Dynamical interpretation of homoclinic Floer homology

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Abstract

We define four versions of Floer homology generated by (certain) homoclinic points and point out their algebraic and dynamical properties. Among others, the rank of filtered homoclinic Floer groups grows linearly with the number of iterations of the underlying symplectomorphism. Moreover, we point out a relation between the (absolute) flux in MacKay & Meiss & Percival [MMP], Mather's [Ma] difference in action $\triangle W$ and homoclinic Floer theory.

1 Introduction

In order to make this survey easily accessible to a broader audience, we repeat some crucial definitions before we delve into the construction and applications of homoclinic Floer homology.

1.1 Notations

A smooth (even dimensional) manifold M is *symplectic* if it admits a closed, nondegenerate 2-form. Such a form is called a *symplectic form* and will (usually) be denoted by ω. A diffeomorphism f on a symplectic manifold (M, ω) is a *symplectomorphism* if f preserves ω, i.e. $f^*\omega = \omega$. The group of symplectomorphisms w.r.t. ω is denoted by $\mathrm{Symp}(M, \omega)$. We abbreviate $\mathrm{Symp}(M)$ if there is no confusion.
Given a smooth function $F : M \times S^1 \to \mathbb{R}$, we set $F_t := F(\cdot, t)$ and define its (nonautonomous) Hamiltonian vector field X_t^F via $\omega(X_t^F, \cdot) = -dF_t(\cdot)$. Then $\dot{z}(t) = X_t^F(z(t))$ is the associated *Hamiltonian equation* and its (nonautonomous) flow is called *Hamiltonian flow*. A *Hamiltonian diffeomorphism* is a symplectomorphisms which can be written as the time-1 map φ_1 of a Hamiltonian flow φ_t. We denote by $\mathrm{Ham}^c(M, \omega)$ the group of compactly supported Hamiltonian diffeomorphisms. Again, we abbreviate $\mathrm{Ham}^c(M)$ if there is no confusion. A Hamiltonian diffeomorphism is called *nondegenerate* if its graph intersects the diagonal in $M \times M$ transversely.

x is a *periodic point* of a diffeomorphism f if there exist $m \in \mathbb{N}$ such that $f^m(x) = x$. In the special case $m = 1$, x is called a *fixed point* and the set of fixed points of f is denoted by $\mathrm{Fix}(f)$. $x \in \mathrm{Fix}(f)$ is called *hyperbolic* if the eigenvalues of the linearization of f in x have modulus different from 1. The *stable manifold* of a hyperbolic fixed point x is given by $W^s(f,x) := \{p \in M \mid \lim_{n\to\infty} f^n(p) = x\}$ and the *unstable manifold* is given by $W^u(f,x) := \{p \in M \mid \lim_{n\to-\infty} f^n(p) = x\}$, briefly W^s and W^u. They are injectively immersed submanifold. If f is symplectic then the stable and unstable manifolds satisfy $\omega|_{W^s} = 0 = \omega|_{W^u}$ and $\dim W^s = \frac{1}{2}\dim M = \dim W^u$, i.e. they are *Lagrangian* submanifolds. Intersection points of the stable and unstable manifold are called *homoclinic points* and we denote the set of homoclinic points by $\mathcal{H} := \mathcal{H}(f,x) := W^s(f,x) \cap W^u(f,x)$. The connected components of $W^s \backslash \{x\}$ resp. $W^u \backslash \{x\}$ are called the *branches* of W^s resp. W^u. A symplectomorphism is called W*-orientation preserving* if it preserves the branches of the (un)stable manifolds. Otherwise it is called W-orientation reversing.

1.2 Motivation

In the 1960s, Arnold conjectured that the number of fixed points of a non-degenerate Hamiltonian diffeomorphism on a closed, symplectic manifold is greater or equal to the sum over the Betti numbers. In order to approach this conjecture, Floer [Fl1, Fl2, Fl3] considered the fixed points as intersection points of the graph of the Hamiltonian diffeomorphism with the diagonal in the symplectic manifold $(M \times M, \omega \oplus (-\omega))$. Since the graph and the diagonal are Lagrangian submanifolds, Floer turned the fixed point problem into a Lagrangian intersection problem. The intersection points can be seen as critical points of the symplectic action functional. This inspired Floer to devise some kind of infinite dimensional Morse theory for the symplectic action functional which is nowadays known as Floer theory. Apart from leading to a proof of Arnold's conjecture, Floer theory gave rise to many other applications in symplectic geometry, dynamical systems and other fields of mathematics and is vividly studied nowadays.

In the study of dynamical systems, homoclinic points are the next more difficult orbit type after fixed points and periodic points. The existence of (transverse) homoclinic points was discovered by Poincaré [Poi1, Poi2] around 1890 when he worked on the n-body problem. In 1935, Birkhoff [Bi] noticed the existence of high-periodic points near homoclinic ones, but it took until Smale's horseshoe formalism in the 1960s to obtain a formal and precise description of the implied dynamics. Since then, homoclinic points have been studied by various means like perturbation theory, calculus of variations and numerical approximation, but many questions are still open.

Up to our knowledge, there are few papers where homoclinic orbits are studied with symplectic methods or means related to Floer theory: Hofer

& Wysocki [HW] use pseudo-holomorphic curves and Fredholm theory. Cieliebak & Séré [CiS] combine variational technics and pseudo-holomorphic curves. Lisi [Li] generalizes Coti Zelati & Ekeland & Séré [CZES] using Lagrangian embedding techniques.

The present survey is based on Hohloch [Ho1, Ho2]. We link homoclinic points to Floer theory by constructing Floer homologies generated by homoclinic points. Then we study their dynamical properties.
More precisely, in Section 2, we construct four versions of homoclinic Floer homology and point out their properties and main differences.
In Section 3, we study — motivated by Polterovich [Pol1, Pol2] — the growth of the rank of homoclinic Floer homology under iteration of the underlying symplectomorphism. Theorem 8 shows the rank of the homoclinic Floer groups to grow linearly.
Section 4 focuses on the dynamical interpretation of homoclinic Floer homology: we introduce the so-called (absolute) flux as defined in MacKay & Meiss & Percival [MMP] and refine their notion of so-called turnstiles. Lemma 10 links the flux to the symplectic action. Proposition 12 interpretes turnstiles in terms of the Floer boundary operator. Corollary 11 deduces linear growth of the flux under iteration of the underlying symplectomorphism. MacKay & Meiss & Percival [MMP] proved the flux to coincide under certain assumptions with Mather's [Ma] difference in action $\triangle W$. Theorem 14 eventually unites the notions of flux, $\triangle W$, (relative) symplectic action and certain immersed disks between homoclinic points.
Section 5 summerizes the discussed homoclinic Floer homologies and their main properties in a table.

Acknowledgements

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2 Types of homoclinic Floer homology

2.1 Primary Floer homology

Let (M, ω) be a symplectic manifold and φ a symplectomorphisms with hyperbolic fixed point x and (un)stable manifolds W^s and W^u. If we think about constructing a Floer homology for the Lagrangian intersection problem $W^s \cap W^u$, we run into the following problem: For $\dim M > 2$, the construction of Floer homology involves (among others) Fredholm analysis and Gromov compactness for moduli spaces of pseudo-holomorphic curves, see e.g. Salamon [Sa]. Since W^s and W^u are highly noncompact, wildly oscillating and not properly embedded the analysis problem is a quite hopeless task. But, for $\dim M = 2$, the analysis can be replaced by combinatorics ([dS], [Fe2], [GaRS]). This works in spite of the chaotic behaviour of W^s and W^u.

Now assume (M, ω) to be $\mathbb{R}^2$ or a closed surface of genus $g \geq 1$ with their resp. volume forms. Consider the set of homoclinic points $\mathcal{H} := W^s \cap W^u$ where we assume the intersection to be transverse. Let p, $q \in \mathcal{H}$ and denote by $[p,q]_i$ the segment between p and q in W^i for $i \in \{s,u\}$. The symplectomorphism φ introduces a $\mathbb{Z}$-action $\mathcal{H} \times \mathbb{Z} \to \mathcal{H}$, $(p,n) \mapsto \varphi^n(p)$. For transversely intersecting $W^s \cap W^u$, the sets $\mathcal{H}$ and $\mathcal{H}/\mathbb{Z}$ are both infinite. Denote by $c_p : [0,1] \to W^u \cup W^s$ a curve with $c_p(0) = x = c_p(1)$ which runs through $[x,p]_u$ to p and through $[p,x]_s$ back to x. We define the homotopy class of p via $[p] := [c_p] \in \pi_1(M,x)$. Then $\mathcal{H}_{[x]} := \{p \in \mathcal{H} \mid [p] = [x]\}$ is the set of *contractible* homoclinic points. It is invariant under the action of φ. When iterating homoclinic points $p \in \mathcal{H}$ we often abbreviate $\varphi^n(p) =: p^n$. Note that in this notation $p^0 = p$ and $p^1 = \varphi(p)$. Analogously to Floer [Fl1], there is a (relative) Maslov index $\mu(p,q) \in \mathbb{Z}$ for p, $q \in \mathcal{H}$ if $[p] = [q]$. In our two-dimensional setting, $\mu(p,q)$ can be seen as twice the tangent winding number of a loop starting in p, running through $[p,q]_u$ to q and through $[p,q]_s$ back to p if we assume the intersections to be perpendicular and if we flip $+90°$ at q and $-90°$ at p. We observe $\mu(p,q) = \mu(p^n,q^n)$ for $n \in \mathbb{Z}$. The (relative) Maslov index yields a *grading* $\mu : \mathcal{H}_{[x]} \to \mathbb{Z}$ via $\mu(p) := \mu(p,x)$ such that for contractible homoclinic points p and q holds

$$\mu(p,q) = \mu(p,x) + \mu(x,q) = \mu(p,x) - \mu(q,x) = \mu(p) - \mu(q).$$

$\mathcal{H}$ and $\mathcal{H}_{[x]}$ are somehow 'too large' sets in order to be used as generators for a Floer chain complex. But there are suitable subsets for which we can construct a Floer homology: We call $p \in \mathcal{H}$ *semi-primary* if $]x,p[_s \cap]x,p[_u = \emptyset$. A contractible $p \in \mathcal{H}_{[x]}$ is called *primary* if $]p,x[_s \cap]p,x[_u \cap \mathcal{H}_{[x]} = \emptyset$ and the set of primary points is denoted by $\mathcal{H}_{pr}$.

Lemma 1 ([Ho1]). *(i) Let φ be W-orientation preserving, p (semi-) primary and denote the branches containing p by W^u_p and W^s_p. Then for every (semi-) primary $q \in (W^u_p \cap W^s_p) \backslash \{p^n \mid n \in \mathbb{Z}\}$ there is a unique $n \in \mathbb{Z}$ such that $q^n \in]p,\varphi(p)[_u \cap]p,\varphi(p)[_s$.*

(ii) For a primary point p holds $\mu(p) = \mu(p,x) \in \{\pm 1, \pm 2, \pm 3\}$.

(iii) If W^s and W^u intersect transversely then $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/\mathbb{Z}$ is finite.

We denote the equivalence class of $p \in \mathcal{H}_{pr}$ in $\tilde{\mathcal{H}}_{pr} = \mathcal{H}_{pr}/\mathbb{Z}$ by $\langle p \rangle$. The homotopy class and the Maslov index μ pass to the quotient via $[\langle p \rangle] := [p]$, $\mu(\langle p \rangle, \langle q \rangle) := \mu(p,q)$ and $\mu(\langle p \rangle) := \mu(p,x)$.

Consider a fixed 2-gon D in $\mathbb{R}^2$ with convex vertices at $(-1,0)$ and $(1,0)$. Denote its lower edge by B_u and its upper edge by B_s. For p, $q \in \mathcal{H}$, we define $\mathcal{M}(p,q)$ to be the space of smooth, immersed 2-gons $v : D \to M$ which are orientation preserving and satisfy $v(B_u) \subset W^u$, $v(B_s) \subset W^s$,

$v(-1,0) = p$ and $v(1,0) = q$. Denote by $G(D)$ the group of orientation preserving diffeomorphisms of D which preserve the vertices. We set $\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/G(D)$ and $m(p,q) := \#\widehat{\mathcal{M}}(p,q)$. For $\langle p\rangle$, $\langle q\rangle \in \tilde{\mathcal{H}}_{pr}$ set $m(\langle p\rangle, \langle q\rangle) := \sum_{n\in\mathbb{Z}} m(p, q^n)$. We define the chain groups and the boundary operator via

$$C_k := C_k(\varphi, x; \mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle p\rangle)=k}} \mathbb{Z}\langle p\rangle, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle, \langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity. We have $\mathrm{rk}_{\mathbb{Z}}(C_k) < \infty$ and, due to Lemma 1, $C_k = 0$ for $k \notin \{\pm1, \pm2, \pm3\}$.

Theorem 2 ([Ho1])**.** *$\partial \circ \partial = 0$, i.e. (C_*, ∂_*) is a chain complex and*

$$H_k := H_k(\varphi, x; \mathbb{Z}) := \frac{\ker \partial_k}{\mathrm{Im}\, \partial_{k+1}}$$

is called primary Floer homology of φ in x and $H_k = 0$ for $k \neq \pm1, \pm2, \pm3$.

Since $\tilde{\mathcal{H}}_{pr}$ and the sum in the definition of ∂ are finite, primary Floer homology is in fact completely determined by (possibly large) compact segments of the (un)stable manifolds centered around the fixed point.

The proofs of the well-definedness of ∂ and of $\partial \circ \partial = 0$ involve the so-called breaking and gluing procedure which mainly relies on the classification of $\mathcal{M}(p,q)$ and of immersions of relative Maslov index 2. Certain parts of the proofs are of combinatorial nature whereas other parts make use of the iteration behaviour of $W^s \cap W^u$ and use classical dynamical results like Palis' λ-Lemma [Pa].

H_* is clearly invariant under conjugation, thus making all constructions natural. But there is also another form of invariance.

Remark 3 ([Ho1])**.** *H_* is invariant under a quite large class of perturbations of the underlying symplectomorphism.*

For a more precise description of the class of perturbations cf. [Ho1]. The proof has to combine analytical and combinatorial arguments since a primary point p might vanish (arise) in two ways: (a) p vanishes as intersection point and (b) p persists as intersection point, but is no longer primary. Generic bifurcations can be considered as second Reidemeister moves and $H_*(\varphi, x)$ is proven to be invariant under them: There are *primary moves* (both vanishing points are primary), *secondary moves* (both vanishing points are nonprimary) or *mixed moves with primary-secondary flips* (one vanishing point primary, the other nonprimary while an odd number of primary points is flipped nonprimary). These phenomena will resurface in the discussion of dynamical properties, cf. Remark 9.

Remark 3 affects the dynamical properties of H_* since the invariance of primary Floer homology is too strong to admit a direct observation of the growth of the rank of the homology groups as it will be done in Theorem 8. One really needs to use the filtered versions in Theorem 8, for more details see [Ho1].

2.2 Semi-primary Floer homology

If we use *contractible semi-primary* points instead of primary points as generators in the previous subsection, we obtain *semi-primary Floer homology* $\tilde{H}_*(\varphi, x)$ (cf. [Ho1]). The construction is analogous to Theorem 2. The differences between $H_*(\varphi, x)$ and $\tilde{H}_*(\varphi, x)$ become apparent when we study the invariance properties: $\tilde{H}_*(\varphi, x)$ is invariant under a much smaller class of perturbations than $H_*(\varphi, x)$. Thus it is much more sensitive w.r.t. the underlying symplectomorphism.

For example (cf. [Ho1] for details), $\tilde{H}_*(\varphi, x)$ notices certain interactions of W^s and W^u with the topology of the manifold whereas primary Floer homology is oblivious to them: Consider the situation in Figure 1. Assume for simplicity that the branches, which do not contain p, do not intersect any other branches. Thus $\langle p \rangle$ and $\langle q \rangle$ are the only primary orbits. We have $\mu(\langle p \rangle) = -1$ and $\mu(\langle q \rangle) = -2$. p and q are primary no matter if $]p, q[_s$ intersects $]q, p^{-1}[_u$ or not. Thus we obtain $H_{-1}(\varphi, x) \simeq \mathbb{Z}$ and $H_{-2}(\varphi, x) \simeq \mathbb{Z}$ and $H_*(\varphi, x) = 0$ else.

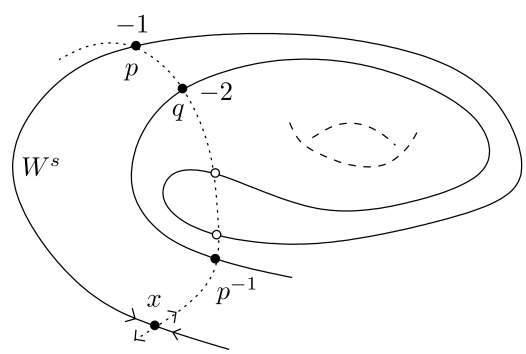

Figure 1: Arising of nontrivial homotopy classes

But if we are considering semi-primary points, the intersection behaviour of $]p, q[_s$ and $]q, p^{-1}[_u$ *does* matter. Whereas p is always semi-primary, q is only semi-primary if $]p, q[_s$ and $]q, p^{-1}[_u$ do not intersect. So if $]p, q[_s$ and $]q, p^{-1}[_u$ do not intersect we obtain $\tilde{H}_{-1}(\varphi, x) \simeq \mathbb{Z}$ and $\tilde{H}_{-2}(\varphi, x) \simeq \mathbb{Z}$ and $\tilde{H}_*(\varphi, x) = 0$ else. But if $]p, q[_s \cap\,]q, p^{-1}[_u \neq \emptyset$ we only have $\tilde{H}_{-1}(\varphi, x) \simeq \mathbb{Z}$ and $\tilde{H}_*(\varphi, x) = 0$ else.

Hockett & Holmes [HH] study the existence and impact of such (semi-primary) homoclinic points on the annulus. If c_p is contractible they call the homoclinic point p *non-rotary.* If c_p winds k times around the hole of the annulus, they call p *k-rotary.* Noncontractible, semi-primary points therefore fit as 1-rotary points in their framework.

Moreover, semi-primary Floer homology distinguishes between φ and φ^n for certain symplectomorphisms φ. For the case $]p,q[_s \cap]q,p^{-1}[_u \neq \emptyset$ in the above example, we obtain (cf. [Ho1] for details)

$$\tilde{H}_{-1}(\varphi^n, x) \simeq \mathbb{Z}^n \quad \text{and} \quad \tilde{H}_m(\varphi^n, x) = 0 \quad \text{for } m \in \mathbb{Z}\backslash\{-1\},\ n \in \mathbb{N}.$$

And if $]p,q[_s \cap]q,p^{-1}[_u = \emptyset$ then

$$\tilde{H}_{-1}(\varphi^n, x) = H_{-1}(\varphi^n, x) \simeq \mathbb{Z} \simeq \tilde{H}_{-2}(\varphi^n, x) = H_{-2}(\varphi^n, x) \quad \text{for } n \in \mathbb{N}$$

and $\tilde{H}_*(\varphi^n, x) = H_*(\varphi^n, x) = 0$ else.

Speaking in terms of the proof of Remark 3, the intersection behaviour of $]p,q[_s$ and $]q,p^{-1}[_u$ can be interpreted as secondary move (since both arising points are not primary). As such, it clearly leaves primary Floer homology invariant. For semi-primary Floer homology, this is obviously not true.

2.3 Cylinder Floer homology

Primary Floer homology as well as semi-primary Floer homology use contractible homoclinic points as generators. But in physical examples, the homoclinic points are often noncontractible. For dynamical systems on the infinite, symplectic cylinder $(\mathcal{Z}, \omega)$, there is a simple idea how to define homoclinic Floer homology based on noncontractible semi-primary points: We identify the cylinder with an annulus in $\mathbb{R}^2$ and 'forget' about the hole of the annulus. In this way, we can use large parts of the homology contruction from Theorem 2. Moreover, we can adjust said construction to 'keep in mind' the original homotopy class of a homoclinic point such that we get meaningful homologies e.g. for the perturbed pendulum and Chirikov's Standard map. We will call this type of Floer homology *cylinder Floer homology.*

We denote by $\mathrm{Symp}_0^c(\mathcal{Z}) := \mathrm{Symp}_0^c(\mathcal{Z}, \omega)$ the group of compactly supported symplectomorphisms isotopic to the identity. For some of the following constructions, we need the symplectomorphisms to be W-orientation preserving. For W-orientation reversing symplectomorphisms φ, consider the W-orientation preserving φ^2 instead. As in the construction of primary Floer homology, we define the homotopy class of a homoclinic point p on the cylinder as $[p] := [c_p] \in \pi_1(\mathcal{Z}, x)$. It holds $[p] = [\varphi(p)]$ for $\varphi \in \mathrm{Symp}_0^c(\mathcal{Z})$.

Let $0 < R_- < R_+ < \infty$ and denote by $\mathcal{Q} := \mathcal{Q}(R_-, R_+)$ the open annulus in $(\mathbb{R}^2, dx \wedge dy)$ centered at the origin with radii R_- and R_+. Let $h : \mathcal{Z} \to \mathcal{Q}$ be an orientation preserving diffeomorphism which identifies the cylinder with the annulus. Given $f \in \mathrm{Symp}_0^c(\mathcal{Z})$, we denote by

$F := F_h := h \circ f \circ h^{-1} \in \mathrm{Diff}(\mathcal{Q})$ its conjugate. If $x \in \mathrm{Fix}(f)$ is hyperbolic so is $\mathfrak{x} := h(x) \in \mathrm{Fix}(F)$. Denote by $\mathcal{H}(f,x) := W^s(f,x) \cap W^u(f,x)$ the set of homoclinic points of f w.r.t. x. Analogously define $\mathcal{H}(F,\mathfrak{x}) := W^s(F,\mathfrak{x}) \cap W^u(F,\mathfrak{x})$ seen as points in $\mathbb{R}^2$, i.e. all of them are considered contractible. Denote by $\mathcal{H}_{pr}(F,\mathfrak{x}) \subset \mathcal{H}(F,\mathfrak{x})$ the set of primary points of F w.r.t. $\mathfrak{x}$ and define $\mathcal{H}_{pr}(f,x) := h^{-1}(\mathcal{H}_{pr}(F,\mathfrak{x}))$. Images under h of homoclinic points $p \in \mathcal{H}(f,x)$ are abbreviated in Gothic print as $\mathfrak{p} := h(p)$ etc. We denote by $\tilde{\mathcal{H}}_{pr}(f,x)$ resp. $\tilde{\mathcal{H}}_{pr}(F,\mathfrak{x})$ the equivalence classes of primary points and set $[\langle p\rangle] := [p]$. We consider $h(p) \in \mathbb{R}^2$ as contractible in $\mathbb{R}^2$ and define the Maslov index of $p \in \mathcal{H}(f,x)$ to be $\mu(p) := \mu_h(p) := \mu(h(p),\mathfrak{x})$. It holds $\mu(\varphi(p),\varphi(q)) = \mu(p,q)$. In order to keep track of the actual homotopy class $[p] \in \pi_1(\mathcal{Z},x)$, we modify the boundary operator via

$$\nu_h(\langle p\rangle,\langle q\rangle) := \nu(\langle \mathfrak{p}\rangle,\langle \mathfrak{q}\rangle) := \begin{cases} m(\langle \mathfrak{p}\rangle,\langle \mathfrak{q}\rangle) & \text{if } \quad \emptyset \neq \mathcal{M}(\mathfrak{p},\mathfrak{q}) \ni v,\ 0 \notin \mathrm{Im}(v), \\ 0 & \text{otherwise.} \end{cases}$$

$B_{R_-}(0) \subset \mathbb{R}^2$ corresponds to the hole of the annulus resp. the S^1-direction of the cylinder. The new signs ensure that only immersions between primary points $\mathfrak{p}$, $\mathfrak{q} \in \mathcal{H}_{pr}(F,\mathfrak{x})$ with $[p] = [q] \in \pi_1(\mathcal{Z},x)$ are counted. As in the previous homoclinic Floer homologies, the chain complex is defined via

$$\mathscr{C}_k(f,x,h) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle p\rangle)=k}} \mathbb{Z}\langle p\rangle, \qquad \mathscr{D}\langle p\rangle := \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} \nu_h(\langle p\rangle,\langle q\rangle)\langle q\rangle.$$

The breaking and gluing construction carries over since $B_{R_-}(0) \subset \mathbb{R}^2$ is untouched by F and can therefore be considered invariant under iteration.

Theorem 4 ([Ho2]). *We have $\mathscr{D} \circ \mathscr{D} = 0$ and $\mathscr{H}_*(f,x,h) := \frac{\ker \mathscr{D}_*}{\mathrm{Im}\, \mathscr{D}_{*+1}}$ is called cylinder Floer homology on $\mathcal{Z}$.*

Since the boundary operator only connects points within the same homotopy class on the cylinder we obtain

Corollary 5 ([Ho2]). *$\mathscr{C}_*(f,x,h)$ and $\mathscr{H}_*(f,x,h)$ split into a direct sum w.r.t. the homotopy classes in $\pi_1(\mathcal{Z},x)$:*

$$\mathscr{C}_*(f,x,h) = \mathscr{C}_*(f,x,h,[\cdot]=1) \oplus \mathscr{C}_*(f,x,h,[\cdot]=0) \oplus \mathscr{C}_*(f,x,h,[\cdot]=-1),$$
$$\mathscr{H}_*(f,x,h) = \mathscr{H}_*(f,x,h,[\cdot]=1) \oplus \mathscr{H}_*(f,x,h,[\cdot]=0) \oplus \mathscr{H}_*(f,x,h,[\cdot]=-1).$$

Clearly, cylinder Floer homology can also be defined using an orientation reversing diffeomorphism h. In fact, cylinder Floer homology only depends on the orientation of the underlying diffeomorphism, i.e. there are two well-defined cylinder Floer homologies $\mathscr{H}_*(f,x,+)$ and $\mathscr{H}_*(f,x,-)$.

2.4 Chaotic Floer homology

Birkhoff [Bi] proved in 1935 that there is an intricate amount of high-periodic points near a homoclinic one. This phenomenon is nowadays explained by Smale's horseshoe and symbolic dynamics. For Hamiltonian systems, Conley's conjecture claims the existence of infinitely many periodic points on certain symplectic manifolds. It has been established for certain classes of manifolds ([Gi], [GiG], [Hi]).

Motivated by those observations, we want to include information about the periodic points 'near' homoclinic ones in homoclinic Floer homology. Assume $\varphi \in \mathrm{Symp}(M)$ and $x \in \mathrm{Fix}(\varphi)$ hyperbolic. Given $n \in \mathbb{Z}$, we assign new signs to primary points $p,\, q \in \mathcal{H}(\varphi^n, x)$ via

$$\nu_n(p,q) := \begin{cases} m(p,q) & \text{if } \emptyset \neq \mathcal{M}(p,q) \ni v,\ \mathrm{Fix}(\varphi^n) \cap \mathrm{Im}(v) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\nu_n(\langle p \rangle, \langle q \rangle) := \sum_{l \in \mathbb{Z}} \nu_n(p, q^l)$ and define the chain complexes $\mathcal{C}_*^{(n)} := C_*(x, \varphi^n; \mathbb{Z})$. The boundary operators are defined on generators via

$$\mathcal{D}^{(n)} : \mathcal{C}_*^{(n)} \to \mathcal{C}_{*-1}^{(n)}, \qquad \mathcal{D}^{(n)}(\langle p \rangle) := \sum_{\substack{\langle q \rangle \in \hat{\mathcal{H}}_{pr}(\varphi^n) \\ \nu_n(\langle p \rangle, \langle q \rangle) = 1}} \nu_n(\langle p \rangle, \langle q \rangle) \langle q \rangle$$

and are extended to $\mathcal{C}_*^{(n)}$ by linearity. Excluding immersions with fixed points in their ranges is compatible with the breaking and gluing procedure. Thus the construction of primary Floer homology carries over and we obtain

Theorem 6 ([Ho1])**.** *$\mathcal{D}^{(n)} \circ \mathcal{D}^{(n)} = 0$ and $\hat{H}_*(x, \varphi^n) := \frac{\ker \mathcal{D}_*^{(n)}}{\mathrm{Im}\, \mathcal{D}_{*+1}^{(n)}}$ is called chaotic Floer homology.*

The main importance of chaotic Floer homology lies in its change under iteration of φ, i.e. the dynamics of $n \mapsto \hat{H}_*(x, \varphi^n)$. For example, we can assign a symplectic zeta function to this sequence via

$$\zeta_{x,\varphi}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\chi(H_*^{\mathrm{Fix}}(x, \varphi^n))}{n} z^n \right)$$

where $\chi(H_*^{\mathrm{Fix}}(x, \varphi^n))$ denotes the Euler characteristic of $H_*^{\mathrm{Fix}}(x, \varphi^n)$. Zeta functions have been studied a lot in number theory, algebraic geometry and dynamical systems. An overview can be found in Fel'shtyn [Fe1, Fe2]. We are investigating the properties of this zeta function and plan to link them to the growth behaviour of symplectomorphisms.

3 Action filtration and growth

In this section, we measure symplectic properties of homoclinic points of symplectomorphisms. Let $(M, \omega = d\alpha)$ stand for the exact manifolds $(\mathbb{R}^2, \omega)$ resp. $(\mathcal{Z}, \omega)$ if not stated otherwise. We assume $f \in \mathrm{Symp}(\mathbb{R}^2)$ resp. $f \in \mathrm{Ham}^c(\mathcal{Z})$ with $x \in \mathrm{Fix}(f)$ hyperbolic.

For $p \in \mathcal{H}$ and $i \in \{s, u\}$, fix a smooth parametrization $\gamma_p^i : [0,1] \to [x,p]_i$ with $\gamma_p^i(0) = x$ and $\gamma_p^i(1) = p$. Define the *symplectic action* of $p \in \mathcal{H}$ via

$$\mathcal{A}(p) := \int\limits_{\bar{\gamma}_p^u \# \gamma_p^s} \alpha$$

with $\bar{\gamma}_p^u(\tau) := \gamma_p^u(1-\tau)$ and where # stands for the concatenation of paths. If $M = \mathbb{R}^2$, Stokes theorem yields $\mathcal{A}(p) = \int_{\bar{\gamma}_p^u \# \gamma_p^s} \alpha = \int_{G(x,p)} \omega$ which is the (signed) symplectic area of the *resonance domain* $G(x,p)$ of p. Back to $M \in \{\mathbb{R}^2, \mathcal{Z}\}$, the *relative action* of p, $q \in \mathcal{H}$ is given by $\mathcal{A}(p,q) := \mathcal{A}(p) - \mathcal{A}(q)$. Since immersions in $\mathcal{M}(p,q)$ are orientation preserving Stokes' theorem yields for such p and q

$$\mathcal{A}(p,q) = \int\limits_v \omega > 0, \quad \text{implying} \quad \mathcal{A}(p) > \mathcal{A}(q). \tag{7}$$

In particular, $\mathcal{A}(p,q)$ is the symplectic area enclosed by $[p,q]_s$ and $[p,q]_u$. $f \in \mathrm{Ham}^c(\mathcal{Z})$ is characterized by $f^*\alpha - \alpha = d\tilde{H}$ for a smooth function $\tilde{H} : \mathcal{Z} \to \mathbb{R}$. This allows us to conclude $\mathcal{A}(p) = \mathcal{A}(f^n(p))$ and $\mathcal{A}(p,q) = \mathcal{A}(f^n(p), f^n(q))$ for all $n \in \mathbb{Z}$ and $f \in \mathrm{Ham}^c(\mathcal{Z})$. The action is also invariant for $f \in \mathrm{Symp}(\mathbb{R}^2)$.

In classical Floer theory, filtration by the action has been used successfully to define and interprete symplectic invariants, see e.g. Schwarz [Sch]. We demonstrate the construction with (positive) cylinder Floer homology $\mathscr{H}_*(f,x,+)$. Let $a \in \mathbb{R}$ and define the *filtered Floer groups* via

$$\mathscr{C}_k^a := \mathscr{C}_k^a(f,x,+) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle p \rangle) = k \\ \mathcal{A}(\langle p \rangle) \le a}} \mathbb{Z}\langle p \rangle.$$

If p, $q \in \mathcal{H}_{pr}(f,x)$ with $\mu(p,q) = 1$ and $\widehat{\mathcal{M}}(p,q) \neq \emptyset$ it follows from (7) that $\mathcal{A}(p) > \mathcal{A}(q)$. Thus the boundary operator $\mathscr{D}$ restricts to $\mathscr{C}_k^a$ and $(\mathscr{C}_*^a, \mathscr{D})$ is a subcomplex of $(\mathscr{C}_*(f,x,+), \mathscr{D})$. For $a < b$, we define $\mathscr{C}_*^{]a,b]} := \mathscr{C}_*^b / \mathscr{C}_*^a$. We identify $\mathscr{C}_*^\infty = \mathscr{C}_*(f,x,+)$ and $\mathscr{C}_*^{]-\infty,a]} = \mathscr{C}_*^a$ and define the *filtered Floer groups* $\mathscr{H}_*^{]a,b]} := \mathscr{H}_*^{]a,b]}(f,x,+)$ as the homology of $\mathscr{C}_*^{]a,b]}$. If $a < a_{min} := \min\{\mathcal{A}(p) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}\}$ and $b > a_{max} := \max\{\mathcal{A}(p) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}\}$ then

$\mathscr{H}_*^{]-\infty,a]} = 0$ and $\mathscr{H}_*^{]b,\infty]} = 0$ such that the homology is concentrated in the interval $]a_{min} - \varepsilon, a_{max}]$ for $\varepsilon > 0$.

In classical Floer theory, the action and (mean) index of periodic Hamiltonian orbits grow linearly under iteration of the underlying symplectomorphism. This fact was used e.g. by Ginzburg & Gürel [GiG] to prove generalizations of the Conley Conjecture ('There are infinitely many periodic Hamiltonian orbits') on certain classes of symplectic manifolds. Polterovich [Pol1] studied growth behaviour of symplectomorphisms under iteration using results by Schwarz [Sch] about the action spectrum. In a subsequent paper, Polterovich [Pol2] used these results to establish a Hamiltonian version of the Zimmer program.
Denote by $\operatorname{Spec}(\varphi, x) := \{\mathcal{A}(\langle p\rangle) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ the homoclinic action spectrum of φ at x and set $\operatorname{gap}(\varphi, x) := \min\{|\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle)| \mid \langle p\rangle, \langle q\rangle \in \tilde{\mathcal{H}}_{pr}\}$.

Theorem 8 ([Ho2]). *Let $\varphi \in \operatorname{Symp}(\mathbb{R}^2)$ resp. $\varphi \in \operatorname{Ham}^c(\mathcal{Z})$. Let $b \in \operatorname{Spec}(\varphi, x)$ and $0 < \varepsilon \leq \frac{1}{2}\operatorname{gap}(\varphi, x)$. Then we obtain*

$$\operatorname{rk} H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = n \operatorname{rk} H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x),$$
$$\operatorname{rk} \mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = n \operatorname{rk} \mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x).$$

Thus filtered homology distinguishes between φ and its iterate φ^n and we have linear growth of the rank. We will see in the next section that the rank is not the only quantity related to homoclinic Floer homology which has linear growth under iteration.

4 Flux, turnstiles and transport

MacKay & Meiss & Percival [MMP] are interested in the long-term behaviour of Hamiltonian systems. A crucial notion in their work is the so-called (absolute) flux: Let c be a simply closed curve in $(\mathbb{R}^2, \omega)$ and let $\varphi \in \operatorname{Symp}(\mathbb{R}^2)$ be W-orientation preserving. Denote by $\operatorname{Int}(c)$ the interior of c and by $\operatorname{Ext}(c)$ its exterior. We define

$$\mathcal{F}lux_\varphi(c) := \operatorname{vol}_\omega(\varphi(\operatorname{Int}(c)) \cap \operatorname{Ext}(c)) = \operatorname{vol}_\omega(\operatorname{Int}(\varphi(c)) \cap \operatorname{Ext}(c))$$

to be the *absolute flux* of φ through c. If φ is W-orientation reversing we set $\mathcal{F}lux_\varphi := \mathcal{F}lux_{\varphi^2}$. Now let c be a curve on $(\mathcal{Z}, \omega)$ with $[c] \in \{\pm 1\} \subset \pi_1(\mathcal{Z})$ without self-intersections and let $\varphi \in \operatorname{Ham}^c(\mathcal{Z})$. The range of c cuts $\mathcal{Z}$ into two connected components. Denote one of them by $\mathcal{Z}_c$. We define the *absolute flux* through c as

$$\mathcal{F}lux_\varphi(c) := \operatorname{vol}_\omega(\mathcal{Z}_{\varphi(c)} \backslash \mathcal{Z}_c).$$

If c is a contractible curve on the cylinder we define $\mathcal{F}lux_\varphi(c)$ similarly as in the plane. We usually call the absolute flux briefly *flux*. Note that the absolute flux differs from the flux homomorphism usually used in symplectic geometry (cf. [McS]). Roughly speaking, the latter considers the difference between $\varphi(\mathrm{Int}(c))\cap\mathrm{Ext}(c)$ and $\varphi(\mathrm{Ext}(c))\cap\mathrm{Int}(c)$. Given a primary orbit $\langle p\rangle$ on the cylinder resp. $\mathbb{R}^2$, we set

$$\mathcal{F}lux_\varphi(\langle p\rangle) := \mathcal{F}lux_\varphi(c_p).$$

$\mathcal{F}lux_\varphi(c)$ measures how much of a barrier the curve c is for the transport: If c is invariant under φ, the flux through c is zero. In that case, c is a complete barrier for the transport of points by φ. Now we investigate certain curves which form a partial barrier and where the 'outlet' only happens along a small part of c: Let p be a homoclinic point. It holds $[x,\varphi(p)]_s \subset [x,p]_s$ and $[x,\varphi(p)]_u \supset [x,p]_u$. Thus the ranges of the curves c_p and $c_{\varphi(p)}$ coincide except in the segments $[p,\varphi(p)]_u$ and $[p,\varphi(p)]_s$. MacKay & Meiss & Percival [MMP] call the resulting picture a turnstile. The name is motivated by its behaviour, cf. Figure 2 (a): One wing sweeps points out of the interior (shaded region), the other one sweeps points in. Let us refine this notion in the following.

φ is called *x-simple* if each pair of intersecting branches contains exactly two primary orbits. If not stated otherwise, assume from now on $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ resp. $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ to be x-simple. Consider primary points $\langle p\rangle$ and $\langle q\rangle$ in a chosen pair of intersecting branches and assume $\{q\} =]p,\varphi(p)[_s \cap]p,\varphi(p)[_u$. The resulting picture is called a *true turnstile with pivot q and frame p and* $\varphi(p)$. The regions enclosed by $[p,\varphi(p)]_s \cup [p,\varphi(p)]_u$ are called the *wings* of the turnstile. An example is sketched in Figure 2 (a). The flux through $\langle p\rangle$ is the area of the shaded region.
If φ is x-simple, but if we assume $\#(]p,\varphi(p)[_s \cap]p,\varphi(p)[_u) = 3$ the resulting picture is called an *overtwisted turnstile* with frame p and $\varphi(p)$ and pivot q. An example is sketched in Figure 2 (b) and the flux through $\langle p\rangle$ is the area of the shaded region.
If we assume a pair of intersecting branches to have k primary orbits $\langle p_1\rangle,\ldots,\langle p_k\rangle$ with $]p_1,\varphi(p_1)[_s \cap]p_1,\varphi(p_1)[_u = \{p_2,\ldots,p_k\}$ we call this picture a *k-generalized turnstile* with frame p_1 and $\varphi(p_1)$ and pivots $p_2,\ldots,p_k$. Note that the *wings* between p_i and p_{i+1} not always have the same symplectic volume. An example is sketched in Figure 2 (c) and the flux through $\langle p_1\rangle$ is the area of the shaded regions.

Remark 9. *Overtwisted turnstiles correspond to mixed moves with primary-secondary flips in the proof of Remark 3, k-generalized turnstiles correspond to primary moves.*

The following statement explains the absolute flux of a primary point in terms of the related turnstile. Denote by c_{pq} a curve which runs from p through $[p,q]_u$ to q and then through $[p,q]_s$ back to p. The wing enclosed by $c_{p,q}$ is called $G(p,q)$.

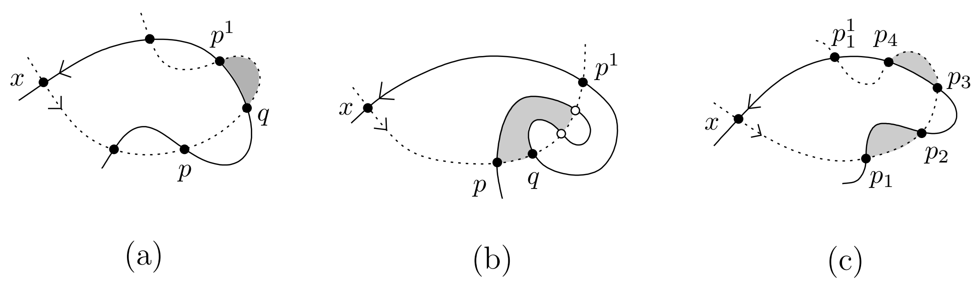

Figure 2: (a) True turnstile, (b) Overtwisted turnstile, (c) Generalized turnstile

Lemma 10 ([Ho2])**.** *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$. Let p be a primary point and pivot of a true turnstile with frame q and $\varphi(q)$. Then we have*

$$\mathcal{F}lux_\varphi(\langle q\rangle) = \Big| \int_{c_{pq}} \alpha \Big| = \Big| \int_{G(p,q)} \omega \Big| = \Big| \int_{G(p,\varphi(q))} \omega \Big| = \Big| \int_{c_{p\varphi(q)}} \alpha \Big|$$

and in particular

$$\mathcal{F}lux_\varphi(\langle p\rangle) = \mathcal{F}lux_\varphi(\langle q\rangle).$$

For a k-generalized turnstile with frame p_1 and $\varphi(p_1)$ and pivots $p_2, \ldots, p_k$ holds

$$\mathcal{F}lux_\varphi(\langle p_1\rangle) = \sum_{i=1}^{\frac{k}{2}} \mid \mathcal{A}(\langle p_{2i-1}\rangle, \langle p_{2i}\rangle) \mid = \sum_{i=1}^{\frac{k}{2}} \mid \mathcal{A}(\langle p_{2i}\rangle, \langle p_{2i+1}\rangle) \mid .$$

For overtwisted turnstiles with frame p and $\varphi(p)$ and pivot q holds

$$\mid \mathcal{A}(\langle p\rangle, \langle q\rangle) \mid > \mathcal{F}lux_\varphi(\langle p\rangle).$$

There are also combinations of generalized and overtwisted turnstiles, but we are mainly interested in a special case of generalized turnstiles. Consider a true turnstile with frame p and $\varphi(p)$ and pivot q. If we iterate φ n times the two primary orbits $\langle p\rangle$ and $\langle q\rangle$ split into $2n$ classes $\langle p^0\rangle, \ldots, \langle p^{n-1}\rangle$ and $\langle q^0\rangle, \ldots, \langle q^{n-1}\rangle$. In particular, we have a $2n$-generalized turnstile with frame p^0 and p^n and pivots $q^0, p^1, \ldots, q^{n-1}$.

Corollary 11 ([Ho2])**.** *For $0 \le i \le n-1$ holds under the above assumptions*

$$\mathcal{F}lux_{\varphi^n}(\langle p^i\rangle) = \sum_{l=0}^{n-1} \mid \mathcal{A}(\langle p^l\rangle, \langle q^{l+1}\rangle) \mid = \sum_{l=0}^{n-1} \mid \mathcal{A}(\langle q^l\rangle, \langle p^l\rangle) \mid$$

and in particular

$$\mathcal{F}lux_{\varphi^n}(\langle p^i\rangle) = n\, \mathcal{F}lux_\varphi(\langle p\rangle).$$

Thus homoclinic orbits forms a partial barrier for the transport of φ with associated turnstile as only in- and outlet. Moreover, the flux grows linearly in n if n is the number of iterations of φ.

Let $\langle p\rangle$ and $\langle q\rangle$ be in the same pair of intersecting branches. Assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$ and $p \in \,]q, \varphi(q)[_s \cap \,]q, \varphi(q)[_u$. Then $\langle p\rangle$ and $\langle q\rangle$ give rise to two distinct (families of) turnstiles, more precisely p is the pivot of a turnstile with frame q and $\varphi(q)$ and q is the pivot of a turnstile with frame $\varphi^{-1}(p)$ and p.

Proposition 12 ([Ho2])**.** *The (true or overtwisted) turnstile with pivot p shows up in the boundary operator ∂ (and also $\mathscr{D}$) via*

$$\partial\langle p\rangle = \langle q\rangle - \langle q\rangle + \sum_{\substack{\langle q\rangle\neq\langle\tilde q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle\tilde q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle\tilde q\rangle)\langle\tilde q\rangle = \sum_{\substack{\langle q\rangle\neq\langle\tilde q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle\tilde q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle\tilde q\rangle)\langle\tilde q\rangle.$$

Thus such turnstiles are annihilated by the boundary operator. If $\mathcal{M}(p,\tilde q) = \emptyset$ for all $\langle q\rangle \neq \langle\tilde q\rangle \in \tilde{\mathcal{H}}_{pr}$ the turnstile with pivot p lies in the kernel of the boundary operator, i.e. the pivot is a cycle.

A *monotone twist map* on a cylinder or annulus with coordinates $(s,t) \in \mathbb{R}\times S^1$ is a volume preserving map f with $f(s,t) = (\tilde s,\tilde t)$ satisfying $\frac{\partial\tilde t}{\partial s} > 0$ for all s and t. Mather [Ma] studies the (non)existence of invariant circles for monotone twist maps using an action functional W and the calculus of variations. He denoted the difference in action between an action maximizing orbit and its associated minimax orbit by $\triangle W$. MacKay & Meiss & Percival [MMP] showed the following relation between turnstiles and $\triangle W$:

Theorem 13 ([MMP])**.** *Let f be a Hamiltonian diffeomorphism on the cylinder which is in addition a monotone twist map. Then, for the periodic, quasiperiodic and heteroclinic orbits of f, holds: Mather's difference in action $\triangle W$ between a maximizing orbit and the associated minimax orbit coincides with the area of one wing of the turnstile, i.e. the flux through the associated curve.*

Now we unite the notions of orientation preserving immersions, (relative) symplectic action of homoclinic points, wings of turnstiles, flux and Mather's difference in action $\triangle W$.

Theorem 14 ([Ho2])**.** *Let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be an x-simple, monotone twist map. Consider a true turnstile with frame p and $\varphi(p)$ and pivot q and assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $v \in \mathcal{M}(p,q) \neq \emptyset$ and*

$$\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle p\rangle,\langle q\rangle) = \int_v \omega = \mathcal{F}lux_\varphi(\langle p\rangle) = \triangle W_{p,q}.$$

Therefore the flux and $\triangle W$ are meaningful quantities for the action filtration of homoclinic Floer homology. Thus everything which is formulated in terms of the symplectic action spectrum can be interpreted in terms of the flux and $\triangle W$. This means that the algebraic notion of homology has a dynamical interpretation and measures dynamical quantities.

5 Table of homoclinic Floer homologies

Let us summarize the main properties of the homoclinic Floer homologies:

	Primary FH	**Semi-pr. FH**	**Cylinder FH**	**Chaotic FH**
Generator	primary, contractible	semi-primary, contractible	‘primary’, noncontr.	primary, contractible
Growth φ vs. φ^n	Filtered FH: linear growth	examples of linear growth	Filtered FH: linear growth	‘subsequence’ $k \mapsto \hat{H}_*(x, (\varphi^n)^k)$
Transport	turnstiles, flux = action	turnstiles not involved: cf. Remark 15	turnstiles, flux = action = $\triangle W$	turnstiles not involved: cf. Remark 15
Number theory				ζ-function

There are different reasons why turnstiles do not play the same role in semi-primary and chaotic Floer homology as they do in primary and cylinder Floer homology:

Remark 15. *1) Consider the example associated to Figure 1 in case* $]p,q[_s \cap\,]q,p^{-1}[_u \neq \emptyset$. *Then none of the pivots of the turnstile with frame* p *and* p^{-1} *is a generator.*

2) Chaotic Floer homology does not count immersions between primary points if there is a certain fixed point in their ranges. Thus it may happen, that one (or maybe both) wings of a turnstile do not show up in the boundary operator in contrast to Proposition 12.

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