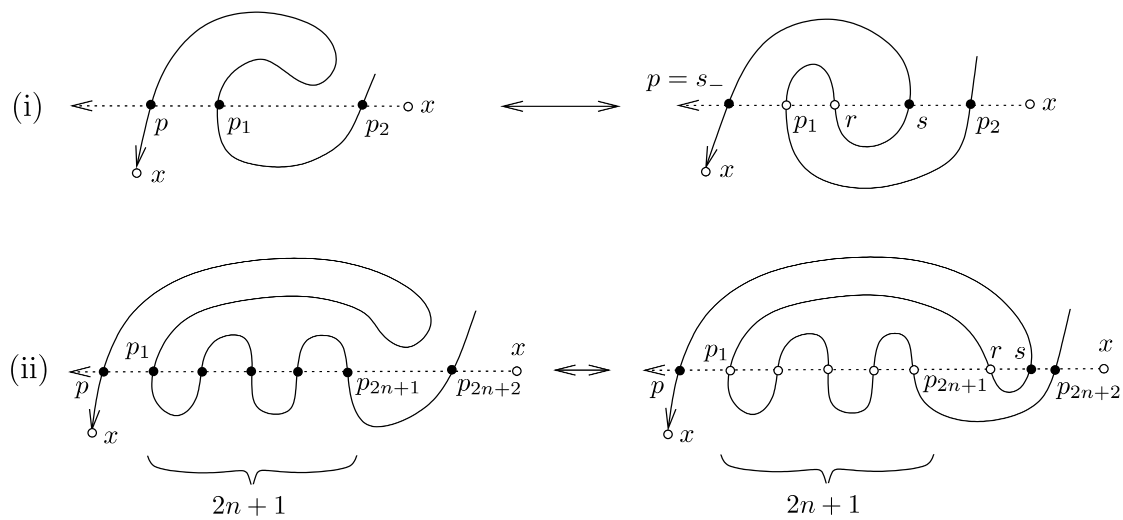


FIGURE 6.6. Mixed (r, s)-moves with one flip in (i) and 2n+1 flips in (ii)

proved in the previous sections. We will work with the lifted tangles on the universal cover.

Now we want to investigate how mixed moves look like. If a (r, s)-move flips a primary points p secondary the segments $]p, x[_0$ and $]p, x[_1$ have to intersect after the move. In particular, r, s and p have to lie in the same pair of branches. Recall the properties of adjacent points from Lemma 4.8 and Lemma 4.9 and the notion of a frame.

LEMMA 6.35. *Let p be a primary points and consider the frame $[p, p^{-1}]_0 \cap [p, p^{-1}]_1$ and denote the primary points p, p_+, $(p_+)_+$, $\ldots$, p^{-1} by $p_0, \ldots, p_n$. Let k be the smallest and l the largest index whose associated points p_k and p_l undergo a primary-secondary flip during the mixed (r, s)-move. Then $]p_l, p_{l+1}[_0 \cap\,]p_{k-1}, p_k[_1 = \{r, s\}$.*

PROOF : (Compare figure 6.6:) Let w.l.o.g. r be secondary and s primary. The primary s has two adjacent points $s_\pm$ connected to s by an embedding of relative index 1. Since there is also an embedding of relative index 1 between s and r the latter has to lie on $]s_-, s[_0$ since it is secondary by assumption. We find $s_- = p_{k-1}$ and $s_+ = p_{l+1}$ and all p_j with $k \leq j \leq l$ perform a primary-secondary flip. For $k = 1$ the situation is sketched in figure 6.6 (i) with $l = 1$ (and thus one flip) and in (ii) with $l = 2n + 1$ (and thus 2n+1 flips). ■

Since the (un)stable manifolds are free of self-intersections a mixed move always takes place within a fixed frame, i.e. the mixed move cannot 'overlap' into another iterate of the frame.

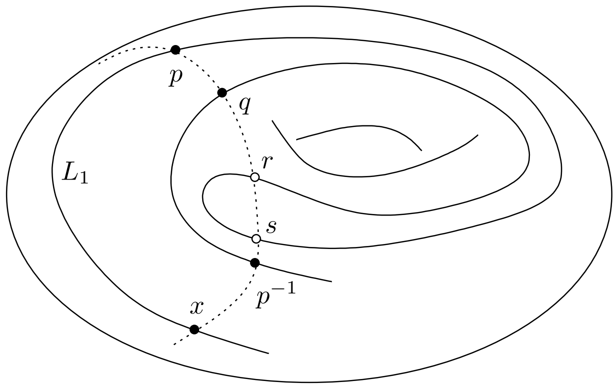


FIGURE 6.7. Arising of nontrivial homotopy classes

DEFINITION 6.36. *Mixed (r, s)-moves with exactly one primary-secondary flip are called* **simple**.

Before we show the invariance of primary homoclinic Floer homology under mixed moves we motivate the condition '$\cdots \cap \mathcal{H}_{[x]}$' in the definition of primary points since it enters here crucially for invariance.

PROPOSITION 6.37. *Without the condition '$\cdots \cap \mathcal{H}_{[x]}$' in the definition of 'primary' the primary homoclinic Floer homology would not be invariant.*

PROOF : Consider the situation of figure 6.7 where a move circles around the hole of the torus. Assume for sake of simplicity that only the branches containing p intersect.
First case: We start with p and q primary and obtain r and s secondary. q remains primary since s is not contractible. We have before and after the move $\mu(p) = -1$ and $\mu(q) = -2$, $C_{-1} = \mathbb{Z}\langle p\rangle$ and $C_{-2} = \mathbb{Z}\langle q\rangle$ with $\partial\langle p\rangle = \langle q\rangle - \langle q\rangle = 0$ and $\partial\langle q\rangle = 0$ such that

$$H_{-1} = \mathbb{Z}\langle p\rangle \qquad \text{and} \qquad H_{-2} = \mathbb{Z}\langle q\rangle$$

and all other homology classes vanish.
Second case: Dropping '$\cdots \cap \mathcal{H}_{[x]}$' is equivalent to using the *contractible* semi-primary points as generators of the chain complex. Before the move p and q are contractible and semi-primary, but after the move q is no longer semi-primary. The generated r is secondary and s semi-primary, but not contractible. Thus it is excluded as generator. Before the move we obtain $H_{-1} = \mathbb{Z}\langle p\rangle$ and $H_{-2} = \mathbb{Z}\langle q\rangle$ and $H_* = 0$ for $n \neq -1, -2$. But after the move there is only p left as generator. Thus $H_{-1} = \mathbb{Z}\langle p\rangle$ and zero otherwise. ∎

We now consider invariance under simple mixed moves.

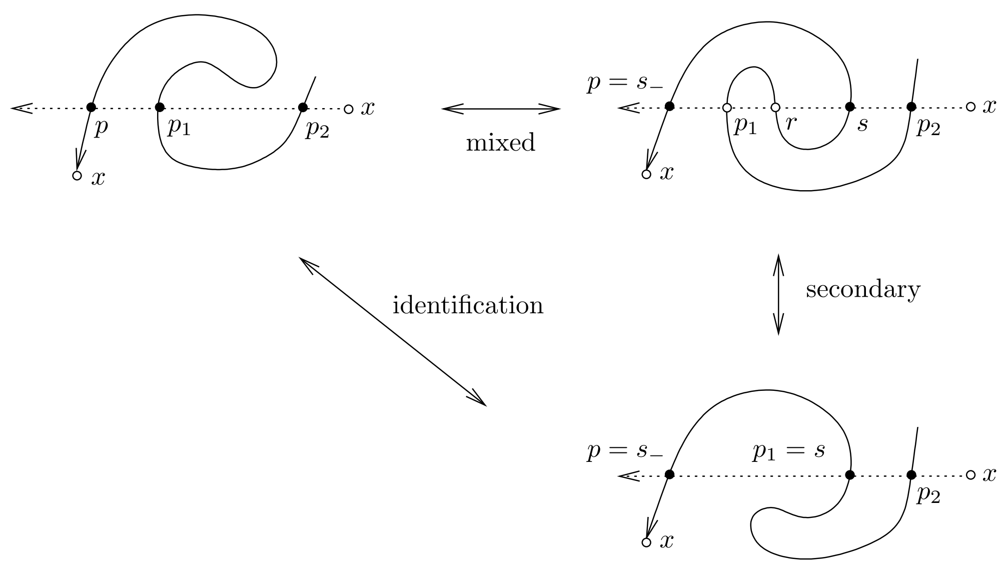


FIGURE 6.8. Invariance under simple mixed moves

PROPOSITION 6.38. *Primary homoclinic Floer homology stays invariant under simple mixed moves.*

PROOF : There is a direct and an indirect proof which we will give both.
Indirect proof: As displayed in figure 6.8 the simple mixed (r, s)-move can be recognized as an identification followed by the secondary (p_1, r)-move. Since both leave the homology invariant so does the simple mixed move.
Direct proof: We construct an explicit chain complex isomorphism keeping the notation from figure 6.6 (i). Figure 6.9 indicates the potential embeddings with flipping vertex p_1 and their correspondence to the new primary point s: Consider the four sectors at p arising from the intersection of the (un)stable manifolds. Now check clockwise the sectors and the change of the associated embedding(s). Since there are no embeddings between iterates of p_1 we can at once divide by the $\mathbb{Z}$-action and set on the generating equivalence classes

$$f : (C_*, \partial) \longrightarrow (C'_*, \partial'), \quad \begin{cases} \langle p_1 \rangle \mapsto \langle s \rangle, \\ a \mapsto a \quad \text{for } a \in \tilde{\mathcal{H}}_{pr} \backslash \{\langle p_1 \rangle\} \end{cases}$$

which is a chain map yielding the desired isomorphism between the chain complexes and thus between the homologies.

∎

Now we consider the invariance under arbitrary mixed moves.

THEOREM 6.39. *Primary homoclinic Floer homology is invariant under mixed moves.*

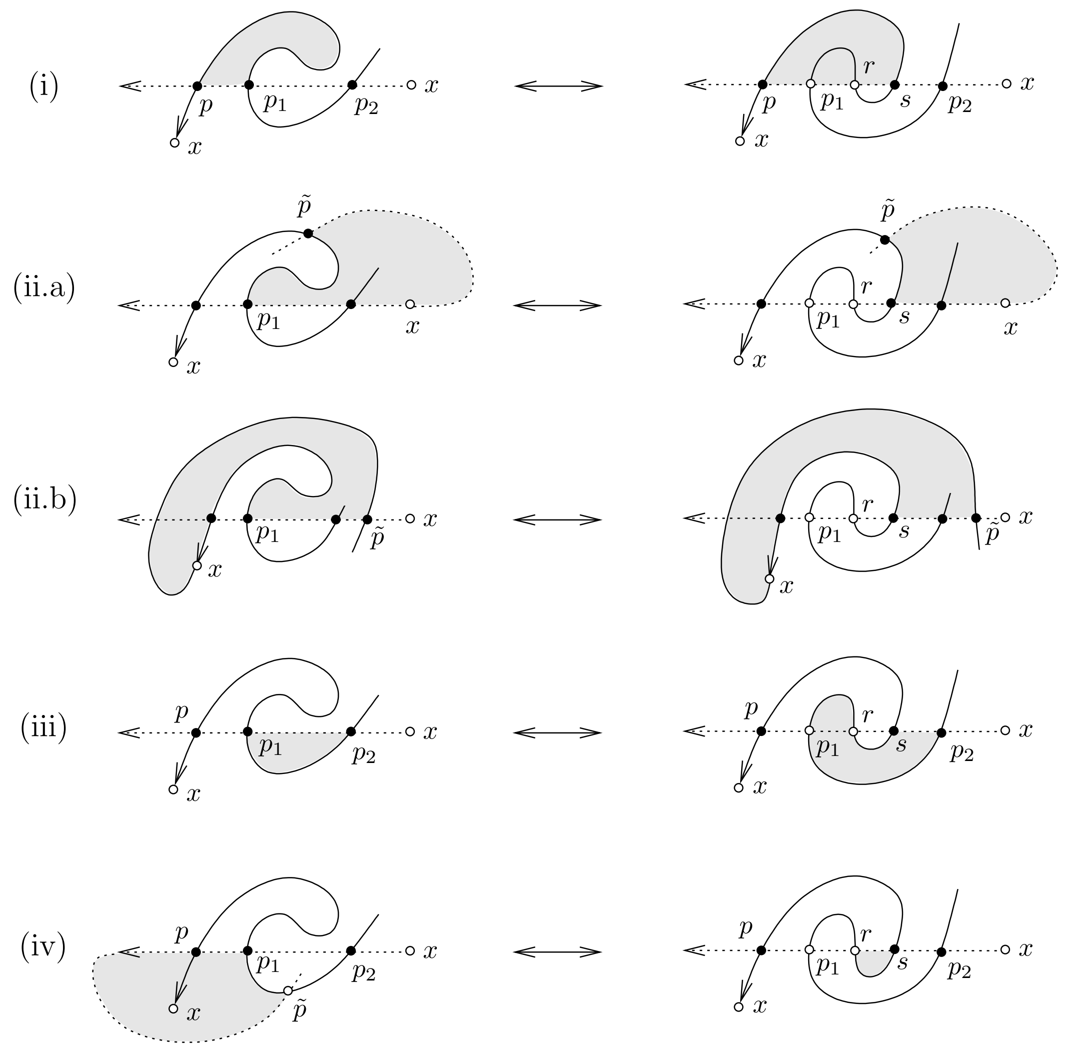


FIGURE 6.9. Corresponding embeddings induced by a simple mixed move

PROOF : For simple mixed moves the claim was already proven in Proposition 6.38. Now consider the case of $2n+1$ flips for $n \geq 1$. The procedure is sketched in figure 6.10 using the conventions of figure 6.6. In the left upper corner the situation before the mixed move is sketched and on the right the situation afterwards where the move has flipped $p_1, \ldots, p_{2n+1}$ secondary and generated the new primary point s. The mixed move can be composed by n primary moves applied successively to $(p_2, p_3), \ldots, (p_{2n-2}, p_{2n-1})$ as sketched downwards on the left side of figure 6.10 followed by the now simple mixed (r, s)-move and finally by n secondary moves restoring $p_2, \ldots, p_{2n+1}$. Since we can identify the sketch in the right upper and lower corner of figure 6.10 and since we know from previous sections that secondary, primary and simple mixed moves leave the homology invariant the same is true for arbitrary mixed moves. ∎

7. The proof of Proposition 6.12

Let $\varphi \in \mathrm{Diff}_\omega(M)$ with hyperbolic $x \in \mathrm{Fix}(\varphi)$. Let (x, φ) be csi and let all primary points be transverse.

Fix a primary point within each pair of branches and consider its frame. The *relative positions of primary and secondary points within the frame* are of combinatorial nature, compare Lemma 4.8 and Lemma 4.9. Thus the results for moves can be generalized to perturbations as long as the primary reference point persists: The combinatorial picture where in the frame primary resp. secondary points can arise and where primary-secondary flips might take place is the same.

First we generalize Lemma 6.17.

LEMMA 6.40. *Let $\hat{\varphi} \in \mathrm{Diff}_\omega(M)$ a small perturbation of φ. Let $\hat{x}$ be the continuation of x. Let $p_\varphi \in \mathcal{H}_{pr}(\varphi)$ be primary and let p_φ persist as transverse homoclinic points $p_{\hat{\varphi}}$, but nonprimary. Then there is $q \in \mathcal{H}_{pr}(\hat{\varphi})$ which is no continuation of any primary point of φ.*

PROOF : We work with the lifted tangles of φ and $\hat{\varphi}$, but we drop the tilde for sake of readability.
According to Theorem 6.1 the segments $[x, p_\varphi]_i$ and $[\hat{x}, p_{\hat{\varphi}}]_i$ are close. Since p_φ is primary $]x, p_\varphi[_0 \cap\,]x, p_\varphi[_1 = \emptyset$, see figure 6.11 (i). But $p_{\hat{\varphi}}$ is nonprimary thus $]\hat{x}, p_{\hat{\varphi}}[_0 \cap\,]\hat{x}, p_{\hat{\varphi}}[_1 \neq \emptyset$. $\hat{x}$ and $p_{\hat{\varphi}}$ remain transverse. Figure 6.11 (ii) – (iv) lists the three basic perturbation types which prevent $p_{\hat{\varphi}}$ to be primary. In all three cases there is a primary $q \in\,]\hat{x}, p_{\hat{\varphi}}[_0 \cap\,]\hat{x}, p_{\hat{\varphi}}[_1$ which has no corresponding point in $]x, p_\varphi[_0 \cap\,]x, p_\varphi[_1$ and thus in $\mathcal{H}_{pr}(\varphi)$. ∎

Thus also in this generalized situation a primary-secondary flip is coupled with the rise of a new primary point.

LEMMA 6.41. *Consider $\varphi \in \mathrm{Diff}_\omega(M)$ with hyperbolic $x \in \mathrm{Fix}(\varphi)$. Let (x, φ) be csi and let all primary points be transverse. Then for sufficiently small perturbations $\hat{\varphi} \in \mathrm{Diff}_\omega(M)$ of φ all primary points remain transverse.*

PROOF : Since all primary points of φ are transverse they persist at least as transverse intersection points for small perturbations. Any primary-secondary flip would require the rise of a new primary point. But the discussion before Lemma 6.40 and the compactness of the frame prevents this for sufficiently small perturbations due to Theorem 6.1. ∎

Now we generalize Proposition 6.19.

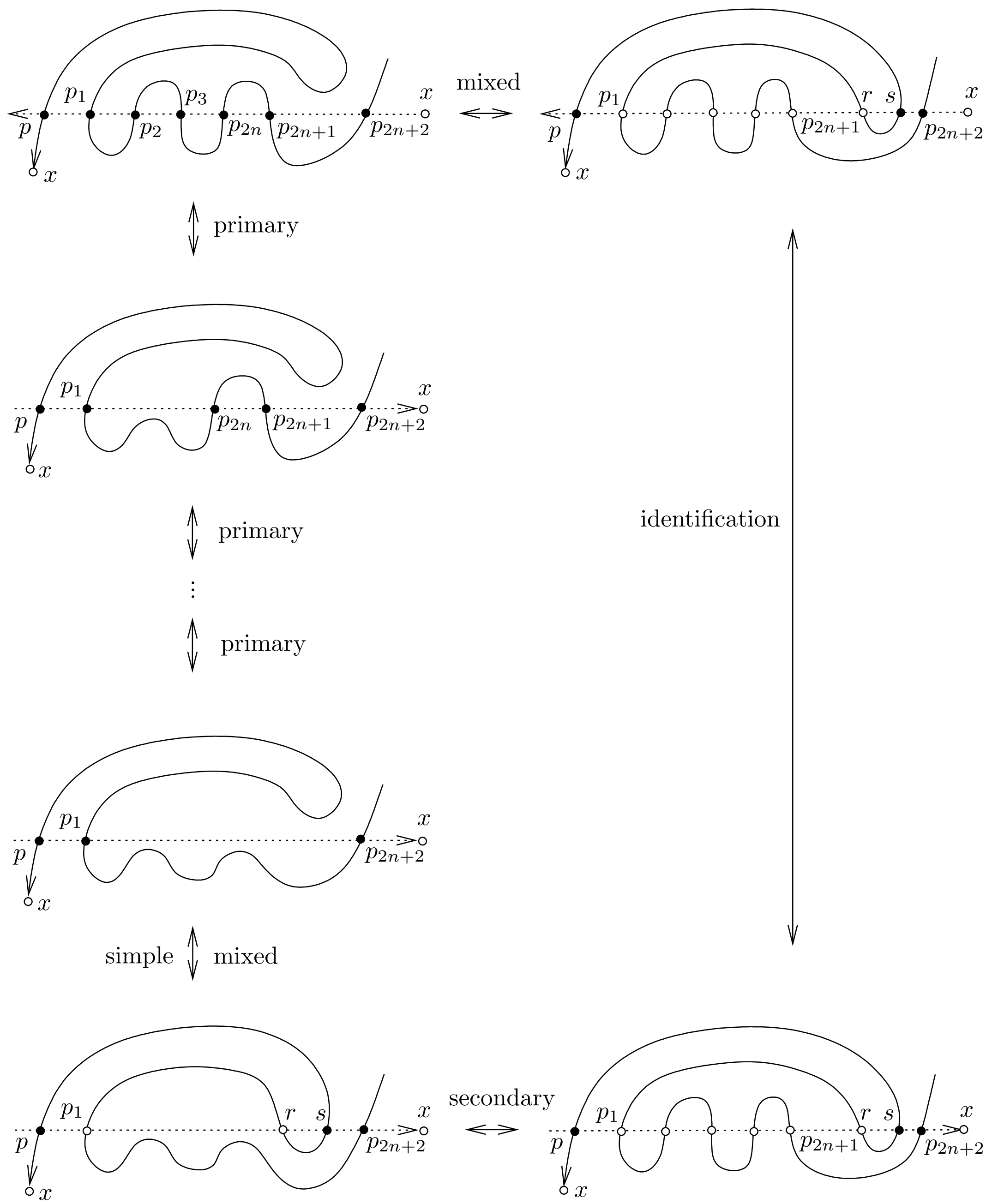


FIGURE 6.10. A mixed (r, s)-move with $2n + 1$ flips seen as the concatenation of n primary, one simple mixed and n secondary moves

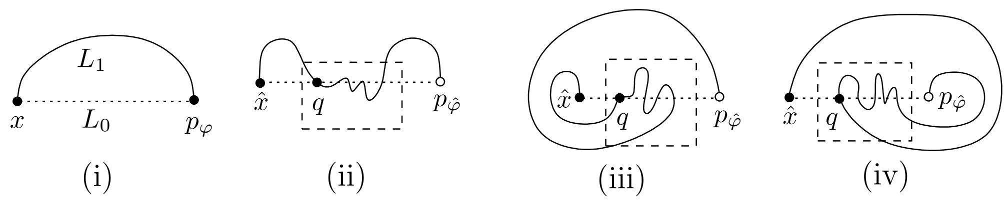


FIGURE 6.11. Causes for primary-secondary flips

LEMMA 6.42. *Let $\varphi \in \mathrm{Diff}_\omega(M)$ be csi with $x \in \mathrm{Fix}(\varphi)$ hyperbolic and all primary points transverse. Let $\hat{\varphi} \in \mathrm{Diff}_\omega(M)$ be small perturbation of φ such that all primary points persist transverse. Consider primary p_φ and q_φ with $\mu(p_\varphi, q_\varphi) = 1$ and denote their continuation by $p_{\hat{\varphi}}$ and $q_{\hat{\varphi}}$. Then $m(p_\varphi, q_\varphi) = m(p_{\hat{\varphi}}, q_{\hat{\varphi}})$.*

PROOF : For simplicity abbreviate $p := p_{\hat{\varphi}}$ and $q := q_{\hat{\varphi}}$ Clearly $\mu(p_\varphi, q_\varphi) = \mu(p, q)$ and if $m(p_\varphi, q_\varphi) \neq 0 \neq m(p, q)$ then their signs coincide. Thus it is enough to show $\mathcal{M}(p_\varphi, q_\varphi) \neq \emptyset$ if and only if $\mathcal{M}(p, q) \neq \emptyset$. We will work on the universal cover with the lifted tangles.
We have to check if the proof of Proposition 6.19 carries over to our more general situation. Let $\mathcal{M}(p_\varphi, q_\varphi) \neq \emptyset$ and assume $\mathcal{M}(p, q) = \emptyset$, i.e. $]p, q[_0 \cap]p, q[_1 \neq \emptyset$.
We just have to check for p_φ and q_φ of figure 4.3 and figure 4.4 all schematic types causing the boundary $[p, q]_0$ and $[p, q]_1$ to intersect apart from p and q.
We only prove the assertion exemplarily in the case of figure 4.3 (b).(-1,2).(i) which is resketched in figure 6.12 (i). The strategy and result for the other cases in figure 4.3 and figure 4.4 are the same.
Consider figure 6.12 (i) and the boundary $[p, q]_0 \cup [p, q]_1$ of the embedding between p and q. We note $[p, q]_0 = [p, x]_0 \cup [x, q]_0$ and check how $[p, x]_0$, $[x, q]_0$ and $[p, q]_1$ can intersect each other. It turns out that Proposition 6.19 generalizes: In figure 6.12 (ii) at least two primary points arise and in (iii), (iv) and (v) at least one. Before the $[p, q]_0$ starting from the situation of sketch (i) can intersect $[p, q]_1$ in (vi) and (vii) it has to pass through $]p, x[_1$. There it generates a primary point in $]p, x[_0 \cap]p, x[_1$ and flips p secondary.
Thus in all cases new primary points arise. But this violates the persistence assumption on the primary points — contradiction. ∎

Now we turn to the proof of Proposition 6.12.
PROOF of Proposition 6.12: For sufficiently small perturbations the primary points persist by Corollary 6.41. Thus the generator set of primary homoclinic chain complex stays unchanged. Moreover the boundary operator persists due to Lemma 6.42. Thus the homology remains unchained. ∎

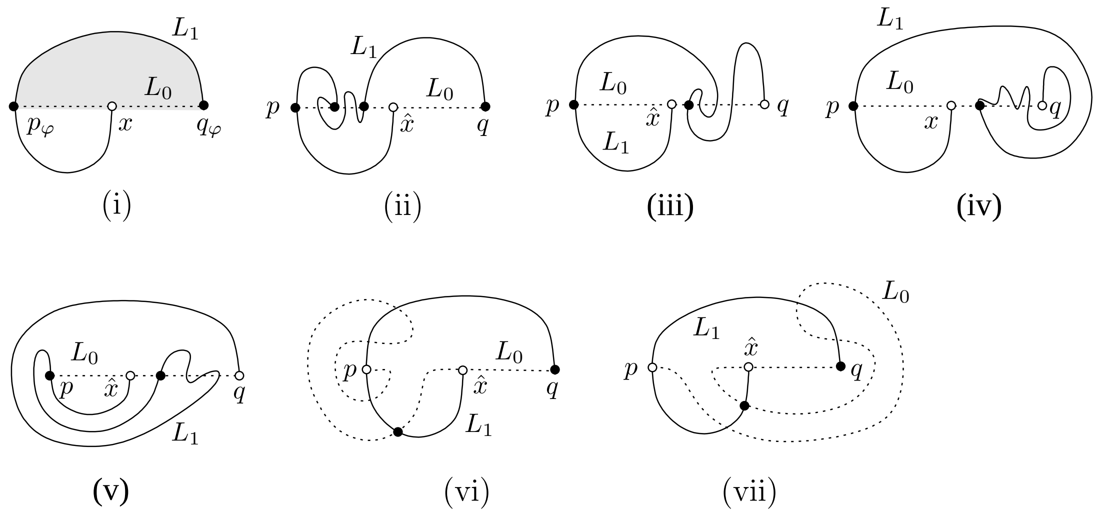


FIGURE 6.12. Embeddings under perturbations

CHAPTER 7

Applications and extensions

In this chapter we extend primary homoclinic Floer homology to non-volume preserving diffeomorphisms. Then we investigate invariance under conjugacy and compare $H_*(x, \varphi)$ and $H_*(x, \varphi^n)$. Moreover we define *chaotic primary homoclinic Floer homology*. We give an alternative sign definition restricted to *primary* points which allows $\mathbb{Z}$-coefficients also in the L-orientation reversing case.. Then we discuss the possibility of differential graded algebras and A_∞-structures generated by (primary) homoclinic points.

1. Non-volume preserving systems

In this section we will deduce a version of primary homoclinic Floer homology for non-volume preserving diffeomorphisms having a homoclinic tangle.

Let M be $\mathbb{R}^2$ or a compact two-dimensional manifold with genus $g \geq 1$ and consider $\varphi \in \mathrm{Diff}(M)$ having a hyperbolic fixed point x. In order to have stable *and* unstable manifolds we have to require $D\varphi(x)$ to have real eigen values λ_1, λ_2 satisfying $|\lambda_1| < 1 < |\lambda_2|$.

Theorem 7.1. *Let $\varphi \in \mathrm{Diff}(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic with real eigenvalues λ_1 and λ_2 satisfying $|\lambda_1| < 1 < |\lambda_2|$.*

(1) *If $\lambda_1\lambda_2 > 0$ then for L-orientation preserving resp. reversing φ the homology $H_*(x, \varphi, \mathbb{Z})$ resp. $H_*(x, \varphi, \mathbb{Z}_2)$ is defined analogously to Definition 4.17 and Definition 4.23.*
(2) *Let $\lambda_1\lambda_2 < 0$ and use Maslov grading in $\mathbb{Z}_2$ and $\mathbb{Z}_2$-coefficients. Then primary homoclinic Floer homology associated to (x, φ) is defined analogously to Definition 4.17.*

Proof : Since $\dim(M) = 2$ and $|\lambda_1| < 1 < |\lambda_2|$ the (un)stable manifolds are one-dimensional due to Theorem 2.1 and therefore Lagrangian. Thus the definition of the Maslov index is still valid, but we have to be careful about invariance w.r.t. the action of φ.

The λ-lemma Theorem 3.21 is the crucial ingredient in the cutting procedure Theorem 3.16. It also holds for nonsymplectic diffeomorphisms such that we only have to worry about Theorem 3.18 and Corollary 3.19. They yield for $\lambda_1\lambda_2 > 0$ the L-orientation perserving and reversing cases depending on the

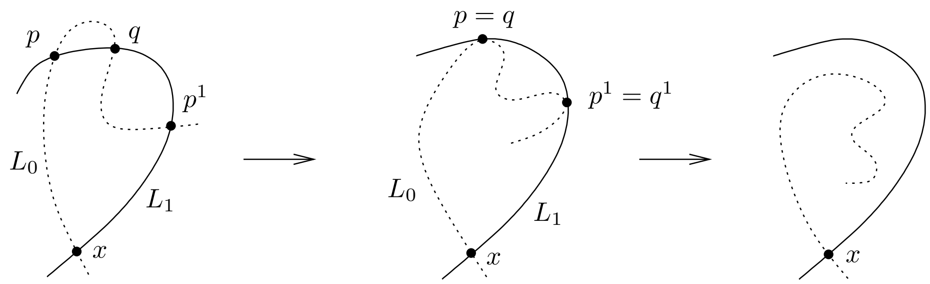


FIGURE 7.1. Volume contracting system

signs of the eigenvalues. For $\lambda_1\lambda_2 < 0$ we obtain the *mixed cases* L_0-orientation preserving and L_1-orientation reversing and vice versa. In all cases the gluing and cutting constructions carry over since the constructions always only consider single branches.
Now consider the case $\lambda_1\lambda_2 > 0$. As for symplectic φ we define the $\mathbb{Z}$- resp. $\mathbb{Z}_2$-signs $m(p,q)$ resp. $m_2(p,q)$, the chain complexes and the boundary operators. Since the Maslov index coincides in the two dimensional situation with the tangent winding number the invariance under the action of φ still is valid although φ is not symplectic. Proposition 4.27 is proven analogously and we can divide by the $\mathbb{Z}$-action.
Now turn to the case $\lambda_1\lambda_2 < 0$. Proposition 4.27 can be proven analogously such that $\{n \in \mathbb{Z} \mid \mathcal{M}(p,q^n) \neq \emptyset\} < \infty$ for primary p and q. But note $\mu(p,q) = -\mu(\varphi(p),\varphi(q))$ such that we have to assume $\mathbb{Z}_2$-values for μ if dividing by the action should be well-defined. For the same reason we have to use the $m_2(p,q)$-signs. ∎

Thus the combinatorial picture of the volume preserving and non-volume preserving situation for $\lambda_1\lambda_2 > 0$ is so far identical. Distinction becomes apparent when we turn to the invariance properties of the homology.

Non-volume preserving systems are much less rigid than volume preserving ones. In partucular Lemma 6.15 does not hold for $\mathrm{Diff}(M)$: There is no 'area balance condition' as in the volume preserving case. Branches can easily be torn apart by a perturbation, no matter how small (see figure 7.1). We summarize

REMARK 7.2. *For non-volume preserving φ Lemma 6.15 fails to be true. If formerly intersecting branches are separated during an isotopy $\tau \mapsto \Phi_\tau \in \mathrm{Diff}(M)$ then the homology might change.*

PROOF : For simplicity assume in figure 7.1 φ to be L-orientation preserving and only the branches containing p and q to intersect. For the picture on the very left we obtain as chain groups $C_2 = \mathbb{Z}\langle q\rangle$ and $C_1 = \mathbb{Z}\langle p\rangle$ with $\partial \equiv 0$ such that

$$H_2 = C_2 = \mathbb{Z}\langle q\rangle \quad \text{and} \quad H_1 = C_1 = \mathbb{Z}\langle p\rangle.$$

But when p and q collaps into each other and vanish as sketched in the middle and on the right then there are no generators of the chain complex left implying $H_* = 0$. ∎

Since Lemma 6.15 fails a perturbation strategy as in the proof of Theorem 6.4 is not at our disposal. But the combinatorial part still is valid.

THEOREM 7.3. *Primary homoclinic Floer homology as defined in Theorem 7.1 is invariant under good isotopies* $(\Phi_\tau)_{\tau\in[0,1]} \in \mathrm{Diff}(M)$.

PROOF : The local picture of a bifurcation is the same as in the volume preserving case, see Newhouse & Palis & Takens [**NePT**] § 2.6, such that the reduction to moves is also valid. For good isotopies thus the combinatorial constructions from Chapter 6 carry over. ∎

2. Invariance under conjugacy

It is natural to inquire the behaviour of primary homoclinic Floer homology under conjugacy. Since we consider purely combinatorial as well as symplectic aspects of primary homoclinic Floer homology within this work we have to distinguish which properties the conjugacy should preserve.

We say that φ, $\psi \in \mathrm{Diff}(M)$ are **topologically** resp. **smoothly** resp. **symplectically conjugate** if there is a homeomorphism resp. diffeomorphism resp. symplectomorphism $h : M \to M$ satisfying $\varphi \circ h = h \circ \psi$. We note

LEMMA 7.4. *Let* φ, $\psi \in \mathrm{Diff}(M)$ *be topologically conjugate by the homeomorphism* h *and* $x \in \mathrm{Fix}(\psi)$ *hyperbolic. Then* $h(x) \in \mathrm{Fix}(\varphi)$ *and* $h(W^s(x,\psi)) = W^s(h(x),\varphi)$ *and* $h(W^u(x,\psi)) = W^u(h(x),\varphi)$.

PROOF : $\varphi(h(x)) = h(\psi(x)) = h(x)$ and thus $h(x) \in \mathrm{Fix}(\varphi)$. Moreover due to the continuity of h we get $\lim_{n\to\pm\infty} \varphi^n(h(p)) = \lim_{n\to\pm\infty} h(\psi^n(p))$ and therefore $h(W^s(x,\psi)) = W^s(h(x),\varphi)$ and $h(W^u(x,\psi)) = W^u(h(x),\varphi)$. ∎

Now we investigate the behaviour of primary homoclinic Floer homology under topological conjugacy.

PROPOSITION 7.5. *Let* φ, $\psi \in \mathrm{Diff}(M)$ *be conjugate by the homeomorphism* h *and* $x \in \mathrm{Fix}(\psi)$ *hyperbolic. If* φ *and* ψ *are not symplectic assume the conditions from Theorem 7.1. Then* $H_*(x,\psi) = H_*(h(x),\varphi)$.
In particular for $\varphi = \psi$ *we obtain* $H_*(x,\varphi) = H_*(h(x), h\circ\varphi\circ h^{-1})$.

PROOF : By attaching 'φ' to a symbol we mark it as associated to φ like $L_0^\varphi := W^u(h(x),\varphi)$ and $[p,q]_0^\varphi \subset L_0^\varphi$ and analogously for ψ. Now fix an orientation on L_0^ψ and on L_0^φ. Since the (un)stable manifolds are injectively immersed real lines

h maps segments to segments, i.e. $h([p,q]_i^{\psi}) = [h(p),h(q)]_i^{\varphi}$ for $i \in \{0,1\}$. Thus we obtain either $m^{\psi}(p,q) = m^{\varphi}(h(p),h(q))$ for *all* p, $q \in \mathcal{H}_{pr}$ or $m^{\psi}(p,q) = -m^{\varphi}(h(p),h(q))$ for *all* p, $q \in \mathcal{H}_{pr}$. h induces an isomorphism $\mathfrak{h}$

$$\mathfrak{h} : C_*(x,\psi) \to C_*(h(x),\varphi), \quad \langle p\rangle \mapsto \langle h(p)\rangle$$

defined on the generators and extended by linearity. We compute

$$\begin{aligned}
\partial^{\varphi}(\mathfrak{h}(\langle p\rangle)) &= \partial^{\varphi}(\langle h(p)\rangle) \\
&= \pm \sum_{\substack{\langle h(q)\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle h(q)\rangle)=\mu(\langle h(p)\rangle)-1}} m^{\varphi}(\langle h(p)\rangle, \langle h(q)\rangle)\langle h(q)\rangle \\
&= \pm\mathfrak{h}\left(\sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m^{\psi}(\langle p\rangle, \langle q\rangle)\langle q\rangle \right) \\
&= \pm\mathfrak{h}(\partial(p))
\end{aligned}$$

which implies the claim. ∎

The converse of Proposition 7.5 is not true:

REMARK 7.6. *Whereas primary homoclinic Floer homology is invariant under primary, mixed and secondary moves described in Chapter 6 conjugacy is destroyed by anyone of them.*

Now what does change if h is a diffeomorphism instead of a homeomorphism? We note

REMARK 7.7. *Let φ, $\psi \in \mathrm{Diff}(M)$ be conjugate by the diffeomorphism h. Let $x \in \mathrm{Fix}(\psi)$ and λ an eigenvalue of $D\psi|_x$ with eigenvector v. Then $Dh|_x.v$ is an eigenvector of $D\varphi|_{h(x)}$ with eigenvalue λ.*

Therefore diffeomorphisms conjugate by a diffeomorphism are bound to have the same eigenvalues which restricts the class of conjugate diffeomorphisms. If we are only interested in the *combinatorial* aspect of primary homoclinic Floer homology conjugation by homoemorphisms yields the more general result than a smooth or even symplectic conjugacy which is without necessarity more restricted.

The situation becomes different if we refine primary homoclinic Floer homology by symplectic (alias volume preserving) aspects as it is done in Chapter 8 by defining the action filtration. There the conjugacy h should leave the action filtration invariant which requires h to be symplectic, see Proposition 8.13.

3. $H_*(x,\varphi)$ and $H_*(x,\varphi^n)$

In this section we compare the primary homoclinic Floer homology of a (symplectic) diffeomorphism φ to the one of φ^n for $n \in \mathbb{Z}$.

Let $\varphi \in \mathrm{Diff}(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic satisfying the assumptions of Theorem 7.1 if φ is not symplectic. Moreover require φ to be L-orientation preserving or reversing.
Denote by $\langle p_1\rangle, \ldots, \langle p_k\rangle$ the generators of $C_*(x,\varphi)$ and set $p_i^j := \varphi^j(p_i)$. For $n \in \mathbb{N}_0$ we have $L_i^{\varphi} = L_i^{\varphi^n}$ for $i \in \{0,1\}$ and $L_i^{\varphi} = L_j^{\varphi^{-n}}$ for $i \neq j \in \{0,1\}$. Note that the number of equivalence classes multiplies: $C_*(x,\varphi^n)$ is generated by $\langle p_1^0\rangle, \ldots, \langle p_k^0\rangle, \langle p_1^1\rangle, \ldots, \langle p_k^{n-1}\rangle$.
Abbreviate $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ and set $\varphi_*^l = \bar{l}$. There is a $\mathbb{Z}_n$-action on the generators via

$$\mathbb{Z}_n \times C_*(x,\varphi^n) \to C_*(x,\varphi^n), \qquad \varphi_*^l.\langle p_i^j\rangle := \langle p_i^{j+l \ mod \ n}\rangle = \langle \varphi^l(p_i^j)\rangle$$

and extend it by linearity to the complex. We notice

$$\varphi_*^l.(\partial\langle p_i^j\rangle) = \partial\langle\varphi^l(p_i^j)\rangle$$

implying the $\mathbb{Z}_n$-action to descend to homology. For L-orientation preserving φ let $\mathbb{K} = \mathbb{Q}$ and $n \in \mathbb{N}_0$. In the orientation reversing case assume $\mathbb{K} = \mathbb{Z}_2$ and $n = 2m+1 \in \mathbb{N}$ odd. Then Theorem 4.24 allows us to treat simultanously also negative exponents when we define

$$\begin{aligned} f &: C_*(x,\varphi^n;\mathbb{K}) \simeq C^{-*}(x,\varphi^{-n},\mathbb{K}) \to C_*(x,\varphi;\mathbb{K}), & f(\langle p_i^j\rangle) &:= \langle p_i\rangle, \\ g &: C_*(x,\varphi;\mathbb{K}) \to C_*(x,\varphi^n;\mathbb{K}) \simeq C^{-*}(x,\varphi^{-n};\mathbb{K}), & g(\langle p_i\rangle) &:= \frac{1}{n}\sum_{j=0}^{n-1}\langle p_i^j\rangle \end{aligned}$$

which are chain maps and we compute

$$f \circ g = \mathrm{Id}_{C_*(x,\varphi;\mathbb{K})}\,.$$

Denote by g_* and f_* the induced maps on the (co)homology.

Theorem 7.8. *Let $\varphi \in \mathrm{Diff}(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic satisfy the conditions of Theorem 7.1 if φ is not symplectic.*

(1) Let φ be L-orientation preserving and $n \in \mathbb{N}_0$. Then g_* is injective and f_* surjective. Thus

$$\dim H_*(x,\varphi;\mathbb{Q}) \leq \dim H_*(x,\varphi^n;\mathbb{Q}) = \dim H^{-*}(x,\varphi^{-n};\mathbb{Q})$$

and the difference is measured by the long exact sequence

$$\cdots \to H_l(\ker f;\mathbb{Q}) \to H_l(x,\varphi^n;\mathbb{Q}) \to H_l(x,\varphi;\mathbb{Q}) \to H_{l-1}(\ker f,\mathbb{Q}) \to \cdots$$

(2) Let φ be L-orientation reversing then φ^2 is L-orientation preserving and the first item applies for φ^2 and $\varphi^{2n} = (\varphi^2)^n$. For $n \in \mathbb{N}_0$ odd g_* is injective and f_* surjective. Thus

$$\dim H_*(x, \varphi; \mathbb{Z}_2) \leq \dim H_*(x, \varphi^n; \mathbb{Z}_2) = \dim H^{-*}(x, \varphi^{-n}; \mathbb{Z}_2)$$

and the difference is measured by the long exact sequence

$$\cdots \to H_l(\ker f; \mathbb{Z}_2) \to H_l(x, \varphi^n; \mathbb{Z}_2) \to H_l(x, \varphi; \mathbb{Z}_2) \to H_{l-1}(\ker f, \mathbb{Z}_2) \to \cdots$$

Proof : *First item:* We drop the coefficient ring $\mathbb{Q}$ in the notation in favour of better readability. $f \circ g = \mathrm{Id}_{C_*(x,\varphi)}$ implies the injectivity of g_* and surjectivity of f_* which yield the dimension estimates. The range of g are the invariants under the $\mathbb{Z}_n$-action and the kernel of f the coinvariants which are both subcomplexes of $C_*(x, \varphi^n)$. We obtain the short exact sequence of chain complexes

$$((\ker f)_*, \partial) \hookrightarrow (C_*(x, \varphi^n), \partial) \twoheadrightarrow \left(\frac{C_*(x, \varphi^n)}{(\ker f)_*}, \bar{\partial} \right) \tag{7.9}$$

where $\bar{\partial}$ is induced by the projection. Moreover

$$h : \left(\frac{C_*(x, \varphi^n)}{(\ker f)_*}, \bar{\partial} \right) \to (\mathrm{Im}(g)_*, \partial), \qquad [c] \mapsto \sum_{\nu=0}^{n-1} \varphi_*^l(c)$$

is an isomorphism and satisfies $h \circ \bar{\partial} = \partial \circ h$, thus an isomorphism of chain complexes. Since also $g : C_*(x, \varphi) \to \mathrm{Im}(g)_*$ is an isomorphism of chain complexes we obtain by means of the long exact sequence of (7.9)

$$\cdots \to H_l(\ker f) \to H_l(x, \varphi^n) \to H_l(x, \varphi) \to H_{l-1}(\ker f) \to \cdots$$

The *second item* follows analogously by observing the exchange of the (un)stable manifolds for negative exponents and Theorem 4.24. ■

In all explicitly calculated examples we obtain in fact $\dim H_*(x, \varphi) = \dim H_*(x, \varphi^n)$.

There have been various approaches coming from classical Floer theory and trying to deal with higher iterates of the symplectomorphism in question or with higher periodic orbits. For example Hutchings & Sullivan [**HuS**] attempt to define a Floer homology generated by higher periodic orbits. Or Fel'shtyn [**Fe**] who considers the Lefschetz and the symplectic zeta function where the latter is defined by

$$\chi_\varphi(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\chi(HF_*(\varphi^n))}{n} z^n \right)$$

where $\chi(HF_*(\varphi^n))$ is the Euler characteristic of $HF_*(\varphi^n)$. In our setting this reads

$$\chi_{x,\varphi}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\chi(H_*(x,\varphi^n;\mathbb{K}))}{n} z^n\right)$$

for the appropriate coefficient ring $\mathbb{K}$ from above.

This leads to the following questions:

(1) Can $\chi(x,\varphi)$ be rational? If yes, for which φ?

(2) Are there applications to Nielsen theory and Reidemeister torsion whose relation to dynamical zeta functions is described in Fel'shtyn [**Fe**]?

4. Chaotic primary homoclinic Floer homology

In contrast to the horseshoe formalism or the notion of entropy who measure the chaos of a dynamical system primary homoclinic Floer homology measures in some way its order. The horseshoe formalism states the existence of periodic points arbitrarily close to the homoclinic ones. In this section we will define a version of primary homoclinic Floer homology who takes them into account. It gathers more information than the homoclinic tangle alone provides and leads to the definition of a symplectic zeta function.

Assume $\varphi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(x)$ hyperbolic. For simplicity assume φ to be L-orientation preserving. The generalization to L-orientation reversing symplectomorphisms (or even diffeomorphisms) is straightforward.

Let $n \in \mathbb{Z}$. We define new signs for primary points p and q associated to φ^n and x via

$$\mathfrak{m}(p,q) := \begin{cases} m(p,q) & \text{if } \emptyset \neq \mathcal{M}(p,q) \ni u,\ \mathrm{Fix}(\varphi^n) \cap \mathrm{Im}(u) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Set $\mathfrak{m}(\langle p\rangle, \langle q\rangle) := \sum_{l\in\mathbb{Z}} \mathfrak{m}(p, q^l)$ and define

$$\begin{aligned} \mathcal{C}_k^n &:= C_k(x,\varphi^n;\mathbb{Z}) \quad \text{for } k \in \mathbb{Z} \\ \mathcal{D} &:= \mathcal{D}^n : \mathcal{C}_*^n \to \mathcal{C}_{*-1}^n \\ \mathcal{D}(\langle p\rangle) &:= \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr}(\varphi^n) \\ \mu(\langle p\rangle,\langle q\rangle)=1}} \mathfrak{m}(\langle p\rangle,\langle q\rangle)\langle q\rangle \end{aligned}$$

on a generator and extend $\mathcal{D}$ by linearity. The signs are cutting and gluing compatible. Thus $\mathcal{D} \circ \mathcal{D} = 0$ follows from the proof of $\partial \circ \partial = 0$. We define **chaotic primary homoclinic Floer homology** to be

$$H_*^{\mathrm{Fix}}(x,\varphi^n) := H_*^{\mathrm{Fix}}(x,\varphi^n;\mathbb{Z}) := \frac{\ker \mathcal{D}_*^n}{\mathrm{Im}\, \mathcal{D}_{*+1}^n}$$

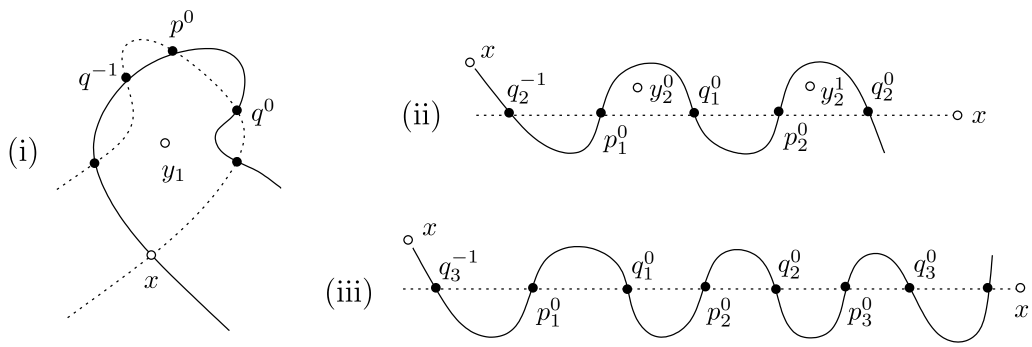


FIGURE 7.2. Chaos near the homoclinic tangle

REMARK 7.10. *Chaotic primary homoclinic Floer homology is invariant under conjugation. But invariance w.r.t. isotopies encounters problems similar to those described in the proof Proposition 6.37.*

Now we want to investigate the dynamics of $n \mapsto H_*^{\mathrm{Fix}}(x, \varphi^n)$.

First consider the following example: Let $\varphi \in \mathrm{Diff}_\omega(M)$ be L-orientation preserving with the homoclinic tangle sketched in figure 7.2 (i). Assume $\mathrm{Fix}(\varphi) = \{x, y_1\}$ and y_2^0 and y_2^1 to be the only 2-periodic points of φ with least period 2. There are no 3-periodic points with least period 3. The homoclinic tangles of φ^2 and φ^3 are drawn in figure 7.2 (ii) and (iii) where we have splitted x into two copies. Assume the positions of y_2^0 and y_2^1 as in figure 7.2 (ii). On L_0 fix the orientation induced by the 'jump direction' of the branch containing the homoclinic points. Set $p_j := p_j^0$ and $q_j := q_j^0$ for the points in the figures. Now we compute the homology $H_*^{\mathrm{Fix}}(x, \varphi^n)$ for $n \in \{1, 2, 3\}$.

For $n = 1$ we obtain

$$\begin{aligned}
&\mathcal{C}^1_{-1} = \mathbb{Z}\langle p \rangle, \quad \mathcal{D}\langle p \rangle = \langle q^{-1} \rangle - \langle q^0 \rangle = 0,\\
&\mathcal{C}^1_{-2} = \mathbb{Z}\langle q \rangle, \quad \mathcal{D}\langle q \rangle = 0,\\
&H_*^{\mathrm{Fix}}(x, \varphi) = \mathcal{C}^1_*.
\end{aligned}$$

For $n = 2$ we obtain

$$\mathcal{C}^2_{-1} = \mathbb{Z}\langle p_1 \rangle \oplus \mathbb{Z}\langle p_2 \rangle \quad \text{and} \quad \mathcal{C}^2_{-2} = \mathbb{Z}\langle q_1 \rangle \oplus \mathbb{Z}\langle q_2 \rangle$$

and as differential

$$\begin{aligned}
\mathcal{D}\langle p_1 \rangle &= \langle q_2 \rangle, \qquad & \mathcal{D}\langle q_1 \rangle &= 0,\\
\mathcal{D}\langle p_2 \rangle &= \langle q_1 \rangle, \qquad & \mathcal{D}\langle q_2 \rangle &= 0
\end{aligned}$$

and thus

$$H_{-1}^{\mathrm{Fix}}(x, \varphi^2) = 0 = H_{-2}^{\mathrm{Fix}}(x, \varphi^2).$$

For $n = 3$ we obtain

$$\mathcal{C}^3_{-1} = \mathbb{Z}\langle p_1\rangle \oplus \mathbb{Z}\langle p_2\rangle \oplus \mathbb{Z}\langle p_3\rangle \quad \text{and} \quad \mathcal{C}^3_{-2} = \mathbb{Z}\langle q_1\rangle \oplus \mathbb{Z}\langle q_2\rangle \oplus \mathbb{Z}\langle q_3\rangle$$

and as differential

$$\begin{aligned} \mathcal{D}\langle p_1\rangle &= \langle q_3\rangle - \langle q_1\rangle, & \mathcal{D}\langle q_1\rangle &= 0,\\ \mathcal{D}\langle p_2\rangle &= \langle q_1\rangle - \langle q_2\rangle, & \mathcal{D}\langle q_2\rangle &= 0,\\ \mathcal{D}\langle p_3\rangle &= \langle q_2\rangle - \langle q_3\rangle, & \mathcal{D}\langle q_3\rangle &= 0 \end{aligned}$$

and thus

$$H^{Fix}_{-1}(x,\varphi^3) \simeq \mathbb{Z} \simeq H^{\mathrm{Fix}}_{-2}(x,\varphi^3).$$

Note that in the above situation for all $n \in \mathbb{N}$ holds

$$H_{-1}(x,\varphi^n) \simeq \mathbb{Z} \quad \text{and} \quad H_{-2}(x,\varphi^n) \simeq \mathbb{Z}. \tag{7.11}$$

This simple example demonstrates the properties of chaotic primary homoclinic Floer homology very well. For the higher iterates we know

$$z \in \mathrm{Fix}(\varphi) \text{ implies } z \in \mathrm{Fix}(\varphi^n),$$
$$z \in \mathrm{Fix}(\varphi^l) \cap \mathrm{Fix}(\varphi^k) \text{ implies } z \in \mathrm{Fix}(\varphi^{k\cdot l}),$$

but additionally new fixed points arise. This relates the homology to number theoretic problems. The dynamical behaviour of $n \mapsto H^{\mathrm{Fix}}_*(x,\varphi^n)$ can be analysed by means of the **chaotic symplectic zeta function**

$$\zeta_{x,\varphi}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\chi(H^{\mathrm{Fix}}_*(x,\varphi^n))}{n} z^n\right)$$

where $\chi(H^{\mathrm{Fix}}_*(x,\varphi^n))$ denotes the Euler characteristic of $H^{\mathrm{Fix}}_*(x,\varphi^n)$.

As in the previous section this leads to

(1) Is there φ such that $\zeta(x,\varphi)$ is rational? If yes, which φ?
(2) Is there a relation to the classical (symplectic) zeta function?
(3) Are there applications to Nielsen theory and Reidemeister torsion whose relation to dynamical zeta functions is described in Fel'shtyn [**Fe**]?

5. Alternative signs

For immersions between *primary* points there is another possible definition of signs different from those in Definition 3.23. They arise from orientations associated to each single branch instead of the whole (un)stable manifold. They admit in the L-orientation preserving case $\mathbb{Z}$-coefficients, but *cannot be generalized to arbitrary homoclinic points.*

For $i \in \{0,1\}$ denote the branches of L_i by L_i^+ and L_i^- and associate *to each branch* its 'jump direction' as orientation and denote it by $o(L_i^+)$ resp. $o(L_i^-)$. Let

p, q be primary with $\mu(p,q) = 1$ and $u \in \mathcal{M}(p,q)$. Associate to $u(B_i) = [p,q]_i$ the orientation induced by the parametrization from p to q and call it o_{pq}. Recall from the classification Lemma 4.25 that $x \notin]p,q[_0 \cap]p,q[_1$ such that there is a branch $L^{pq} \in \{L_0^+, L_0^-, L_1^+, L_1^-\}$ containing both p and q.

Definition 7.12. *Let p, q be primary. We set*

$$n(p,q) := \begin{cases} 1 & \text{if } \mu(p,q) = 1,\ \mathcal{M}(p,q) \neq \emptyset,\ o(L^{pq}) = o_{pq}, \\ -1 & \text{if } \mu(p,q) = 1,\ \mathcal{M}(p,q) \neq \emptyset,\ o(L^{pq}) \neq o_{pq} \\ 0 & \text{otherwise.} \end{cases}$$

If there are two branches L_0^{pq} and L_1^{pq} then p and q are adjacent and $o(L_0^{pq}) = o_{pq} = o(L_1^{pq})$. Thus $n(p,q)$ is well-defined.
There are nonprimary homoclinic points p and q with $\mathcal{M}(p,q) \neq \emptyset$ and $x \in]p,q[_0 \cap]p,q[_1$. Thus the definition does not generalize to arbitrary homoclinic points.

The analogon of Lemma 3.24 is true:

Lemma 7.13. *Let p and r be primary with $\mu(p,r) = 2$ and $w \in \widehat{\mathcal{N}}(p,r)$. For $i \in \{0,1\}$ assume the existence of q_i with $\mu(p,q_i) = 1 = \mu(q_i,r)$ and $u_i \in \widehat{\mathcal{M}}(p,q_i)$ and $v_i \in \widehat{\mathcal{M}}(q_i,r)$ such that $v_i \# u_i = w$. Then*

$$n(p,q_0) \cdot n(q_0,r) = -n(p,q_1) \cdot n(q_1,r).$$

Proof : Checking in figure 4.7 the eight possible $w = v_i \# u_i \in \widehat{\mathcal{N}}(p,r)$ sketched in the left and right column yields the claim. ∎

Thus the signs $n(p,q)$ are gluing and cutting compatible. Moreover we do not need to distinguish the cases L-orientation preserving and reversing. $\mathbb{Z}$-coefficients are possible in both cases since $n(p,q) = n(p^l,q^l)$ for all $l \in \mathbb{Z}$. Analogously to $C_*(x,\varphi;\mathbb{Z})$, ∂ and $H_*(x,\varphi;\mathbb{Z})$ define

$$\tilde{C}_* := \tilde{C}_*(x,\varphi;\mathbb{Z}), \quad \tilde{\partial} \quad \text{and} \quad \tilde{H}_* := \tilde{H}_*(x,\varphi;\mathbb{Z})$$

based on the signs $n(p,q)$.

Theorem 7.14. *The invariance theorems from Chapter 6 hold for $\tilde{H}_*(x,\varphi;\mathbb{Z})$.*

Proof : The formula from Theorem 6.25 is true with the new signs. Thus the invariance under primary moves carries over. And since x never lies within a frame the new signs are also compatible with mixed moves. For invariance under secondary moves the signs were not relevant. Thus the invariance proof carries over. ∎

If we compute $\tilde{H}_*$ for the 'figure eight' tangle from figure 5.2 we note $\tilde{H}_* \simeq H_*$, but the generators of the homology groups are different:

$$\begin{aligned}
\tilde{\partial}\langle p\rangle &= \langle q_1\rangle - \langle q_1^{-1}\rangle + \langle q_2\rangle - \langle \tilde{q}_2^2\rangle = \langle q_2\rangle - \langle \tilde{q}_2\rangle,\\
\tilde{\partial}\langle \tilde{p}\rangle &= -\langle \tilde{q}_1\rangle + \langle \tilde{q}_1^{-1}\rangle + \langle \tilde{q}_2\rangle - \langle q_2^4\rangle = -\langle q_2\rangle + \langle \tilde{q}_2\rangle,\\
\tilde{\partial}\langle q_1\rangle &= -\langle r\rangle + \langle \tilde{r}^3\rangle = -\langle r\rangle + \langle \tilde{r}\rangle,\\
\tilde{\partial}\langle q_2\rangle &= \langle r\rangle - \langle r^{-1}\rangle = 0,\\
\tilde{\partial}\langle \tilde{q}_1\rangle &= \langle r^3\rangle - \langle \tilde{r}\rangle = \langle r\rangle - \langle \tilde{r}\rangle,\\
\tilde{\partial}\langle \tilde{q}_2\rangle &= -\langle \tilde{r}\rangle + \langle \tilde{r}^1\rangle = 0,\\
\tilde{\partial}\langle r\rangle &= 0,\\
\tilde{\partial}\langle \tilde{r}\rangle &= 0.
\end{aligned}$$

and

$$\begin{aligned}
\tilde{H}_{-1} &= \mathbb{Z}(\langle p\rangle + \langle \tilde{p}\rangle) \simeq \mathbb{Z}(\langle p\rangle - \langle \tilde{p}\rangle) = H_{-1},\\
\tilde{H}_{-2} &= \frac{\mathbb{Z}\langle q_2\rangle \oplus \mathbb{Z}\langle \tilde{q}_2\rangle \oplus \mathbb{Z}(\langle q_1\rangle + \langle \tilde{q}_1\rangle)}{\mathbb{Z}(\langle q_2\rangle - \langle \tilde{q}_2\rangle)} = H_{-2}\\
\tilde{H}_{-3} &= \frac{\mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle}{\mathbb{Z}(\langle r\rangle - \langle \tilde{r}\rangle)} \simeq \frac{\mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle}{\mathbb{Z}(\langle r\rangle + \langle \tilde{r}\rangle)} = H_{-3}.
\end{aligned}$$

We note the same phenomenon concerning the 'tilted figure eight'. The homology groups are isomorphic, but have different generators.

If for L-orientation preserving φ always $H_* \simeq \tilde{H}_*$ is an open question.

6. DGAs and A_∞-structures

We will show that DGA's and A_∞-structures are not well-defined using *primary* points as generators since the cutting construction of polygons fails. For homoclinic points they are well-defined. Apart from that the Maslov index does not yield the proper degree for a differential when polygons are involved which have more than two vertices.

A **differential graded algebra (DGA)** (A, d) is a graded algebra A together with a differential d of degree -1 satisfying $d(ab) = d(a)b + (-1)^{-\deg(a)}ad(b)$. Its **homology** $H(A, d)$ is the graded algebra $H(A, d) := \ker d / \operatorname{Im} d$.

How might a DGA arise in our framework? Therefore we briefly recall Chekanov's important work [**Che**]. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x, y, z) = (x, y)$ and let L be a Legendrian knot. Assume $\pi(L)$ to have only transverse double crossings which are denoted by $p_1, \ldots, p_n$ where $n \geq 0$. Denote by $A := T(p_1, \ldots, p_n)$ the free associative unital algebra over $\mathbb{Z}_2$ with generators $p_1, \ldots, p_n$. It is graded by the

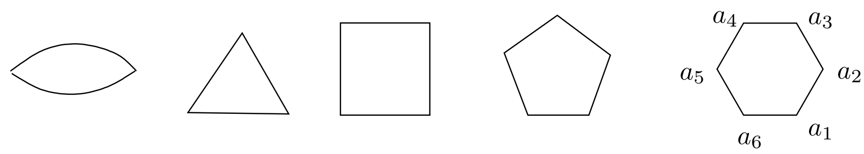


FIGURE 7.3. Convex standard n-gons

monomials. Each p_j divides $\pi(L)$ into two closed curves whose tangent winding number induces the degree $\deg(p_j)$ of p_j in $\mathbb{Z}/m(L)\mathbb{Z}$ where $m(L)$ is the Maslov number of L.
Denote by D_n the **convex standard n-gon** sketched in figure 7.3. Let its vertices be $a_1, \ldots, a_n$ as sketched for the 6-gon. The edge between a_i amd a_{i+1} is denoted by $[a_i, a_{i+1}]$. Let $W_k := W_k(\pi(L))$ be the set of smooth orientation preserving immersions $u : D_k \to \mathbb{R}^2$ with $u(\partial D_k) \subset \pi(L)$ (which implies $u(a_j) \in \{p_1, \ldots, p_n\}$). Denote by $G_K \subset \mathrm{Diff}(D_k)$ the subgroup of orientation preserving diffeomorphisms of D_k fixing the vertices. Then $\hat{W}_k := W_k/G_k$ is discrete. Chekanov introduces some notion of positivity (negativity) of a vertex a_i w.r.t. $u \in \hat{W}_k$ and sets $W_k^+ := \{u \in \hat{W}_k \mid a_1 \text{ positive}, a_2, \ldots, a_k \text{ negative for } u\}$ and $W_k^+(p_j) := \{u \in W_k^+ \mid u(a_1) = p_j\}$.
Recall $A = \bigoplus_{i \geq 0} A_i$ where $A_0 = \mathbb{Z}_2$, $A_1 = \{p_1, \ldots, p_n\} \otimes \mathbb{Z}_2$ and $A_i = (A_1)^{\otimes i}$ and define $\partial := (\partial_i)_{i \geq 0}$ with $\partial_k(A_i) \in A_{k+i-1}$ by

$$\partial_{k-1}(p_j) := \sum_{u \in W_k^+(p_j)} u(a_2) \cdots u(a_k) \tag{7.15}$$

and extend it by linearity and the Leibniz rule. Chekanov shows that (7.15) is well-defined and satisfies $\deg(\partial) = -1$. And by a gluing and cutting construction for polygons he obtains $\partial \circ \partial = 0$ which enables to pass to homology. By means of the homology he deduces the existence of Legendrian knots which are not Legendrian isotopic, but have the same classical invariants.

Now we try to adapt these notions to our framework. Let $\varphi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed point x and (un)stable manifolds L_0 and L_1. Let the degree of a homoclinic point be its Maslov index and replace $\pi(L)$ by $L_0 \cup L_1$ and the crossings of $\pi(L)$ by the (primary) homoclinic points. Redefine $W_k := W_k(L_0 \cup L_1)$ in the obvious way. We note

- Only immersed polygons with even number of vertices arise since L_0 and L_1 do not have self-intersections and the edges have to alternate between L_0 and L_1.

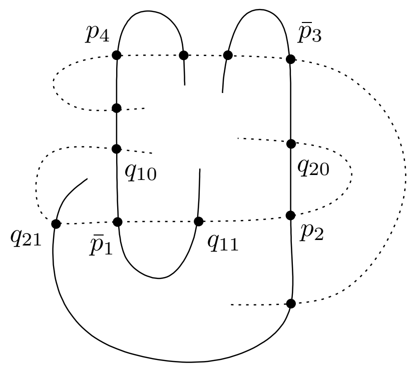


FIGURE 7.4. Positive and negative vertices

- A vertex p_i of $u \in \hat{W}_k$ is called **positive** if $u([a_{i-1}, a_i]) = [p_{i-1}, p_i]_0$ and $u([a_i, a_{i+1}]) = [p_i, p_{i+1}]_1$. Otherwise p_i is called **negative**. We set $\sigma(p) = +1$ if p is positive and $\sigma(p) = -1$ otherwise.
- Intuitively applying the Floer differential $\mathfrak{d}$ to a positive resp. negative vertex resembles a homology resp. cohomology boundary operator: In figure 7.4 $\mathfrak{d}p_2$ contains (among others) q_{20} and q_{21} and both satisfy $\mu(p_2, q_{20}) = 1 = \mu(p_2, q_{21})$. But for the negative p_1 we obtain in $\mathfrak{d}p$ the vertices q_{10} and q_{11} and $\mu(p_1, q_{10}) = -1 = \mu(p_1, q_{11})$.

Chekanov obtains for the degrees of the vertices of an immersed polygon P and the number of its positive vertices (compare [**Che**], lemma 6.3)

$$\sum_{\substack{p \in Vertex(P) \\ p \text{ positive}}} \deg(p) - \sum_{\substack{p \in Vertex(P) \\ p \text{ negative}}} \deg(p) = 2 - \#\{p \in Vertex(P) \mid p \text{ positive}\}. \tag{7.16}$$

By admitting only one positive vertex in a polygon (7.16) becomes

$$\deg(p) - 1 = \sum_{\substack{q \in Vertex(P) \\ q \text{ negative}}} \deg(q)$$

which implies degree -1 for Chekanov's differential.
The winding number of the standard polygon D_n is $w(D_n) = 1 - \frac{n}{4}$. In our framework (proven similarly to (7.16)) we obtain

$$\sum_{i=1}^{2n} \sigma(p_i)\mu(p_i) = \pm 2w(D_{2n}) = \pm(2 - n) \tag{7.17}$$

for the vertices $p_1, \ldots, p_{2n}$ of $u \in \hat{W}_{2n}$. But in our framework an element of $\hat{W}_{2n}$ always has exactly n positive and n negative vertices. Thus we obtain *only for*

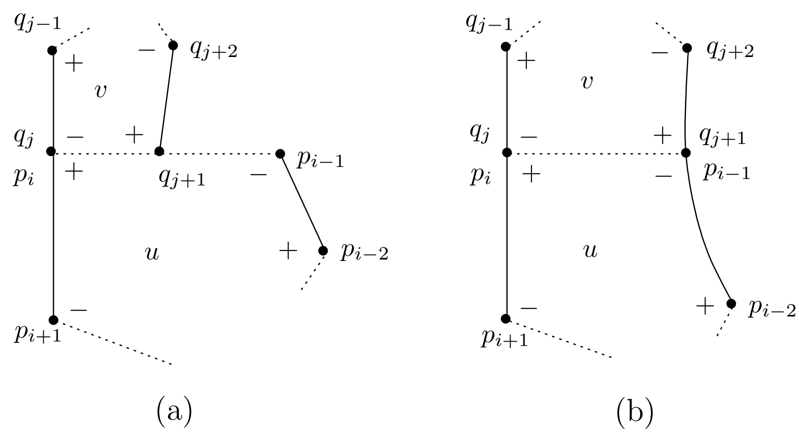


FIGURE 7.5. Gluing and cutting of polygons

the case $n = 1$ the relation

$$\mu(p) - 1 = \mu(q)$$

which we already used for primary homoclinic Floer homology. Therefore using the Maslov index analogously to Chekanov does not admit the correct index difference for a differential. *In this sense Floer homology is sharp w.r.t. the combinatorics of homoclinic tangles.* Any DGA approach has to use another index.

In order to prove $\partial \circ \partial = 0$ Chekanov used a cutting and gluing construction of polygons. The gluing construction is similar to Theorem 3.14 and carries over. Now we turn to the cutting construction. Denote by D'_n the standard n-gon with a_1 as concave and $a_2, \ldots, a_n$ as convex vertices. Based on $u : D'_n \to \mathbb{R}^2$ define V_n and $\hat{V}_n$ analogously to W_n and $\hat{W}_n$. The winding number Ind from Definition 3.7 with its property Remark 3.8 easily generalizes to immersed polygons.

THEOREM 7.18 (cutting). *Let L_0 and L_1 be strongly intersecting and $w \in V_{2n}$ with vertices in $\mathcal{H}$. Then there is $m \in \{1, \ldots, n\}$ and an immersed convex $2m$-gon u and an immersed convex $2(n-m)$-gon v such that $w = u \# v$. A convex immersed $2(n+m-2)$-gon can be cutted into a convex $2n$-gon and a convex $2m$-gon (see figure 7.5).*

PROOF : The idea of the proof is similar to the one of Theorem 3.16 relying on the λ-lemma 3.21. Unfortunately the 'injectivity' property near vertices as described in Proposition 3.13 is not true for $2n$-gons if $n \geq 2$. So we have to find other significant points where neighbourhood sectors lie in the exterior of the polygon.
Denote by γ the curve starting in a_1 of the standard $2n$-gon D'_{2n} and running counterclockwise through the edges back to a_1. We locate segments of the curve

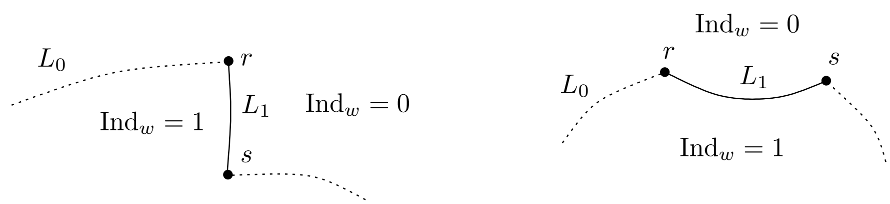


FIGURE 7.6. Special points

$w \circ \gamma$ which separate components with $\mathrm{Ind}_w = 0$ from those with $\mathrm{Ind}_w = 1$. Since L_0 and L_1 do not have self-intersections there are at least two points r, s joined by such a segment and having the desired property as sketched in figure 7.6 (the possible case of two concave vertices in r and s is omitted in the sketch).
A small ball around r and s consists of two sectors, one with $\mathrm{Ind}_w = 1$ and one with $\mathrm{Ind}_w = 0$.
Now we proceed as in the proof of Theorem 3.16 moving r or s resp. via φ close to x and use the λ-lemma Theorem 3.21 and the strong intersection property in order to obtain the existence of the cutting points. ■

Unfortunately is the cutting construction of polygons not compatible with the restriction to primary points:

REMARK 7.19. *If we require the vertices of the immersed polygons to be primary the cutting construction fails.*

PROOF : Consider the shadowed concave 4-gon in figure 7.7 with vertices $\tilde{p}^{-4}$, q_2, r and $\tilde{p}^{-3}$ which are all primary. One cut is along $[r, \tilde{q}^{-3}]_1$ and yields the 4-gon with vertices $\tilde{p}^{-4}$, q_2, r, $\tilde{q}_1^{-3}$ and the 2-gon with vertices $\tilde{p}^{-3}$ and $\tilde{q}_1^{-3}$. All appearing vertices are primary.
The other cut is along $[r, z]_0$ and yields the 4-gon with vertices $\tilde{p}^{-4}$, z, r, $\tilde{p}^{-3}$ and the 2-gon with vertices q_2 and z. But z is not primary. ■

If we try to define a DGA admitting all homoclinic points as generators we encounter similar problems as in Floer homology, see Chapter 1 or Chapter 9 §3.

$\mathcal{A}_\infty$-structures are some kind of dualized DGAs, for the exact definition see Seidel [**Se1**]. Thus we deduce

REMARK 7.20. *Defining A_∞-structures for homoclinic tangles encounters problems which are analogous to those appearing in the DGA approach.*

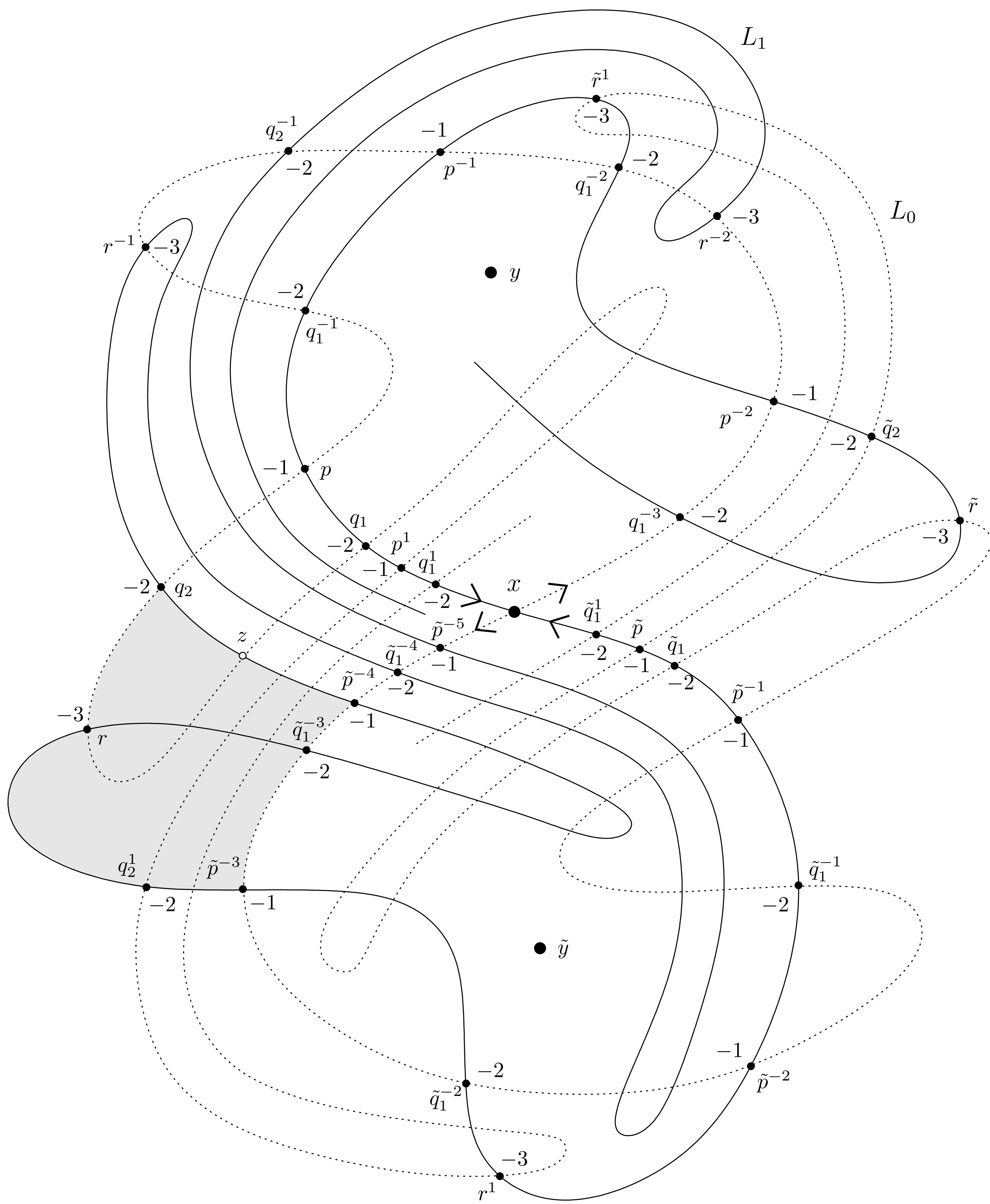


FIGURE 7.7. Cutting of polygons is not well-defined within the set of primary points

CHAPTER 8

Action spectrum and action filtration

In this chapter we define the action spectrum and the action filtration of primary homoclinic Floer homology and discuss its isotopy properties. Since $\widetilde{M}$ and L_i have vanishing homology there is no analogon to the constructions of Schwarz [**Sch3**] and Leclercq [**Le**] for a continuous section of the action spectrum bundle asigned to a given (co)homology class. As discussed in Chapter 6 the homotopy argument for invariance is not at our disposal. Since moreover an isotopy Φ_τ is not *applied* to the noncompact Lagrangians, but intrinsically related via $L_i^\tau = W^i(x_\tau, \Phi_\tau)$ analysing isotopies becomes much more difficult. Nevertheless for Melnikov and Lazutkin systems we can analyse the action spectrum depending on the isotopy parameter.

1. The action functional

Let $\varphi \in \mathrm{Diff}_\omega(M)$ with $x \in \mathrm{Fix}(\varphi)$ hyperbolic inducing a homoclinic tangle. See x as constant path in $\mathcal{P}(L_0, L_1)$ and denote by $\mathcal{P}_x(L_0, L_1)$ the component containing x.

Definition 8.1. *Set $Q := [0,1]^2$ and define for $v \in \mathcal{P}_x(L_0, L_1)$ the curve $\hat{v} \in C^\infty([0,1]^2, M)$ satisfying $\hat{v}(s,\cdot) \in \mathcal{P}_x(L_0, L_1)$ for all $s \in [0,1]$ and $\hat{v}(0,\cdot) = v$ and $\hat{v}(1,\cdot) = x$. The* **action functional** *is defined by*

$$\mathcal{A} : \mathcal{P}_x(L_0, L_1) \to \mathbb{R}, \quad \mathcal{A}(v) := \mathcal{A}(v, x) := \int_Q \hat{v}^*\omega.$$

The components of $\mathcal{P}(L_0, L_1)$ apart from the one containing x are uninteresting for us. $\mathcal{A}(v)$ is independent of the chosen path $\hat{v}$ since $\pi_2(M) = 0$ and $\pi_1(L_i) = 0$ for $i \in \{0, 1\}$ w.r.t. the topology provided by the immersions $\mathbb{R} \to L_i$. Given $v_1,\ v_2 \in \mathcal{P}_x(L_0, L_1)$ we define $\hat{w}$ to be the concatenation of $\hat{v}_1$ and $\hat{v}_2(1-\cdot,\cdot)$ reparametrized to Q as domain of definition. Then the **relative action** is given by

$$\mathcal{A}(v_1) - \mathcal{A}(v_2) = \mathcal{A}(v_1, v_2) := \int_Q \hat{w}^*\omega \tag{8.2}$$

and does not depend on the chosen path $\hat{w}$. This implies $\mathcal{A}(v_1, v_2) = -\mathcal{A}(v_2, v_1)$ and corresponds to changing the reference path from x to v_2, i.e. $\mathcal{A}(v_1, x) =$

$\mathcal{A}(v_1, v_2) + \mathcal{A}(v_2, x)$. The (relative) action is invariant w.r.t. the $\mathbb{Z}$-action of φ on $L_0 \cap L_1$, i.e.

$$\mathcal{A}(\varphi^n(p)) = \mathcal{A}(p) \qquad \text{and} \qquad \mathcal{A}(\varphi^n(p), \varphi^n(q)) = \mathcal{A}(p, q),$$

such that $\mathcal{A}$ can be defined for homoclinic equivalence classes via

$$\mathcal{A}(\langle p \rangle) := \mathcal{A}(p) \quad \text{and} \quad \mathcal{A}(\langle p \rangle, \langle q \rangle) := \mathcal{A}(\langle p \rangle) - \mathcal{A}(\langle q \rangle).$$

The tangent space of $\mathcal{P}_x(L_0, L_1)$ in v is

$$T_v \mathcal{P}_x(L_0, L_1) = \{\xi \in \Gamma^\infty(v^* TM) \mid \xi(0) \in T_{v(0)} L_0,\ \xi(1) \in T_{v(1)} L_1\}$$

and we compute

$$d\mathcal{A}(v).\xi = \int_0^1 \omega(\dot{v}(t), \xi(t)) dt.$$

We note that $v \in \mathrm{Crit}(\mathcal{A})$ implies v constant and therefore v can be seen as homoclinic point $v \in L_0 \cap L_1$. And isolated homoclinic points can be seen as critical points of $\mathcal{A}$.

Given two transverse homoclinic points p, q with $\mu(p, q) = 1$ and $u \in \mathcal{M}(p, q)$ we can determine the sign of $\mathcal{A}(p, q)$: Choose a smooth $h : Q \to D$ with $h(0, \cdot) = (-1, 0)$, $h(1, \cdot) = (1, 0)$ and $h(\cdot, i) = B_i$ for $i \in \{0, 1\}$ mapping $\mathrm{Int}(Q)$ diffeomorphically to $\mathrm{Int}(D)$. Since the relative action does not depend on the chosen path between p and q we consider $\hat{v} := u \circ h$ and obtain

$$\mathcal{A}(p, q) = \int_Q \hat{v}^* \omega = \int_Q h^*(u^* \omega) = \int_D u^* \omega > 0 \tag{8.3}$$

since u is orientation preserving. $\mathcal{A}(p, q) = \mathcal{A}(p) - \mathcal{A}(q)$ and (8.3) imply

$$\mathcal{A}(p) > \mathcal{A}(q) \tag{8.4}$$

imitating the negative L^2-gradient flow of the action functional along the pseudo-holomorphic strips in classical Lagrangian Floer theory.

There is another possibility to express the action functional using the universal cover $\tau : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ with $\tau^* \omega = \tilde{\omega}$. Fix $\tilde{x} \in \tau^{-1}(x)$ and lift the tangle according to Notation 4.5. For $v \in \mathcal{P}_x(L_0, L_1)$ denote by $\tilde{v}$ the according lift and by $\tilde{\hat{v}}$ the one of $\hat{v}$. We compute $\tilde{\hat{v}}^* \tilde{\omega} = \tilde{\hat{v}}^* \tau^* \omega = \hat{v}^* \omega$ and therefore

$$\mathcal{A}(v) = \int_Q \hat{v}^* \omega = \int_Q \tilde{\hat{v}}^* \tilde{\omega} =: \mathcal{A}(\tilde{v}). \tag{8.5}$$

Analogously we define $\mathcal{A}(\tilde{v}_1, \tilde{v}_2)$ for v_1, $v_2 \in \mathcal{P}_x(L_0, L_1)$ and deduce

$$\mathcal{A}(v_1, v_2) = \mathcal{A}(\tilde{v}_1, \tilde{v}_2).$$

Since $\tilde{M} \simeq \mathbb{R}^2$ is contractible the Poincaré lemma yields the existence of a 1-form $\tilde{\lambda}$ such that $\tilde{\omega} = d\tilde{\lambda}$. A simple calculation proves

Lemma 8.6. *For $\tilde{p}_i \in \tilde{L}_i$ let $\tilde{\gamma}_{\tilde{p}_i} : [0,1] \to \tilde{L}_i$ be smooth with $\tilde{\gamma}_{\tilde{p}_i}(0) = \tilde{x}$ and $\tilde{\gamma}_{\tilde{p}_i}(1) = \tilde{p}_i$ for $i \in \{0,1\}$. Then*

$$S_i : \tilde{L}_i \to \mathbb{R}, \quad S_i(\tilde{p}_i) := \int_{\tilde{\gamma}_{\tilde{p}_i}} \tilde{\lambda}$$

is well-defined and satisfies $dS_i = \tilde{\lambda}|_{\tilde{L}_i}$. S_i is called **generating function** *of $\tilde{L}_i$.*

For $c \in \mathbb{R}$ also $S_i + c$ is a generating function; S_i satisfies $S_i(\tilde{x}) = 0$. Note that in case of compact M the closed ω cannot be exact. Therefore the above construction only works on $\tilde{M}$.

Using Stokes we can express the relative action of homoclinic points by means of the generating functions S_i.

Lemma 8.7. *Let $p,\ q \in L_0 \cap L_1$ be contractible and isolated and see them as constant paths $p,\ q \in \mathcal{P}_x(L_0, L_1)$. Then*

$$\begin{aligned} \mathcal{A}(p,q) &= S_0(\tilde{q}) - S_0(\tilde{p}) + S_1(\tilde{p}) - S_1(\tilde{q}) \\ &= (S_1 - S_0)(\tilde{p}) - (S_1 - S_0)(\tilde{q}), \\ \mathcal{A}(p) &= (S_1 - S_0)(\tilde{p}). \end{aligned}$$

2. Action spectrum and action filtration

For $\varphi \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic the **primary action spectrum** of (x, φ) is defined by

$$\Sigma_{x,\varphi} := \{\mathcal{A}(\langle p \rangle) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}(x,\varphi)\}$$

and since $\tilde{\mathcal{H}}_{pr}$ is finite so is $\Sigma_{x,\varphi} \subset \mathbb{R}$. Thus the difference of the action levels is bounded from below by a small positive constant. Since $\Sigma_{(x,\varphi)}$ only depends on x and φ different Hamiltonians generating the same time-1 map induce the same spectrum.

Inspired by Leclercq [**Le**] we define a symplectic invariant estimating the difference of the action of adjacent primary points.

Given a transverse homoclinic point p then there is $\varepsilon > 0$ and an embedding $e_\varepsilon^p : B_\varepsilon(0) \subset \mathbb{R}^2 \to M$ such that

$$\begin{cases} e_\varepsilon^p(0) = p \quad \text{and} \quad (e_\varepsilon^p)^*\omega = dx \wedge dy, \\ (e_\varepsilon^p)^{-1}(L_0) = B_\varepsilon(0) \cap (\mathbb{R} \times \{0\}), \\ (e_\varepsilon^p)^{-1}(L_1) = B_\varepsilon(0) \cap (\{0\} \times \mathbb{R}). \end{cases} \tag{8.8}$$

Definition 8.9. *The* **primary radius** $r(p)$ **of** $p \in \mathcal{H}_{pr}$ *is the supremum over* $\varepsilon > 0$ *such that*

$$\begin{cases} e^p_\varepsilon \text{ satisfying (8.8) is defined,} \\ e^q_\varepsilon \text{ satisfying (8.8) is defined for all } q \in \mathcal{H}_{pr} \text{ with } m(p,q) = 1, \\ \operatorname{Im}(e^p_\varepsilon) \cap \operatorname{Im}(e^q_\varepsilon) = \emptyset \text{ for all } q \in \mathcal{H}_{pr} \text{ with } m(p,q) = 1. \end{cases}$$

Since φ *is symplectic* $r(\langle p \rangle) := r(p)$ *is well-defined and we define the* **primary radius of** (L_0, L_1) *to be*

$$r := r(L_0, L_1) := \min\{r(\langle p \rangle) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}\}.$$

We note

Corollary 8.10. *Let* p *and* q *be primary with* $\mu(p,q) = 1$ *and* $\mathcal{M}(p,q) \neq \emptyset$. *Then*

$$\mathcal{A}(p) - \mathcal{A}(q) = \mathcal{A}(p,q) \geq \tfrac{1}{4}\pi(r^2(p) + r^2(q)) \geq \tfrac{1}{2}\pi r^2,$$

thus

$$\sqrt{\tfrac{2}{\pi}\mathcal{A}(p,q)} \geq r.$$

Using $\sqrt{\frac{2}{\pi}\mathcal{A}(p,q)} \geq r$ Corollary 8.20 will give explicit upper estimates for r for certain classes of dynamical systems.

Although $\tilde{\mathcal{H}}_{pr}$ is finite and φ symplectic we cannot dispose of the condition $m(p,q) = 1$: The oscillations assured by the λ-lemma Theorem 3.21 would render $r(p) = 0$. Thus the action difference of two nonadjacent points only can be estimated using r if there is a sequence of adjacent points in between.

Now we approach the action filtration of primary homoclinic Floer homology. In case of L-orientation preserving φ we define for $a \in \mathbb{R}$

$$C^a_k := C^a_k(x, \varphi, \mathbb{Z}) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(p) = k \\ \mathcal{A}(p) \leq a}} \mathbb{Z}\langle p \rangle$$

and analogously $C^a_k := C^a_k(x, \varphi, \mathbb{Z}_2)$ for L-orientation reversing φ.

If p and q are primary with $\mu(p,q) = 1$ and $\mathcal{M}(p,q) \neq \emptyset$ then (8.4) states $\mathcal{A}(p) > \mathcal{A}(q)$. Therefore the boundary operator ∂ restricts to C^a_k and (C^a_*, ∂) is a subcomplex of (C_*, ∂). As in Schwarz [**Sch3**] we define for $a < b$

$$C^{]a,b]}_* := C^b_* / C^a_*$$

and there is the short exact sequence of chain complexes

$$0 \to C^a_* \xrightarrow{i} C^b_* \xrightarrow{j} C^{]a,b]}_* \to 0 \quad \text{for } a < b \leq \infty.$$

We identify $C_*^\infty = C_*$ and $C_*^{]-\infty,a]} = C_*^a$ and setting

$$H_*^{]a,b]} := H_*(C_*^{]a,b]}, \partial)$$

we obtain for $-\infty \le a < b < c \le \infty$ the long exact sequence

$$\dots \to H_{k+1}^{]b,c]} \to H_k^{]a,b]} \xrightarrow{i_*} H_k^{]a,c]} \xrightarrow{j_*} H_k^{]b,c]} \to H_{k-1}^{]a,b]} \to \dots$$

of **filtered primary homoclinic Floer homology groups.**

REMARK 8.11. *For $a \in R$ small enough and for $b \in \mathbb{R}$ large enough holds*

$$H_*^{]-\infty,a]} = 0 \quad \textit{and} \quad H_*^{]b,\infty]} = 0.$$

PROOF : Choose $a < \min\{\mathcal{A}(p) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and $b > \max\{\mathcal{A}(p) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$. ∎

For certain homology classes we know exactly their critical levels in the action filtration.

REMARK 8.12. *Set $I := \{n \in \{\pm 1, \pm 2, \pm 3\} \mid C_{n-1} = 0\}$ and for $k \in I$ consider $c \in C_k$ with $c = \sum c_l \langle p_l\rangle$. Then c represents a homology class and lives in $H^{]a,b]}$ for $a < \min_l A(\langle p_l\rangle)$ and $\max_l \mathcal{A}(\langle p_l\rangle) \le b$. In particular 1, $-3 \in I$.*

If we consider cohomology the analogon is true for $J := \{n \in \mathbb{Z} \mid C^{n+1} = 0\}$, in particular -1, $3 \in J$.

Now we ask for the invariance properties of the action spectrum and the filtered primary homoclinic Floer homology. We start with the analogon of Proposition 7.5.

PROPOSITION 8.13. *Let φ, ψ, $h \in \mathrm{Diff}_\omega(M)$ and $x \in \mathrm{Fix}(\psi)$ hyperbolic and let φ and ψ be conjugate by h, i.e. $\varphi \circ h = h \circ \psi$. Then $\Sigma_{x,\psi} = \Sigma_{h(x),\varphi}$ and for all $-\infty \le a < b \le +\infty$*

$$H_*^{]a,b]}(x,\psi) = H_*^{]a,b]}(h(x),\varphi).$$

PROOF : This is proven analogously to Proposition 7.5, but we have to check that the (relative) action remains unchanged. To p, $q \in \mathcal{H}_{pr}(\psi)$ and $u \in \mathcal{M}(p,q)$ corresponds under the conjugacy $h(p)$, $h(q) \in \mathcal{H}_{pr}(\varphi)$ and $h \circ u \in \mathcal{M}(h(p),h(q))$. Since $h^*\omega = \omega$ we obtain $\mathcal{A}(p) = \mathcal{A}(h(p))$ and

$$\mathcal{A}(p,q) \overset{(8.3)}{=} \int_D u^*\omega = \int_D (h\circ u)^*\omega \overset{(8.3)}{=} \mathcal{A}(h(p),h(q))$$

which yields the claim. ∎

Moreover we note

REMARK 8.14. *$\Sigma_{x,\varphi} = \Sigma_{x,\varphi^n}$ for $n \in \mathbb{Z}$ since iteration only produces artifically new equivalence classes without changing the tangle, compare the discussion before Theorem 7.8.*

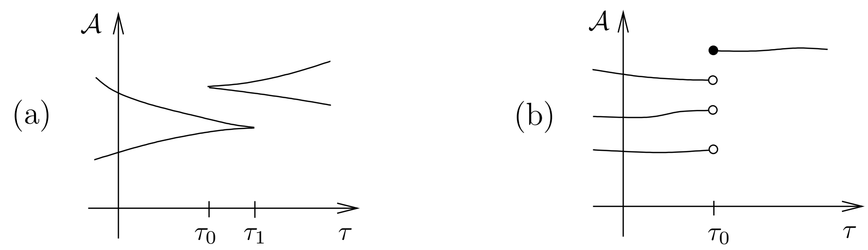


FIGURE 8.1. Changing of the action spectrum under primary (a) and mixed moves (b)

Whenever primary points vanish the spectrum and the action filtration might change. In figure 8.1 the behaviour of the action spectrum w.r.t. primary and mixed moves is sketched. Assume for simplicity all homoclinic equivalence classes to have different action and consider an isotopy Φ_τ. Figure 8.1 (a) displays at τ_0 and τ_1 primary moves vanishing resp. generating two primary points. The local model from Remark 6.13 implies the cusps to behave like $\tau \mapsto (\tau \pm \tau_0)^{\frac{3}{2}}$. A mixed move destroys a certain number of primary points and generates one, compare figure 6.6. And the modulus of the action of the new primary point is larger than those of the destroyed ones as sketched in figure 8.1 (b).

Nevertheless we will see in Corollary 8.19 that the 'rough picture' of the action spectrum stays invariant under certain isotopies.

REMARK 8.15. *There is no analogon to the constructions done by Schwarz* [**Sch3**] *and Leclercq* [**Le**] *who assign to a given homology class of M resp. L_i a section of the action bundle.*

PROOF : Schwarz [**Sch3**] and Leclerc [**Le**] exploit some versions of the PSS-isomorphism which in our framework does not exist. Moreover anyway $H_*^{sing}(L_i) = 0$ for $n \neq 0$ w.r.t. topology induced by the immersions $\mathbb{R} \to L_i$. ■

3. Isotopy invariants

Now we consider an isotopy Φ_τ and ask for changes of the action spectrum and the filtered primary Floer homology.

Schwarz [**Sch3**] and Leclercq [**Le**] use the well-known homotopy technic in order to answer the change of their invariants under changing the Hamiltonian function resp. the Lagrangian submanifold. But we already mentioned in Chapter 6 that the homotopy argument is not at our disposal. Even if we could use the homotopy argument we would need some generalization of the Hofer distance to noncompact Lagrangian manifolds depending *implicitly* via $L_0^\tau = W^u(x_\tau, \Phi_\tau)$

and $L_1^\tau = W^s(x_\tau, \Phi_\tau)$ on the isotopy. Moreover generally always *both* Lagrangians change under an isotopy.
Proposition 8.13 implies that conjugacies cannot be used in order to model isotopies since conjugacies do not admit bifurcations. They only can be used in order to straighten *compact* segments of one invariant manifold w.r.t. the other.

Briefly, the lack of compactness of L_i and the fact that we *do not apply isotopies to the Lagrangians, but that they depend implicitly on the isotopy* Φ_τ makes the problem much more difficult than in the approaches of Schwarz and Leclercq.

We deduce from Lemma 8.7

REMARK 8.16. *Let Φ_τ be an isotopy from (x^φ, φ) to (x^ψ, ψ). Denote by S_i^φ resp. S_i^ψ for $i \in \{0,1\}$ the associated generating functions. Let $p \in \mathcal{H}_{pr}(\varphi)$ let $p_\tau \in \mathcal{H}_{pr}(\Phi_\tau)$ be its continuation. Then*

$$\mathcal{A}(p_0) - \mathcal{A}(p_1) = (S_1^\varphi - S_0^\varphi)(\tilde{p}_0) - (S_1^\psi - S_0^\psi)(\tilde{p}_1).$$

and analogously for $\mathcal{A}(p_0, q_0) - \mathcal{A}(p_1, q_1)$.

Theorem 6.1 implies for isotopies with Φ_0 and Φ_1 close to each other that compact segments of L_i^φ and L_i^ψ around the fixed point are close to each other, but it says nothing about the exact relative positions w.r.t. each other. At least it assures the signed areas $\mathcal{A}(p_0)$ and $\mathcal{A}(p_1)$ to be close.

Since (filtered) primary homoclinic Floer homology is already determined by compact segments of L_i^φ and L_i^ψ as explained after Definition 4.23 we can evaluate Remark 8.16 as accurately as wished using a good computer program for *explicit* examples.

Apart from numerical approaches there are not many theoretical technics for comparing *explicitly* the (un)stable manifolds and their intersection points. Most of the technics dealing with homoclinic points in discrete dynamical systems are interested in genericity and stability results and thus only deal locally with tiny perturbations. More or less there are only Melnikov's method (sketched in Appendix A) and Lazutkin's invariant (see Appendix B) which provide explicit estimates for the (relative) action depending on the perturbation. Both methods concentrate on primary homoclinic or heteroclinic points.

Melnikov's result Theorem A.4 applies to a Hamiltonian system arising from

$$H_\varepsilon : \mathbb{R} \times M \to \mathbb{R}, \quad H_\varepsilon(t,x,y) := H_0(x,y) + \varepsilon H_1(t,x,y)$$

and yields the existence of primary homoclinic points for the isotopy

$$\varepsilon \mapsto \Phi_\varepsilon^M := \varphi_1^{H_\varepsilon}$$

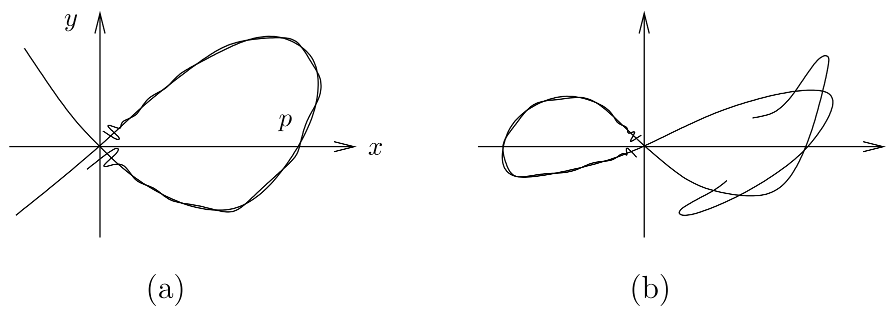


FIGURE 8.2. Typical homoclinic tangle of the quadratic map (a) and the asymmetric cubic map (b)

(for $\varepsilon > 0$ sufficiently small) if the Melnikov function is independent from ε. Note that in many physically relevant cases the Melnikov function is given explicitly. For more details and the according references see Appendix A.

If the splitting of the (un)stable manifolds is very small it might be to tiny to be detected by Melnikov's method. This is for instance the case for Chirikov's standard map

$$F_\varepsilon : T^2 \to T^2, \quad (x, y) \mapsto (x_1, y_1), \quad y_1 := y + \varepsilon \sin(x), \quad x_1 := x + y_1.$$

as sketched by Lazutkin [**Laz**] and finally proven by Gelfreich [**Ge1**]. For more details and the according references see Appendix B. F_ε is related to the pendulum equation and thus has either *noncontractible* homoclinic points or *heteroclinic* points when considered on $\mathbb{R}^2$. Thus it is uninteresting for primary homoclinic Floer homology. But some types of the so called *generalized standard map*

$$G_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x_1, y_1), \quad y_1 := y + \varepsilon f(x), \quad x_1 := x + y_1,$$

as considered in Gelfreich & Lazutkin [**GeL**] and Gelfreich & Simó [**GeS**] admit contractible homoclinic points. Unfortunately up to now their theory has been made rigorous only for the *area-preserving Hénon map* where $f(x) = x - x^2$. In other cases like $f(x) = x + rx^2 - x^3$ with $r \in \mathbb{R}$ only numerical experiments suggest the validity of analogous results. Thus

$$\varepsilon \mapsto \Phi^L_\varepsilon := G_\varepsilon$$

is the considered isotopy w.r.t. Lazutkin's theory. The homoclinic tangle for the Hénon map is sketched in figure 8.2 (a) and for asymmetric (i.e. $r \neq 0$) cubic f in (b).

A priori Melnikov's method inquires primary points lying on a pair of branches which arose from a homoclinic loop of the unperturbed system. The primary points which Lazutkin's method analyses arise from symmetries of the map G_ε. Such primary points we will call **direct**.

THEOREM 8.17 ([**Me**], [**GeL**], [**GeS**]). *(1) Let $p = p_\varepsilon$, $q = q_\varepsilon \in \mathcal{H}_{pr}(\Phi_\varepsilon^M)$ be direct and adjacent with $\mu(p,q) = 1$. Let $\mathfrak{M}$ be the Melnikov function and s_p and s_q zeros associated to p and q. Then*

$$\mathcal{A}(p,q) = \varepsilon \int_{s_q}^{s_p} \mathfrak{M}(s)ds + O(\varepsilon^2).$$

(2) Let $p = p_\varepsilon$, $q = q_\varepsilon \in \mathcal{H}_{pr}(\Phi_\varepsilon^L)$ be direct and adjacent with $\mu(p,q) = 1$ and a_0 and $h \approx \varepsilon^2$ taken from (B.2). Then

$$|\mathcal{A}(p,q)| \approx \frac{4\pi a_0}{h^6} e^{-\frac{2\pi^2}{h}} (1 + O(h^4 e^{-\frac{\pi^2}{h}})) \approx O\left(\frac{e^{-\frac{2\pi^2}{\sqrt{\varepsilon}}}}{\varepsilon^3}\right).$$

Moreover Melnikov's and Lazutkin's theory imply

REMARK 8.18. *(1) If p and q are adjacent, but not direct $\mathcal{A}(p,q)$ is of order $O(\varepsilon)$ resp. $O(e^{\sqrt{\varepsilon}})$.*
(2) For adjacent primary p and q the modulus of the action $\mathcal{A}(p)$ and $\mathcal{A}(q)$ is huge in comparison with $\mathcal{A}(p,q)$.

PROOF : *First item:* Under the *chaotic layer* we understand the region visited by the homoclinic tangle, see figures 8.2 and 2.1. For $\varepsilon > 0$ not to large this region looks like a tubular neighbourhood of the homoclinic loop of the unperturbed system. We call its maximal distance relative to the homoclinic loop *width* of the chaotic layer. Either KAM theory or Zaslavsky's [**Za**] explicit estimates show that the width of the chaotic layer is of order $O(\varepsilon)$ resp. $O(e^{\sqrt{\varepsilon}})$. Thus for adjacent primary points we can estimate the relative action by the area of the chaotic layer which is of order $O(\varepsilon)$ resp. $O(e^{\sqrt{\varepsilon}})$ — but in fact their relative action is much smaller.
Second item: $A(p)$ approximates the area enclosed by the homoclinic loop(s) of the unperturbed system. ∎

COROLLARY 8.19. *(1) Let p be primary and $Q := \mathcal{H}_{pr} \cap [p, p^1[_0 \cap [p, p^1[_1$. Then $\frac{1}{\#Q}\sum_{q\in Q} A(q) \approx \mathcal{A}(p) + O(\varepsilon)$ resp. $\approx \mathcal{A}(p) + O(e^{\sqrt{\varepsilon}})$ can be seen as isotopy invariant.*
(2) For $c = \sum_l c_l \langle p_l \rangle$ from Remark 8.12 with $a < \min_l \mathcal{A}(p_l)$ and $\max_l \mathcal{A}(p_l) \le b$ we have $|a - b| \approx O(\varepsilon)$ resp. $\approx O(e^{\sqrt{\varepsilon}})$.

Figure 8.3 pictures the action spectrum of the tangle of figure 8.2 and sketches Corollary 8.19.

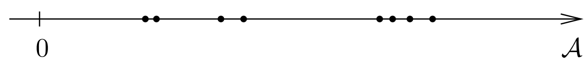


FIGURE 8.3. Action spectrum of the tangle from figure 8.2 (b)

Using Corollary 8.10Theorem 8.17 yields an upper bound for the primary radius $r = r(L_0^\varepsilon, L_1^\varepsilon)$:

COROLLARY 8.20. *For $(L_0^\varepsilon, L_1^\varepsilon)$ arising from Φ_ε^M resp. Φ_ε^L the primary radius $r = r(L_0^\varepsilon, L_1^\varepsilon)$ is bounded from above by*

$$\sqrt{\frac{2\varepsilon}{\pi} \int\limits_{s_q}^{s_p} \mathfrak{M}(s) ds + O(\varepsilon^2)}$$

resp.

$$\frac{2\sqrt{2a_0}}{h^3} e^{-\frac{\pi^2}{h}} \sqrt{(1 + O(h^4 e^{-\frac{\pi^2}{h}}))} \approx O\left(\frac{e^{-\frac{\pi^2}{\sqrt{\varepsilon}}}}{\varepsilon\sqrt{\varepsilon}}\right).$$

REMARK 8.21. *Whereas for Φ_ε^L direct primary points have exponentially small relative action Gelfreich and Simó* [**GeS**] *discovered that for cubic asymmetric f the ratio is exponentially different when we compare the relative action associated to points on the left loop and the relative action on the right loop as sketched in figure 8.2 (b).*

CHAPTER 9

Applications and outlook

In this chapter we sketch generalizations and further applications of (primary) homoclinic Floer homology.

1. Invariance under Hamiltonian isotopies

We sketch a proof strategy in order to show invariance of primary homoclinic Floer homology under Hamiltonian isotopies.

CONJECTURE 9.1. *Let (M, ω) be a closed two-dimensional symplectic manifold with genus $g \geq 1$. Let φ and ψ be strongly intersecting Hamiltonian diffeomorphisms with $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ both hyperbolic. Let all primary points be transverse. If a pair of branches does not admit primary points assume the semi-primary ones transverse. Assume there is a Hamiltonian isotopy Φ between (x, φ) and (y, ψ). Then*

$$H_*(x, \varphi) = H_*(x, \psi).$$

Our idea for a proof is the following:

(1) Show that Φ can be perturbed to a strongly intersecting Hamiltonian isotopy with generic bifurcations for all $\tau \in [0, 1]$.
(2) Recall the $k \in \mathbb{N}_0$ from Remark 4.3. Show for Hamiltonian diffeomorphisms $k = 0$ and that the existence of primary points of a pair of branches implies the existence of a semi-primary contractible one within this pair.
(3) For each pair of intersecting branches of the isotopy show

$$B := \{\tau \in [0, 1] \mid \exists \text{ primary points}\}$$

and B^c are open. Thus one of them is empty.
(4) Now the proof technics of Theorem 6.4 carry over.

The *first item* is difficult. Usually genericity results for *paths* of diffeomorphisms concerning dynamical behaviour are highly nontrivial as described for instance in the survey article by Newhouse [**Ne2**].
'Strongly intersecting' is a generic property for Hamiltonian diffeomorphisms on closed surfaces in the C^r-topology with $1 \leq r \leq \infty$, see Xia [**Xia3**]. Moreover Hamiltonian system on compact two-dimensional manifolds have very strong

existence and rigidity properties, see the recent results by Xia [**Xia1, Xia2**], Burns & Weiss [**BW**] and others.
For those reasons Xia [**Xia4**] and the author think the first item true. Unfortunately the proof from Xia [**Xia3**] does not generalize to paths. Note that for volume preserving isotopies which are not Hamiltonian the first item is very likely wrong, compare Xia [**Xia2, Xia3**] and Remark 6.6 on $\mathbb{R}^2$.

The *second item* will be a tricky combination of combinatorial and analytical aspects. We need it for the proof of the third item.

The *third item* is proven by a flux argument applied to a (semi-)primary point p: Consider the loop from c_p from p to x via $[p,x]_0$ and back to p via $[p,x]_1$. Since the flux of $c_p - c_{p^1}$ is zero we conclude the existence of (semi-)primary points in $]p,x[_0 \cap\,]p,x[_1$ which persist under small perturbations. The second item together with the knowledge where primary points can arise in a (semi-)primary frame allows to conclude B and B^c open.

2. Application to Birkhoff invariants

In this section we motivate and explain

Conjecture 9.2. *Filtered primary homoclinic Floer homology of a hyperbolic periodic point in the KAM-picture reflects coefficients of the Birkhoff normal form.*

Motivation: Consider the 2-dimensional KAM picture around an elliptic point with its chain of islands, see for instance Arnold & Avez [**ArA**]. Such a chain consists of $2n$ periodic points alternatingly hyperbolic and elliptic with the symplectomorphism φ acting on them. The heteroclinic tangles generated by the hyperbolic periodic points in the chain give rise to a homoclinic tangle of each hyperbolic periodic point by means of the λ-lemma Theorem 3.21. Let x be one of those hyperbolic periodic points and consider $m \in \mathbb{Z}$ such that $x \in \mathrm{Fix}(\varphi^m)$. As discussed in the proof of Theorem 7.8 the (un)stable manifolds of φ and φ^m coincide as sets, but differ in the number of homoclinic equivalence classes. The homoclinic tangle of x looks schematically like the one of figure 5.3 by considering y as the elliptic fixed point in the center of the KAM picture and placing at the position $\tilde{y}$ the other periodic points of the chain.

Call the primary orbits encircling the elliptic fixed point, but not the island chain **inner** primary points and those encircling the island chain **outer** ones. In figure 5.3 $\langle p\rangle$ and $\langle q\rangle$ correspond to inner primary points.

Gelfreich & Simó [**GeS**] consider this situation under the point of view of splitting of the (un)stable manifolds when passing from the integrable to the perturbed situation (compare Appendix B). They announce that the splitting of inner and outer (primary) homoclinic points differs due to the changes of the rotation number (of an integrable approximation) along the resonance zone and that sufficiently close to the elliptic fixed point those changes are related to the second coefficient of the Birkhoff normal form around the elliptic fixed point. For details they refer to the not yet published preprint of Simó & Vieiro [**SiV**].

Now different splitting means different angle at the arising primary points and therefore different relative action, compare Appendix B.

Since the action filtration is sensitive to the relative action which represents the difference of (certain) critical levels it notices different splitting angles. And since the splitting behaviour is related to the Birkhoff coefficients so is the *filtered* primary homoclinic Floer homology. This indicates a way to express the Birkhoff invariants by means of Floer homology.

3. Generalization to nonprimary points

Let $\varphi \in \mathrm{Diff}_\omega(M)$ with $x \in \mathrm{Fix}(\varphi)$ hyperbolic and $\mathcal{H} := L_0 \pitchfork L_1$ transverse. Let L_0 and L_1 be strongly intersecting. The cutting and gluing construction from Theorem 3.14 and Theorem 3.16 are valid for arbitrary homoclinic points which suggests attempts to enlarge the generator set of the chain complex. If we want to generalize $\mathfrak{d}$ resp. ∂ to $\mathcal{H}$ resp. $\tilde{\mathcal{H}}$ we have to investigate for $p \in \mathcal{H}$

- $\#\{q \in \mathcal{H} \mid m(p,q) \neq 0\} < \infty$?
- $\#\{n \in \mathbb{Z} \mid m(p,q^n) \neq 0\} < \infty$ for $q \in \mathcal{H}$?
- $\#\{\langle q\rangle \in \tilde{\mathcal{H}} \mid m(p,q^n) \neq 0 \text{ for some } n \in \mathbb{Z}\} < \infty$?

As already mentioned in Chapter 1 all three can be infinite: For x in figure 9.1 holds $m(x,p^n) \neq 0$ for all $n \in \mathbb{Z}$. There are also tangles with p, $q \in \mathcal{H}$ and $m(p,q^n) \neq 0$ for $n \in \mathbb{Z}^{>n_o}$ or $n \in \mathbb{Z}^{<n_0}$ for some $n_0 \in \mathbb{Z}$. And for p in figure 9.1 holds $m(p,s_n) \neq 0 \neq m(p,r_n)$ and $\langle s_n\rangle$ and $\langle r_n\rangle$ are all mutually distinct for $n \in \mathbb{N}$.

The natural idea is to find a *filtration* $(\mathcal{H}^c)_{c\in\mathbb{R}} \subset \mathcal{H}$ satisfying

(1) $\mathcal{H}^c \subset \mathcal{H}^{c'}$ for $c < c'$ and $\varphi(\mathcal{H}^c) = \mathcal{H}^c$ and $\#\tilde{\mathcal{H}}^c < \infty$ for all c
(2) The cutting and gluing construction are well-defined within $\mathcal{H}^c$ for all c.

This turns out to be a rather tricky task. We will discuss the following approaches:

- Filtration by action sublevel sets
- Exhaustion of L_0 and L_1
- Winding numbers
- Generalized primary points
- Structure index

3.1. The action filtration. We already sketched parts of the following in Chapter 1. Filtration by the sup- or suplevel sets of the action functional $\mathcal{A}$ is the most natural idea due to the 'negative gradient flow' property (8.4). In a similar situation it has been successfully used by Abbondandolo & Schwarz [**AbS**]. Unfortunately in our case neither the sup- nor sublevel are finite:

LEMMA 9.3. *Consider $p \in \mathcal{H}$ in figure 9.1. There exist $(s_n)_{n\in\mathbb{N}} \in \mathcal{H}$ and $(r_n)_{n\in\mathbb{N}} \in \mathcal{H}$ with*

(1) $\langle s_n \rangle$ and $\langle r_n \rangle$ mutually distinct for $n \in \mathbb{N}$,
(2) $m(p, s_n) \neq 0 \neq m(p, r_n)$ for all $n \in \mathbb{N}$,
(3) $\mathcal{A}(s_n) < -2\mathcal{A}(p) < \mathcal{A}(r_n)$ for all n,
(4) $\lim_{n\to\infty} \mathcal{A}(s_n) = -2A(p) = \lim_{n\to\infty} \mathcal{A}(r_n)$.

PROOF : Consider the symmetric 'figure eight' tangle of figure 9.1, locate the points p, r_n and s_n. Due to symmetry we have $\mathcal{A}(p^n) = \mathcal{A}(\tilde{p}^m)$ for all n, $m \in \mathbb{Z}$. Let Q_n be the polygon with vertices p, r_n, $\tilde{p}^{n-1}$ and x oriented counterclockwise. Choose p close enough to x. The λ-lemma Theorem 3.21 implies that for $n \to \infty$ the edge $[\tilde{p}^{n-1}, r_n]_1$ of Q_n converges to $[x, p]_1$ while the length of the other two edges tends to zero. Therefore $\lim_{n\to\infty} \int_{Q_n} \omega = 0$ and since $\mathcal{A}(p) = \mathcal{A}(\tilde{p}^n)$ for all n we get

$$\lim_{n\to\infty} \mathcal{A}(p, r_n) = -\mathcal{A}(p) + \lim_{n\to\infty} \int\limits_{Q_n} \omega = -\mathcal{A}(p)$$

approaching $-\mathcal{A}(p)$ from above. Analogously the s_n yield an approximation of $-\mathcal{A}(p)$ from below which yields the claim due to $\mathcal{A}(p, q) = \mathcal{A}(p) - \mathcal{A}(q)$ for all q. ■

3.2. Exhaustion. If we exhaust L_0 and L_1 by considering segments $[p_n, q_n]_i \subset [p_{n+1}, q_{n+1}]_i$ for $i \in \{0, 1\}$ centered around x and use $\mathcal{H}(n) := [p_n, q_n]_0 \pitchfork [p_n, q_n]_1$ as generator sets of the chain complex the cutting construction fails.

3.3. Winding numbers. A filtration by the 'winding number' around fixed points, i.e. considering $[p] \in \pi_1(M\backslash\{y\})$ in figure 9.1, does not lead to finite sup- or sublevel sets.

3.4. Generalized primary points. Generalize the definition of primary points to

$$\mathcal{H}(n) := \{p \in \mathcal{H} \mid \#([p, x[_0 \pitchfork \ [p, x[_1 \cap \mathcal{H}_{pr}) = n\} \text{ for } n \in \mathbb{N}.$$

Generally the cutting construction is not well-defined. But if we choose the connecting immersions carefully we can define a homology whose chain groups are generated by $\mathcal{H}(n)$.

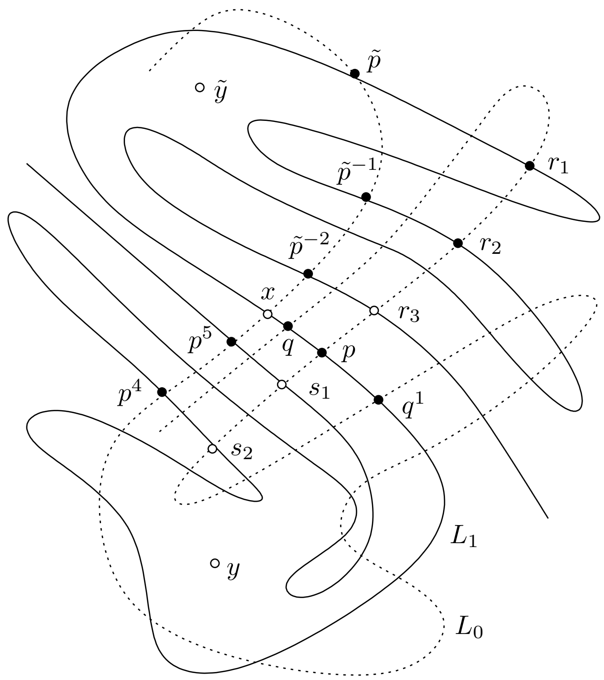


FIGURE 9.1. The differential of p is infinite mod $\mathbb{Z}$ with converging action

- For primary points $p \in \mathcal{H}_{pr}$ use the differential $\mathfrak{d}$ resp. ∂ after dividing by the action.
- For $p \in \mathcal{H}(n) \backslash \mathcal{H}_{pr}$ and $q \in \mathcal{H}$ set

$$\tilde{m}(p,q) = \begin{cases} m(p,q) & \text{if } \mathcal{M}(p,q) = \emptyset \\ m(p,q) & \text{if } u \in \mathcal{M}(p,q) \neq \emptyset,\ \mathrm{Im}(u) \cap \mathcal{H}_{pr} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

These signs lead to a well-defined boundary operator and thus to a homology.

However, this homology seems a bit unnatural and no real generalization of primary homoclinic Floer homology since the primary and generalized primary points do not interact.

3.5. Structure index. The *structure index* defined in Easton [**E**] and Hocket & Holmes [**HH**] motivates the following approach: Let p be primary and define

$$\tilde{\mathcal{H}}(n) := \{\langle p \rangle \mid p \in \{[p, p^1]_0 \cap [p^l, p^{l+1}]_1 \mid 0 \leq l \leq n\}\}.$$

In all computed examples it was well-defined, but up to now there is no proof for the validity of the cutting construction within $\tilde{\mathcal{H}}(n)$.

4. Generalization to higher dimensions

So far primary homoclinic Floer homology is defined for homoclinic tangle in a two-dimensional situation. Now we discuss the generalization to higher dimensional manifolds.

First note that we have to require the diffeomorphism to be symplectic in order to have *Lagrangian* (un)stable manifolds needed for the definition of the Maslov index. Thus in higher dimension homoclinic Floer homology (if defined) is automatically a *symplectic invariant*. (In the two-dimensional situation primary homoclinic Floer homology also is defined for diffeomorphisms, but invariance is only natural within the class of symplectomorphisms.)

Let (M^{2n}, ω, J) be a symplectic $2n$-dimensional manifold with (time dependent) almost complex structure J compatible with ω. Let $\varphi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed point x and strongly intersecting (un)stable manifolds L_0 and L_1. Let $\mathcal{H} := L_0 \pitchfork L_1$ be transverse and graded by the Maslov index.
We consider for $p, q \in \mathcal{H}$ with $\mu(p,q) = 1$ the space $\mathcal{M}(p,q)$ of $u : \mathbb{R} \times [0,1] \to M$ satisfying

(1) $\partial_s u + J\partial_t u = 0$,
(2) $u(\cdot, 0) \subset L_0$ and $u(\cdot, 1) \subset L_1$
(3) $\lim_{s\to-\infty} u(s,\cdot) = p$ and $\lim_{s\to+\infty} u(s,\cdot) = q$.

Dividing by the $\mathbb{R}$-action $(\sigma.u)(s,t) = u(s+\sigma, t)$ we define $\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/\mathbb{R}$. Let $\mu(p,q) = 1$. Now we have to worry about

- $\#\widehat{\mathcal{M}}(p,q) < \infty$?
- $\#\{q \in \mathcal{H} \mid \widehat{\mathcal{M}}(p,q) \neq \emptyset\} < \infty$?
- $\#\{n \in \mathbb{Z} \mid \widehat{\mathcal{M}}(p,q^n) \neq \emptyset\} < \infty$?
- Are Theorem 3.14 and Theorem 3.16 satisfied?

Already the first question is a highly nontrivial Fredholm problem since the Lagrangians are noncompact. The sets in the second and third item are already infinite in the two-dimensional setting. And the last item is even a more difficult Fredholm and compactness problem than the first one.

Even if the first question can be answered positively (under certain assumptions) we still have to find a 'good' set of homoclinic points which allow a well-defined differential. Primary points have no natural generalization to higher dimensions. Thus the situation becomes similarly difficult to the two-dimensional one when looking for a filtration of $\mathcal{H}$.

APPENDIX A

Melnikov's perturbation method

This appendix briefly sketches Melnikov's [**Me**] perturbation method which yields (under certain assumptions) from a time independent system having a homoclinic trajectory a slightly time dependent system having a homoclinic tangle. This section summarizes Guckenheimer & Holmes [**GH**], §4.5, §4.6. where all proofs and details carefully are written down. Helpful figures can be found in Arrowsmith & Place [**ArP**] §3.8.

Let $H_0 : \mathbb{R}^2 \to \mathbb{R}$ be a time independent Hamiltonian function and $X := X^{H_0}$ its Hamiltonian vector field. Let $Y : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a periodic time dependent vector field and assume X and Y to be at least C^2 and to be bounded on bounded sets. Consider the time independent integrable Hamiltonian system

$$\dot{z}(t) = X(z(t)) \tag{A.1}$$

and for $\varepsilon > 0$ its time dependent perturbation

$$\dot{z}(t) = X(z(t)) + \varepsilon Y(t, z(t)) \tag{A.2}$$

which yields for $\varepsilon = 0$ the original system. From now on we assume (A.2) to have a hyperbolic saddle point x_0 and an orbit q_0 homoclinic to x_0, i.e. $\lim_{t\to\pm\infty} q_0(t) = x_0$, which is sketched in figure 1.1 (a). We set $\Gamma_0 := \{q_0(t) \mid t \in \mathbb{R}\} \cup \{x_0\}$.

The idea is to use (computable) global solutions of the integrable system (A.1) in order to obtain solutions of the perturbed system (A.2). More precisely we expect that the perturbation will break the homoclinic $\mathbb{R}$-family $q_0(\cdot + r)$ of the unperturbed system and will gain from it homoclinic solutions of the perturbed system.

Consider the global cross section $\Sigma_{t_0} := \{t_0\} \times \mathbb{R}^2 \subset S^1 \times \mathbb{R}^2$ with the Poincaré map (first return map) $P^\varepsilon_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0}$ associated to the perturbed system (A.2). Hyperbolicity and the implicit function theorem imply for sufficiently small ε the existence of a unique hyperbolic periodic orbit $t \mapsto \gamma_\varepsilon(t)$ of the perturbed system (A.2) which is $O(\varepsilon)$ close to x_0. Thus the associated Poincaré map $P^\varepsilon_{t_0}$ has a unique hyperbolic saddle point $x^\varepsilon_{t_0}$ which is $O(\varepsilon)$ close to x_0.

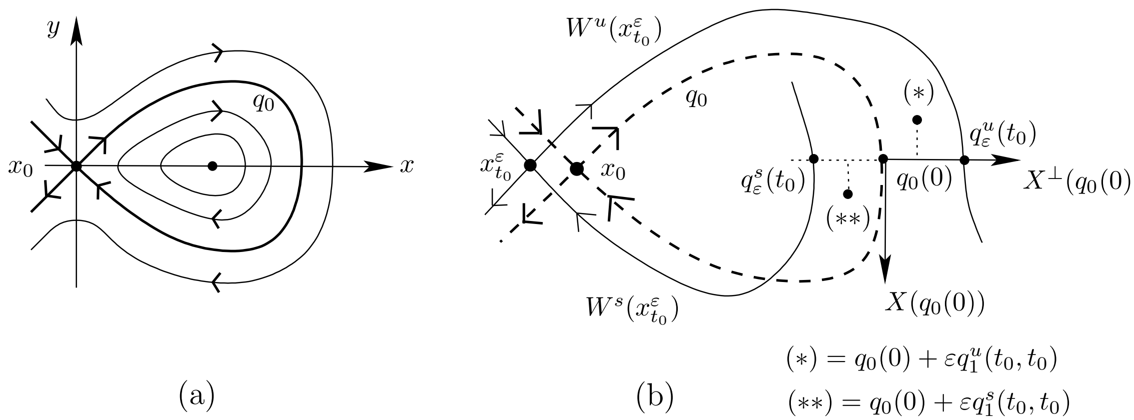


FIGURE 1.1. The Melnikov construction

For $t \geq t_0$ we define q^s_1 to be the solution of the first variational equation with initial time t_0

$$\dot{q}^s_1(t, t_0) = DX(q_0(t - t_0)).q^s_1(t, t_0) + Y(t, q_0(t - t_0))$$

and analogously q^u_1 for $t \leq t_0$.

If X and Y are C^r with $r \geq 2$ it can be shown that the local stable and unstable manifolds $W^s_{loc}(\gamma_\varepsilon)$ and $W^u_{loc}(\gamma_\varepsilon)$ are C^r-close to those of the unperturbed periodic orbit $S^1 \times \{x_0\}$. Moreover, orbits $q^s_\varepsilon(t, t_0)$ and $q^u_\varepsilon(t, t_0)$ lying in $W^s(\gamma_\varepsilon)$ resp. $W^u(\gamma_\varepsilon)$ associated to Σ_{t_0} can be (uniformely) written in the indicated intervals as

$$\begin{aligned} q^s_\varepsilon(t, t_0) &= q_0(t - t_0) + \varepsilon q^s_1(t, t_0) + O(\varepsilon^2) \quad \text{for } t \in [t_0, \infty[, \\ q^u_\varepsilon(t, t_0) &= q_0(t - t_0) + \varepsilon q^u_1(t, t_0) + O(\varepsilon^2) \quad \text{for } t \in]-\infty, t_0]. \end{aligned} \tag{A.3}$$

Now we are able to start with the actual construction which is sketched in figure 1.1 (b). The points $q^u_\varepsilon(t_0) := q^u_\varepsilon(t_0, t_0)$ and $q^s_\varepsilon(t_0) := q^s_\varepsilon(t_0, t_0)$ denote the unique points on $W^u(x^\varepsilon_{t_0})$ resp. $W^s(x^\varepsilon_{t_0})$ 'closest' to $x^\varepsilon_{t_0}$ and lying on the normal to Γ_0 in $q_0(0)$ spanned by $X^\perp(q_0(0)) := (-X^2(q_0(0)), X^1(q_0(0)))^T$ where $X = (X^1, X^2)$. We define the 'distance' of the (un)stable manifolds $W^u(x^\varepsilon_0)$ and $W^s(x^\varepsilon_0)$ on the global cross section Σ_{t_0} at the point $q_0(0)$ to be

$$d(t_0) := q^u_\varepsilon(t_0) - q^s_\varepsilon(t_0).$$

If we define the wedge product of two vectors $a = (a_1, a_2)^T$ and $b = (b_1, b_2)^T$ to be $a \wedge b := a_1 b_2 - a_2 b_1$ then the C^r-closeness of $W^u(x^\varepsilon_{t_0})$ and $W^s(x^\varepsilon_{t_0})$ to Γ_0 and the expansion (A.3) imply

$$d(t_0) = \frac{X(q_0(0)) \wedge (q^u_1(t_0, t_0) - q^s_1(t_0, t_0))}{|X(q_0(0))|} \cdot \varepsilon + O(\varepsilon^2)$$

The wedge product $X \wedge (q_1^u - q_1^s)$ is nothing else than the projection of $(q_1^u - q_1^s)$ onto $X^\perp$. The **Melnikov function** is defined as the integral

$$\mathfrak{M}(t_0) := \int_{-\infty}^{\infty} X(q_0(t - t_0)) \wedge Y(t, q_0(t - t_0))dt$$

which simplies for a Hamiltonian vector field $Y = Y^{H_1}$ induced by the Hamiltonian H_1 nicely to

$$\mathfrak{M}(t_0) := \int_{-\infty}^{\infty} \{H_0(q_0(t - t_0)), H_1(t, q_0(t - t_0))\}dt$$

where $\{H_0, H_1\} = \partial_x H_0 \partial_y H_1 - \partial_y H_0 \partial_x H_1$ denotes the Poisson bracket. The geometric importance of the Melnikov function lies in its relation to the 'distance' d via

$$d(t_0) = \frac{\varepsilon\, \mathfrak{M}(t_0)}{|X(q_0(0))|} + O(\varepsilon^2),$$

i.e. the 'distance' d is up to scaling with ε and normalization by $|X(q_0(0))|$ approximatively given by the Melnikov function. The ability of the Melnikov function to provide homoclinic intersection points is described in the following theorem.

Theorem A.4. *If $\mathfrak{M}(t_0)$ is independent of ε and if it has (as function of t_0) simple zeros then for small enough $\varepsilon > 0$ the (un)stable manifolds $W^u(x^\varepsilon_{t_0})$ and $W^s(x^\varepsilon_{t_0})$ intersect transversely. If $\mathfrak{M}(t_0)$ is bounded away from zero then $W^u(x^\varepsilon_{t_0})$ and $W^s(x^\varepsilon_{t_0})$ do not intersect.*

If the Melnikov function can be computed explicitly then the question about the existence of homoclinic points can accuratly be answered.

The 'distance' d is sometimes also called 'primary distance function' since it measures the distance between $W^u(x^\varepsilon_{t_0})$ and $W^s(x^\varepsilon_{t_0})$ perpendicular to the homoclinic trajectory q_0 where the reference points $q^u_\varepsilon(t_0)$ and $q^s_\varepsilon(t_0)$ are the *first* intersection in elapsed time.
In order to deal with homoclinic points arising from 'later' intersections Rom-Kedar [**RK2**] used the so called 'secondary Melnikov function' having similar properties as the original one.

Given two adjacent primary points $p = p_\varepsilon$ and $q = q_\varepsilon$ the Melnikov method provides a formula for the relative action $\mathcal{A}(p, q)$ from (8.2). The following result may be found in Kovačič [**Ko**].

Let Hamiltonian be of the form $H(t,x,y) := H_0(x,y) + \varepsilon H_1(t,x,y)$ and 2π-periodic in time. Then

$$\mathfrak{M}(\tau) := \int_{\mathbb{R}} \{H_0, H_1\}(\tau + t, q_0(t))dt$$

and denoting the zeros of $\mathfrak{M}$ corresponding to p and q by τ_p and τ_q we obtain

$$\mathcal{A}(p,q) = \varepsilon \int_{\tau_q}^{\tau_p} \mathfrak{M}(\tau)d\tau + O(\varepsilon^2). \tag{A.5}$$

Kaper & Wiggins [**KaW**] deduce analogously in case of *adiabatic* Hamiltonian system

$$\dot{x} = \partial_y H(z,x,y), \quad \dot{y} = -\partial_x H(z,x,y), \quad \dot{z} = \varepsilon$$

(and also for more general z-dependence) the *adiabatic* Melnikov function $\mathfrak{M}_A(z)$ and $\mathcal{A}(p,q) = \int_{z_0}^{z_1} \mathfrak{M}_A(z)dz + error$ where the error vanishes at least as fast as ε. Moreover they distinguish between **locally action minimizing homoclinic orbits** if $\mathfrak{M}_A(z) = 0$ and $\frac{d}{dz}\mathfrak{M}_A(z) > 0$ and **locally action maximizing homoclinic orbits** if $\mathfrak{M}_A(z) = 0$ and $\frac{d}{dz}\mathfrak{M}_A(z) < 0$.

There is ample literature about the Melnikov method and its application among mathematicians and physicists. We already mentioned the text books by Guckenheimer & Holmes [**GH**] and Arrowsmith & and Place [**ArP**] to which we add Wiggins [**W**]. Good intuition motivated by physical applications yields Zaslavsky [**Za**].

APPENDIX B

Lazutkin's homoclinic invariant

This appendix summarizes the construction proposed by Lazutkin [**Laz**] and finally proved by Gelfreich [**Ge1**] of Lazutkin's homoclinic invariant for the so called **(Chirikov) standard map**

$$F_\varepsilon : T^2 \to T^2, \quad (x,y) \mapsto (x_1, y_1), \quad y_1 := y + \varepsilon \sin(x), \quad x_1 := x + y_1$$

for $T^2 := \mathbb{R}^2/(2\pi\mathbb{Z})$ and $\varepsilon > 0$ small. The standard map F_ε appears as integrator of the pendulum equation

$$\dot{x} = y, \quad \dot{y} = \sin x,$$

i.e. the difference $F_\varepsilon(x,y) - \Phi_\varepsilon(x,y)$ lies in $O(\varepsilon^2)$ where Φ_t denotes the flow of the system. The (un)stable manifolds associated to the pendulum system coincide, but there are analytic obstructions preventing coinciding in case of F_ε. The transition from the integrable situation with coinciding (un)stable manifolds to noncoinciding invariant manifolds is also called **splitting**. Often splitting is analysed by the Melnikov method (see Appendix A), but in case of the standard map and most of its generalizations the splitting is to small to be detected by Melnikov's method.

F_ε has one hyperbolic fixed point in $(0,0)$ and one elliptic in $(\pi, 0)$ and $DF_\varepsilon(0,0) = \left(\begin{smallmatrix} 1+\varepsilon & 1 \\ \varepsilon & 1 \end{smallmatrix}\right)$ with eigenvalues λ and λ^{-1} where $\lambda = 1 + \frac{1}{2}\varepsilon + \sqrt{\varepsilon + \frac{\varepsilon^2}{4}}$. Figure 2.1 sketches the upper half of the now *heteroclinic* tangle on the universal cover $\mathbb{R}^2$. Unlike our usual notation the unstable manifold is drawn by a non-dotted line since the oscillations are to small for dots.

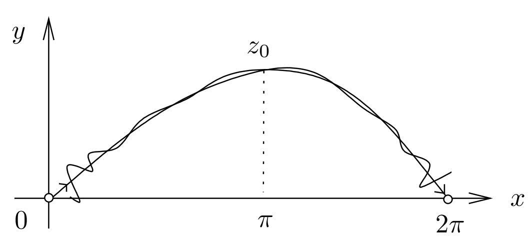


FIGURE 2.1. Heteroclinic tangle of the standard map

Instead of ε-dependence it is more convenient to consider $h := \log\lambda$ which is related to ε via $\varepsilon = 4\sinh^2(\frac{h}{2})$ implying $\varepsilon \approx h^2$. Solutions of the finite difference system

$$x(t+h) = x(t) + y(t+h),$$
$$y(t+h) = y(t) + \varepsilon \sin(x(t))$$

yield parametrizations of the branches sketched in figure 2.1 via $z^- := (x^-, y^-)$ for the unstable branch and $z^+ := (x^+, y^+)$ for the stable one under the conditions

$$\lim_{t\to-\infty} x^-(t) = 0 \qquad \text{and} \qquad x^-(0) = \pi,$$
$$(x^+, y^+)(t) := (2\pi - x^-(-t), y^-(-t) + \varepsilon \sin x^-(-t)).$$

x^+ satisfies $\lim_{t\to\infty} x^+(t) = 0$ and $x^+(0) = \pi$. Note that the boundary conditions do not yield unique solutions. We find $(x^-(0), y^-(0)) = (x^+(0), y^+(0)) =: z_0$ as heteroclinic point and define **Lazutkin's homoclinic invariant** of z_0 by

$$\Lambda := \det \begin{pmatrix} \dot{x}^-(0) & \dot{x}^+(0) \\ \dot{y}^-(0) & \dot{y}^+(0) \end{pmatrix}$$

which yields the same value for all $F_\varepsilon^n(z_0)$ for $n \in \mathbb{Z}$ and is invariant under symplectic coordinate changes. Geometrically Λ is the value of the symplectic form $dx \wedge dy$ evaluated on the tangent vectors on the parametrizations in z_0, i.e. the area of the parallelogram given by those vectors. If α is the splitting angle at z_0 then

$$\alpha = \sin^{-1}\left(\frac{\Lambda}{|\tilde{z}^-(0)||\tilde{z}^+(0)|}\right).$$

In the literature Lazutkin's invariant usually is denoted by ω, but since ω stands in our framework for the symplectic form we avoid confusion by using Λ.
There is exactly one other second primary heteroclinic equivalence class $\langle z_1\rangle$ different from $\langle z_0\rangle$ and they have up to sign the same aymptotic expansion in the following theorem.

Theorem B.1. *Λ has the asymptotic expansion*

$$\Lambda \overset{as}{=} \frac{4\pi}{h^2} e^{-\frac{\pi^2}{h}} \left(\sum_{n=0}^{\infty} h^{2n}\Lambda_n\right)$$

where $\overset{as}{=}$ means that the absolute value of the error when considering only the sum for $n \leq N$ can be estimated from above by $O(h^{2N-2}e^{-\frac{\pi^2}{h}})$. We have $\Lambda_0 = 1118.827706\ldots$ and $\Lambda_1, \ldots, \Lambda_4$ are also known with high accuracy.

For all sufficiently small $\varepsilon > 0$ the (un)stable manifolds of F_ε intersect transversely in z_0 and the splitting angle is given by

$$\alpha \stackrel{as}{=} \frac{\pi}{h^2} e^{-\frac{\pi^2}{h}} \left(\sum_{n=0}^{\infty} h^{2n} c_n \right)$$

where the coefficients c_n can be expressed via the coefficients Λ_n, in particular

$$\Lambda_0 = c_0, \qquad \Lambda_1 = c_1 + \tfrac{c_0}{4}, \qquad \Lambda_2 = c_2 + \tfrac{c_1}{4} + \tfrac{25}{72} c_0.$$

The relative action between z_0 and adjacent primary heteroclinic points in $\langle z_1 \rangle$ is asymptotically given by

$$\frac{2}{\pi} e^{-\frac{\pi^2}{h}} \left(\sum_{n=0}^{\infty} h^{2n} \Lambda_n \right).$$

Now we consider the **generalized standard map**

$$G_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x_1, y_1), \quad y_1 := y + \varepsilon f(x), \quad x_1 := x + y_1,$$

as introduced in Gelfreich & Lazutkin [**GeL**] and Gelfreich & Simó [**GeS**]. Gelfreich & Simó [**GeS**] study numerical methods for f being a polynomial, a trigonometrical polynomial or a meromorphic or rational function, but only when G_ε is the area-preserving Hénon-map, i.e. $f(x) = x - x^2$, the results are (up to now) mathematically rigorous.

The Hénon-map has two symmetry lines, namely $\{y = 0\}$ and $\{y = -\frac{\varepsilon}{2}(x - x^2)\}$, and their 'first' intersection points with the (un)stable manifolds are representatives $p = p_\varepsilon$ and $q = q_\varepsilon$ of the (exactly) two primary equivalence classes. The according homoclinic tangle is sketched in figure 8.2 (a) and the homoclinic invariant of p is given by

$$\Lambda(p) \stackrel{as}{=} \frac{4\pi}{h^6} e^{-\frac{2\pi^2}{h}} \left(\sum_{k \geq 0} a_k h^{2k} + e^{-\frac{2\pi^2}{h}} \sum_{k \geq 0} b_k h^{2k} \right).$$

p and q have (up to sign) a common asymptotic expansion, but numerically $\Lambda(p)$ and $-\Lambda(q)$ differ by $\frac{8\pi}{h^6} e^{-\frac{4\pi^2}{h}} \sum_{k \geq 0} c_k h^{2k}$. The relative action between adjacent representants of $\langle p \rangle$ and $\langle q \rangle$ is up to an exponentially small quantity of higher order given by

$$\text{(B.2)} \qquad \frac{\Lambda h^2}{2\pi^2} \approx \frac{4\pi a_0}{h^6} e^{-\frac{2\pi^2}{h}} (1 + O(h^4 e^{-\frac{\pi^2}{h}})).$$

If $f(x) = x + rx^2 - x^3$ is a cubic polynomial then the constructions and expansions as performed for the standard map are not yet proven rigorously. Nevertheless high-precision numerical experiments suggest their validity. They yield the homoclinic tangle sketched in figure 8.2 (b). The splitting of the (un)stable manifolds is exponentially small w.r.t. h, but for $r \neq 0$ the tangle is not symmetric w.r.t. the y-axis and the splitting on the right hand side is exponentially large compared with the one on the left hand side.

Bibliography

[AbS] Abbondandolo, A.; Schwarz, M.: *On the Floer homology of cotangant bundles* Comm. Pure Appl. Math. **59** (2006), no. 2, 254 – 316

[AmB] Ambrosetti, A.; Badiale, M.: *Homoclinics: Poincaré-Melnikov type results via a variational approach* Ann. Inst. Henri Poincaré, Vol. **15**, No. 2, 233 – 252

[Ar1] Arnold, V.I.: *Sur une propriété des applications globalement canonique de la méchanique classique*, C. R. Acad. Sci., Paris, **261** (1965), 3719 – 3722

[ArA] Arnold, V.I.; Avez, A.: *Problèmes Ergodiques de la Mécanique Classique* Gauthier-Villars, Paris 1967

[ArGZV] Arnold, V.I.; Gusein-Zade, S.M.; Varchenko, A.N.: *Singularities of differentiable maps, Volume I: The classification of critical points, caustics and wave fronts* Birkhäuser 1985

[ArP] Arrowsmith, D.; Place, C.: *An introduction to dynamical systems* Cambridge University Press 1990

[Bi] Birkhoff, G. D.: *Nouvelles recherches sur les systèmes dynamiques* Mem. Pont. Acad. Sci. Nov. Lyncaei, 1935, **53**, 85 – 216

[Br] Bredon, G.: *Topology and geometry* Springer 1993

[BW] Burns, K.; Weiss, H.: *A geometric criterion for positive topologcal entropy* Commun. Math. Phys. **172** (1995), 95 –118

[Che] Chekanov, Y.: *Differential algebra of Legendrian links* Invent. math. **150** (2002), 441 – 483

[CiS] Cieliebak, K.; Séré, E.: *Pseudo-holomorphic curves and multiplicity of homoclinic orbits* Duke Math. J. **77** No. 2 (1995), 483 – 518

[CP] Contopoulos, G.; Polymilis, C.: *Geometrical and dynamical properties of homoclinic tangles in a simple Hamiltonian system* Physical review E, Vol. 46, no.3 (March 1993), 1546 – 1557

[CZES] Coti Zelati, V.; Ekeland, I.; Séré, E.: *A variational approach to homoclinic orbits in Hamiltonian systems* Math. Ann. **288**, 133 – 160 (1990)

[dS] de Silva, S.: *Products in the symplectic Floer homology of Lagrangian intersections* Thesis, Merton College, Oxford 1998

[E] Easton, R.W.: *Trellises formed by stable and unstable manifolds in the plane* Trans. Am. Math. Soc. **294, no. 2** (1986), 719 – 732

[EGH] Eliashberg, Y.; Givental, A.; Hofer, H.: *Introduction to symplectic field theory* Geom. Funct. Anal. (special volume, part II) (2000), 560 – 673

[ENS] Etnyre, J.; Ng, L.; Sabloff, J.: *Invariants of Legendrian knots and coherent orientations* J. Symplectic Geom. 1 (2002), no. 2, 321 – 367

[Fe] Fel'shtyn, A.: *Dynamical Zeta functions and Floer homology* Contemporary Mathematics, Vol. 385 (2005), 187 – 203

[Fl1] Floer, A.: *A relative Morse index for the symplectic action* Comm. Pure Appl. Math., Vol. **41** (1988), 393 – 407

[Fl2] Floer, A.: *The unregularized gradient flow of the symplectic action* Comm. Pure Appl. Math., Vol. **41** (1988), 775 – 813

[Fl3] Floer, A.: *Morse theory for Lagrangian intersections* J. Diff. Geom. **28** (1988), 513 – 547

[FO_3] Fukaya, K.; Oh, Y.-G.; Ohta, H., Ono, K.: *Lagrangian intersection Floer theory — anomaly and obstruction* Preprint

[GauRS] Gautschi, R.; Robbin, J.; Salamon, D.; *Heegard splittings and Morse-Smale flows* Preprint ETH Zürich October 2002

[Ge1] Gelfreich, V.: *A proof of the exponentially small transversality of the separatrices for the standard map* Comm. Math. Phys. 201 (1999), no. 1, 155 – 216

[Ge2] Gelfreich, V.: *Numerics and Exponential Smallness* Handbook of Dynamical Systems, Vol. 2, Ed. by B. Fiedler, 256 – 312, Elsevier 2002

[GeL] Gelfreich, V.; Lazutkin, V.: *Splitting of separatrices: perturbation theory and exponentially smallness* Russian Math. Surveys **56**: 3, 499 – 558 (2001) (or Uspekhi Mat. Nauk **56**:3, 79 – 142)

[GeS] Gelfreich, V.; Simó, C.: *High-precision computations of divergent asymptotic series and homoclinic phenomena*, Preprint 2006

[GH] Guckenheimer, J.; Holmes, P.: *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* Springer, third edition 1990

[HH] Hocket, K.; Holmes, P.: *Josephson's junction, annulus maps, Birkhoff attractors, horseshoes and rotation sets* Ergod. Th. Dynam. sys. **6** (1986), 205 – 239

[HuS] Hutchings, M.; Sullivan, M.: *The periodic Floer homology of a Dehn twist* Alg. Geom. Topology **5** (2005), 301 – 354

[KaW] Kaper, T.; Wiggins, S.: *Lobe area in adiabatic Hamiltonian systems* Physica D 51 (1991), 205 – 212

[KH] Katok, A.; Hasselblatt, B.: *Introduction to the modern theory of dynamical systems* Cambridge University Press 1995

[KiS] Kirchgraber, U.; Stoffer, D.: *Chaotic behaviour in simple dynamical systems* SIAM Review, Vol. **32**, No. 3 (Sep., 1990), 424 – 452

[Ku] Kupka, I.: *Contribution à la théorie des champs génériques* Contributions to differential equations, vol. 2 (1963), 457 – 484 and Addendum and Corrections vol. 3, no. 4 (1964), 411 – 420

[Ko] Kovačič, G.: *Lobe area via action formalism in a class of Hamiltonian systems* Physica D 51 (1991), 226 – 233

[KZ] Kuznetsov, L.; Zaslavsky, G.: *Scaling invariance of the homoclinic tangle* Physical Review E **66**, 046212 (2002)

[Lau] Laudenbach, F.: *Engouffrement symplectique et intersections lagrangiennes* Comment. Math. Helvetici **70** (1995), 558 – 614

[Laz] Lazutkin, V.: *Splitting of separatrices for Chirikov's standard map (russian)* VINITI no. 6372 (84) 1984, revised english translation in Math. and Physics Prepint Archive 98-421, www.ma.utexas.edu/mp_arc

[Le] Leclercq, R.: *Spectral invariants in Lagrangian Floer theory* Preprint

[Li] Lisi, S.: *Homoclinic orbits and Lagrangian embeddings* arXiv:math.SG/0608801 v1 31 Aug 2006

[McS1] McDuff, D.; Salamon, D.: *Introduction to Symplectic Topology* Clarendon Press, Oxford 1998

[McS2] McDuff, D.; Salamon, D.: *J-holomorphic Curves and Quantum Cohomology* AMS University Lecture series 6, Providence, Rhode Island, 1994

[Me] Melnikov, V. K.: *On the stability of the center for time periodic perturbations* Trans. Moscow Math. Soc. **12** (1963), 1– 57

[Mi] Milnor, J.: *Topology from the differentiable viewpoint* Princeton University Press, Princeton, New Jersey 1997

[Mo] Moser, J.: *Stable and random motions in dynamical systems* Princeton University Press 1973

[Ne1] Newhouse, S.: *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms* Publ. Math. IHES **50** (1979), 101 – 151

[Ne2] Newhouse, S.: *On the mathematical contributions of Jacob Palis* in: Astérisque **286** (2003), Geometric methods in dynamics I, Volume in honor of Jacob Palis, 1 – 24

[NePT] Newhouse, S.; Palis, J.; Takens, F.: *Bifurcations and stability of families of diffeomorphisms* Publ. Math. IHES **57** (1983), 5 – 71

[Ng] Ng, L.: *Computable Legendrian invariants* Topology **42** (2003), no. 1, 55 – 82

[Ol] Oliveira, F.: *On the C^∞ genericity of homoclinic orbits* Nonlinearity **13** (2000), 653 – 662

[Pa] Palis, J.: *On Morse-Smale dynamcal systems* Topology, Vol. 8 (1969), 385 – 405

[PaT1] Palis, J.; Takens, F.: *Hyperbolicity and the creation of homoclinic orbits* Annals of Math. **125** (1987)

[PaT2] Palis, J.; Takens, F.: *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations* Cambridge University Press 1993

[Po1] Poincaré, H.: *Sur le problème des trois corps et les équations de la dynamique* Acta Mathematica, 1890, **13**, 1 – 271

[Po2] Poincaré, H.: *Les méthodes nouvelles de la mechanique céleste* Gauthier-Villars, Paris 1899

[Ra1] Rabinowitz, P.: *Homoclinic and heteroclinic orbits for a class of Hamiltonian systems* Calc. Var. Partial Diff. Equ. 1, No. 1 (1993), 1 – 36

[Ra2] Rabinowitz, P.: *Some critical point theorems and applications to semilinear elliptic partial differential equations* Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4) **5** (1978), no. 1, 215 – 223

[RK1] Rom-Kedar, V.: *Homoclinic tangles — classification and applications* Nonlinearity **7** (1994), 441-473

[RK2] Rom-Kedar, V.: *Secondary homoclinic bifurcation theorems* Chaos **5 (2)**, 1995

[R] Robbin, J.: *Heegard splittings and Floer homology* Preprint September 2000

[Ro] Robinson, R. C.: *Generic properties of conservative systems* Am. J. Math. **92**, no. 3 (1970), 562 – 603

[Sa] Salamon, D.: *Lectures on Floer homology* held at the IAS/ Parc City graduate summer school 1997.

[Sch1] Schwarz, M.: *Morse Homology* Birkhäuser 1993

[Sch2] Schwarz, M.: PhD-thesis ETH Zürich 1995

[Sch3] Schwarz, M.: *On the action spectrum for closed symplectically aspherical manifolds* Pacific J. of Math., Vol. 193, No. 2, 2000

[Se1] Seidel, P.: *Fukaya categories and Picard-Lefschetz theory* Preprint June 2006

[Sér] Séré, E.: *Existence of infinitely many homoclinic orbits in Hamiltonian systems* Math. Z. **209**, 27 – 42 (1992)

[Sh] Shil'nikov, L.P.: *Homoclinic trajectories: From Poincaré to the present* translated by R. Cooke in *Mathematical events of the twentieth century*, Springer 2006

[SiV] Simó, C.; Vieiro, A.: *Some relevant aspects of area preserving maps and the planar conservative Hénon map*, Preprint

[Sm1] Smale, S.: *A structurally stable differentialbe homeomorphism with an infinite number of periodic points* In: Proceedings of the International Symposium on nonlinear Oscillatons, Izdat. Akad. Nauk Ukrain SSR, Vol. 2, Kiev, 1963, 365 – 366

[Sm2] Smale, S.: *Diffeomorphisms with many periodic points* Differential and combinatorial topology, Princeton Univ. Press, Princeton N.J., 1965, 63 – 80

[Sp] Spanier, E.: *Algebraic topology*, Springer 1966

[Su] Sullivan, M.: *K-theoretic invariants for Floer homology* GAFA, Geom. funct. anal. **12** (2002), 810 – 872

[Ta] Takens, F.: *Homoclinic points in conservative systems* Invent. Math. **18** (1972), 267 – 292

[V] Viterbo, C.: *Intersection de sous-variété lagrangiennes, fonctionnelles d'action et indice des systèmes hamiltoniens* Bull. Soc. Math. France **115** (1987) no. 3, 361–390

[W] Wiggins, S.: *Global dynamics, phase space transport, orbits homoclinic to resonances, and applications*, AMS, Providence, Rhode Island 1993

[Xia1] Xia, Z.: *Homoclinic points in symplectic and volume-preserving diffeomorphisms* Commun. Math. Phys. **177** (1996), 435 – 449

[Xia2] Xia, Z.: *Homoclinic points and intersections of Lagrangian submanifolds* Discr. Cont. Dyn. Sys. **6**, no. 1 (2000), 243 – 253

[Xia3] Xia, Z.: *Homoclinic points for area preserving surface diffeomorphisms* Preprint 2006

[Xia4] Xia, Z.: *Private communications*

[Za] Zaslavsky, G.: *Hamiltonian chaos and Fractional dynamics*, Oxford University Press 2005

[Ze] Zehnder, E.: *Homoclinic points near elliptic fixed points* Comm. pure appl., **26**, (1973) 131 – 182

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

(Ort, Datum)

(Unterschrift)

Wissenschaftlicher Werdegang des Verfassers

- Abitur 1997 (Note 1,0) am Gymnasium am Ostring, Bochum
- WS 1997/98 – SS 1999 Mathematikstudium an der Albert-Ludwigs-Universität Freiburg und Aufnahme in die Studienstiftung des Deutschen Volkes; Vordiplom im Sommer 1999 (Note 'sehr gut')
- WS 1999/00 – SS 2000 Mathematikstudium an der Ecole Normale Supérieure de Lyon
- WS 2000/01 – WS 2002/03 Mathematikstudium an der Albert-Ludwigs-Universtät Freiburg
- März 2003 Diplom (Note 'sehr gut'), Diplomarbeit 'Optimale Massebewegung im Monge-Kantorovich-Transportproblem' (Betreuer V. Bangert)
- Juni 2003 – Juni 2006 Promotionsstudent, Stipendiat des Graduiertenkollegs 'Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften' an der Universität Leipzig, ab Juni 2006 DFG-Drittmittel SCHW892/2-2
  Titel der Dissertation 'Floer homology for homoclinic tangles' (Betreuer M. Schwarz)

Bibliographische Daten

Hohloch, Sonja
Floer homology for homoclinic tangles

Dissertation, Universität Leipzig 2007
153 Seiten, 50 Abbildungen, 82 Literaturangaben

Mathematical subject classification (2000): 53D40, 57R58, 37C29, 37D10, 37E30, 37J10, 37J45, 70K44
Keywords: Floer homology, homoclinic points, (un)stable manifolds, symplectic dynamical systems