

VERTICAL ALMOST TORIC SYSTEMS

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ABSTRACT. This paper introduces vertical almost-toric systems, a generalization of semi-toric systems (introduced by Vũ Ngọc and classified by Pelayo and Vũ Ngọc), that provides the language to develop surgeries on almost-toric systems in dimension 4. We prove that vertical almost-toric systems are natural building blocks of almost-toric systems. Moreover, we show that they enjoy many of the properties that their semi-toric counterparts do.

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1. INTRODUCTION

A driving problem in Hamiltonian mechanics and symplectic geometry is to classify integrable systems up to a suitable notion of equivalence. An *integrable system* is a triple (M, ω, Φ) , where (M, ω) is a $2n$ -dimensional symplectic manifold and $\Phi : (M, \omega) \rightarrow \mathbb{R}^n$ is a smooth map whose components are in involution and functionally independent almost everywhere on M . This paper introduces vertical almost-toric systems, a category of integrable systems on 4-dimensional symplectic manifolds that generalizes toric and semi-toric systems and lays the foundation for studying almost-toric systems. A key feature of vertical almost-toric systems is that they behave well under a process of taking appropriate subsystems, a fact that facilitates development of precise language to define, for vertical almost-toric systems, integrable surgeries in the sense of Zung [51].

In general, classification of integrable systems becomes a tractable problem only under assumptions that restrict the topology of fibers of the system. Intuitively, the greatest challenge arises from non-compactness of the group action. Accordingly, full classifications were first established for toric systems in which the \mathbb{R}^n action is replaced by an \mathbb{T}^n -action. Building upon the foundational results of Atiyah [1] and Guillemin & Sternberg [14], Delzant [7] classified toric systems on closed manifolds. More recently, Karshon & Lerman [23] have extended Delzant's classification to non-compact toric manifolds, relying upon the local normal forms of Guillemin & Sternberg [15] and Marle [26].

Once one allows non-compactness of the group acting on the total space, complexity of both the fibers and of the total space can be reasonably controlled by restricting the singularities of the moment map. Symington [40] and Vũ Ngọc [46] have proposed a notion of almost-toric systems on 4-dimensional symplectic manifolds that includes toric systems but also allows for so-called focus-focus fibers, which can be thought of as the Lagrangian analog of the nodal fibers that arise in Lefschetz fibrations. The diffeomorphism types of closed manifolds that support an almost-toric system has been determined (cf. Leung & Symington [25]), and recently, almost-toric systems have proved to be of independent interest in symplectic topology (cf. Vianna [41, 42, 43]).

While the classification problem of almost-toric systems has not been settled, even in the compact case, an important subclass of almost-toric systems has been completely understood: Pelayo & Vũ Ngọc

([34, 35]) have classified semi-toric systems, which were initially introduced by Vũ Ngọc in [46]. An integrable system $(M, \omega, \Phi = (J, H))$ is semi-toric if it is almost-toric and if J is a proper moment map of an effective Hamiltonian S^1 -action. Semi-toric systems, whose total spaces may be non-compact, share many fundamental properties with closed symplectic toric manifolds, like connectedness of the fibers of the moment map, but their classification is significantly more involved as the presence of focus-focus fibers introduces more data (see Pelayo & Vũ Ngọc [34]). While semi-toric systems appear naturally both in symplectic topology and in (quantum) Hamiltonian mechanics (cf. Eliashberg & Polterovich, Le Floch & Pelayo & Vũ Ngọc, Pelayo & Vũ Ngọc [11, 24, 36]), the properness condition excludes some familiar almost-toric integrable systems, such as the spherical pendulum (cf. Duistermaat [9]). For this reason, Pelayo & Ratiu & Vũ Ngọc ([32, 33]) introduce a family of almost-toric systems that share some of the main properties of semi-toric systems, like connectedness of the fibers of the moment map, while allowing enough freedom to include examples such as the spherical pendulum. These systems are called generalized semi-toric. In such an almost-toric system $(M, \omega, \Phi = (J, H))$ the moment map Φ is proper and J is the moment map of an effective Hamiltonian S^1 -action that satisfies some constraints on the sets of singular points and values (see Definition 1.3 in Pelayo & Ratiu & Vũ Ngọc [33]).

Vertical almost-toric systems, defined and studied in this paper, can be viewed as an extension of generalized semi-toric systems to the non-compact setting but were defined with different purposes in mind. For instance, vertical almost-toric systems provide a category that can accommodate non-compact systems that are convenient local models and building blocks for almost-toric systems. The essential difference between vertical almost-toric systems and generalized semi-toric systems is that the moment maps of the former are merely required to be proper onto their image while the moment map of a generalized semi-toric system must be proper. (The fibers of the moment maps of both types of systems are connected; for vertical almost-toric systems this connectedness is imposed by definition, while for semi-toric and generalized semi-toric systems it can be proven as a result of the other properties.)

The definition of a vertical almost-toric system is crafted so that appropriately chosen subsystems are again vertical almost-toric. Specifically, given an open, connected subset U of the moment map image of a vertical-almost toric system, if the intersection of U with any vertical line is either empty or connected then restricting the moment map to the preimage of U yields a vertical almost-toric system. In fact, such a subsystem of a (generalized) semi-toric system is vertical almost-toric.

In a forthcoming paper, [20], the process of taking such subsystems is an essential ingredient in the definition of surgeries in the category of vertical almost-toric systems. Those surgeries are going to be applied to the determination of which Hamiltonian S^1 -spaces underlie compact semi-toric systems ([21], forthcoming), thus completing the work started in Hohloch & Sabatini & Sepe [19]. The language of vertical almost-toric systems also allows one to have a more conceptual understanding of the local-to-global arguments in the classification of semi-toric systems (cf. Pelayo & Vũ Ngọc [34, 35]); this is also going to be explored in a separate paper.

The main results of this paper are as follows:

- (A) A connected component of a fiber of an almost-toric system admits an open neighborhood that is isomorphic to a vertical almost-toric system (cf. Proposition 4.9);
- (B) Using terminology analogous to that for (generalized) semi-toric systems (cf. Pelayo & Ratiu & Vũ Ngọc [33]), vertical almost-toric systems possess *cartographic homeomorphisms* (see Theorem 4.24). These are homeomorphisms of the moment map image onto subsets of \mathbb{R}^2 that encode the induced *\mathbb{Z} -affine structures* (cf. Section 2.3). In particular, the monodromy introduced by focus-focus fibers is encoded via *vertical cuts*.
- (C) The space of all cartographic homeomorphisms of a given vertical almost-toric system is described (cf. Theorem 4.36), generalizing the analogous result for semi-toric system (cf. Vũ Ngọc [46]). This can be used to construct an invariant of the isomorphism class of a vertical almost-toric system analogous to the *semi-toric polygon* of Pelayo & Vũ Ngọc [34, Definition 4.5].
- (D) Given a vertical almost-toric system $(M, \omega, \Phi = (J, H))$ and a cartographic homeomorphism $f : \Phi(M) \rightarrow \mathbb{R}^2$, the composition $f \circ \Phi$ may lack the smoothness required of a moment map. We provide a method for smoothing $f \circ \Phi$ to obtain a vertical almost-toric system isomorphic to $(M, \omega, \Phi = (J, H))$ whose moment map image equals $f(\Phi(M))$ on the complement of arbitrarily small neighborhoods of the cuts used to define f (cf. Theorem 4.48).

Result (A) establishes vertical almost-toric systems as building blocks for almost-toric systems. While it is probably known to experts in the area, we could not find a complete, self-contained proof of this fact and decided to include it, along with proofs of basic topological facts leading up to it (cf. Section 3.2).

Results (B) and (C) are not surprising in light of the work in Pelayo & Ratiu & Vũ Ngọc, Vũ Ngọc [33, 46] and, in fact, their proofs are

heavily influenced by that work. However, we provide explicit proofs for several reasons:

- There is ambiguity in the literature with regards to the notion of isomorphism in the category of semi-toric systems. It seems that, in spite of what is stated in Pelayo & Vũ Ngọc [34, 35], isomorphisms of semi-toric systems must induce orientation-preserving diffeomorphisms on their moment map images in order for the semi-toric invariants in Pelayo & Vũ Ngọc [34, 35] to be well-defined (cf. [37]). The restriction to orientation-preserving homeomorphisms has been made explicit in the definition of generalized semi-toric systems (cf. Pelayo & Ratiu & Vũ Ngọc [33, Definition 2.4]). Deeming this restriction unnecessary, we allow orientation-reversing cartographic homeomorphisms, and hence provide a proof that accommodates such homeomorphisms.
- Our alternative proof of the existence of cartographic homeomorphisms in the case in which the defining cuts disconnect the moment map image allows us to avoid the ‘homotopy argument’ of Pelayo & Ratiu & Vũ Ngọc [33, Step 5 of the proof of Theorem B].
- The description of the set of cartographic homeomorphisms of a vertical almost-toric system is analogous to that of a semi-toric system. However, the potential for infinitely many focus-focus points in a vertical almost-toric system gives a richer behavior, as can be seen by comparing Section 4.4 with Vũ Ngọc [46, Section 4].

Result (D), which ensures the existence of η -cartographic systems in the isomorphism class of a vertical almost-toric system, is tailored for applications. In particular, it plays an important role in defining surgeries of vertical almost-toric systems ([20], forthcoming). Also, it allows one to make precise the notion that the image of a cartographic homeomorphism is a limit of moment map images (cf. Proposition 4.52).

Structure of the paper. Section 2 defines and explains notions that we use throughout the paper, while also establishing notation. While the section should serve as a self-contained primer to guide readers unfamiliar with the subject through this paper (and the forthcoming [20]), it may be of interest to experts in the field as well, for a few ideas which do not appear in many other places. For instance, we introduce the notion of a *faithful* integrable system, which, intuitively speaking, is one whose moment map image reflects the topology of the partition of the total space into the connected components of the fibers of the moment map (cf. Section 2.2). Moreover, in Section 2.7, we elaborate on the notion of *cartographic homeomorphisms* that is

introduced in Pelayo & Ratiu & Vũ Ngọc [33] for generalized semi-toric systems and establish some general properties for these objects. Note that the section should not be taken as an exhaustive reference for either topological or symplectic aspects of integrable systems.

In Section 3, almost-toric systems are defined and their basic properties are explored. In particular, the neighborhood of a connected component of a fiber is described (cf. Section 3.2) and, in preparation for the next section on vertical almost-toric systems, we describe properties of systems that are both faithful and almost-toric (Section 3.3).

Section 4 is the heart of the paper: It contains the definition of vertical almost-toric systems as well as all the main results described above. Section 4.1 contains main result (A) as Proposition 4.9, characterizations of the moment map image of vertical almost-toric systems (Corollary 4.5), and a useful criterion to determine which saturated subsystems of vertical almost-toric systems are vertical almost-toric (Proposition 4.7). Section 4.2 explores the consequences of the presence of the S^1 -action. The existence of cartographic homeomorphisms (Theorem 4.24) is proved in Section 4.3, which also establishes some useful topological properties of the complements of the cuts needed to define cartographic homeomorphisms. Section 4.4 describes the set of cartographic homeomorphisms associated to a given vertical almost-toric system, paying particular attention to the subtleties that arise from allowing infinitely many focus-focus points (see Theorem 4.36). Finally, Section 4.5 proves that, in some sense, cartographic homeomorphisms can be made smooth everywhere by modifying them on arbitrarily small neighborhoods of the defining cuts. This is the content of Theorem 4.48, which can be used to establish the existence of η -cartographic vertical almost-toric systems in any given isomorphism class (Theorem 4.51).

Notation and conventions.

Topological conventions.

- A subset of a topological space is endowed with the subspace topology unless otherwise stated.
- A *pair* of topological spaces (Y, Z) consists of a topological space Y together with a subset $Z \subset Y$ endowed with the relative topology. A topological embedding of pairs of topological spaces $(Y_1, Z_1), (Y_2, Z_2)$ is a topological embedding $\chi : Y_1 \rightarrow Y_2$ that restricts to a topological embedding of Z_1 into Z_2 . A homeomorphism between pairs of topological spaces is a topological embedding onto the target.

- A map $f : X \rightarrow Y$ between topological spaces X and Y is *proper onto its image* if the map $f : X \rightarrow f(X)$ is proper.

Smoothness conventions.

- Following Joyce [22], consider the subspace $[0, +\infty[^n \subset \mathbb{R}^n$ and let M be a topological space. An *n-dimensional smooth atlas with corners* on M is a set $\mathcal{A} := \{(U_i, \chi_i)\}$, where
 - the set $\{U_i\}$ is an open cover of M ;
 - for each i , there is an open set $V_i \subset [0, +\infty[^n$ such that the map $\chi_i : U_i \rightarrow V_i$ is a homeomorphism; and
 - for all i, j with $U_i \cap U_j \neq \emptyset$, the map $\chi_j \circ \chi_i^{-1} : \chi_i(U_i \cap U_j) \rightarrow \chi_j(U_i \cap U_j)$ is a diffeomorphism.

A *smooth manifold with corners* of dimension n is a Hausdorff, paracompact, second countable topological space together with an n -dimensional smooth atlas with corners. A *smooth structure with corners* on a topological space is an equivalence class of smooth atlases with corners, where two atlases are deemed equivalent if their union is again a smooth atlas with corners of a given dimension.

- A smooth atlas with corners is *\mathbb{Z} -affine* (or *integral affine*) if the transition maps $\chi_j \circ \chi_i^{-1} : \chi_i(U_i \cap U_j) \rightarrow \chi_j(U_i \cap U_j)$ of the atlas are of the form

$$x \mapsto Ax + b,$$

for some $(A, b) \in \text{AGL}(n; \mathbb{Z}) = \text{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n$, where n is the dimension of the atlas. A *\mathbb{Z} -affine manifold with corners* of dimension n is a Hausdorff, paracompact, second countable topological space together with an n -dimensional \mathbb{Z} -affine atlas with corners. And a *\mathbb{Z} -affine structure with corners* on a topological space is an equivalence class of \mathbb{Z} -affine atlases with corners, where two atlases are deemed equivalent if their union is again a \mathbb{Z} -affine atlas with corners of a given dimension.

Smooth atlases and *\mathbb{Z} -affine atlases*, without corners, (and the corresponding manifold structures) are defined as above, with the stipulation that the images of the coordinate charts are subsets of \mathbb{R}^n . Also, note that, because the transition maps of are smooth, a \mathbb{Z} -affine atlas (with or without corners) is also a smooth atlas, and hence defines a unique smooth structure.

- Let $A \subset \mathbb{R}^n$ be a subset. A map $f : A \rightarrow \mathbb{R}^m$ is said to be *smooth* if for all $x \in A$ there exists an open neighborhood $U_x \subset \mathbb{R}^n$ of x and a smooth map $f_x : U_x \rightarrow \mathbb{R}^m$ that is a local extension of f ;
- A map $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *smooth embedding* if it is a diffeomorphism onto its image.

- Manifolds are assumed to be without boundary or corners unless otherwise stated.

Boundary conventions. Two types of boundaries of subsets $X \subset \mathbb{R}^n$ are dealt with in this paper whenever X is a smooth manifold with corners embedded in \mathbb{R}^n . The topological boundary, the closure of $X \subset \mathbb{R}^n$ minus its interior, is denoted ∂X . Meanwhile, its boundary as a manifold with corners, $X \cap \partial X$, is denoted $\partial_\infty X$. For instance if $X = \{(x, y) \mid |x| < 1 \text{ and } |y| \leq 1\}$, then

$$\partial X = \{(x, y) \mid |x| = 1, |y| \leq 1 \text{ or } |y| = 1, |x| \leq 1\}$$

and

$$\partial_\infty X = \{(x, y) \mid |y| = 1, |x| < 1\}.$$

Group conventions. Throughout the paper, the identification $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ is used tacitly.

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2. PRIMER ON INTEGRABLE SYSTEMS

This section introduces the basic notions regarding integrable systems that are used throughout the paper. Section 2.1 presents the category of integrable systems and defines the coarsest topological invariant: the *leaf space* of an integrable system (see Definition 2.4). Systems whose leaf spaces can be identified with the moment map images play an important role in this paper and are studied in Section 2.2; we call such systems *faithful*. In Sections 2.3 – 2.6, we endow large subsets of the leaf space of an integrable system with an \mathbb{Z} -*affine structure*. First, following Duistermaat [9], we show how the part of the leaf space corresponding to regular leaves inherits such a structure in Section 2.3. Second, we identify a class of systems that are isomorphic to systems equipped with Hamiltonian torus actions of maximal dimension: these are called *weakly toric*, are related to symplectic toric manifolds and are studied in Section 2.5. Third, Section 2.6 extends the \mathbb{Z} -affine structure on the regular part of the leaf space to include singular leaves that admit a neighborhood supporting a Hamiltonian

torus action of maximal dimension. In Section 2.4, we recall a fundamental property that \mathbb{Z} -affine structures enjoy, namely that they can be *developed*. Finally, following Pelayo & Ratiu & Vũ Ngọc [33], we introduce the notion of *cartographic homeomorphisms*, which, intuitively, can be thought of a way to encode the above \mathbb{Z} -affine structure in a way that is compatible with singular orbits of the system.

2.1. Completely integrable Hamiltonian systems. We begin by introducing the category of integrable systems.

Definition 2.1. For any $n \geq 1$, the *category of completely integrable Hamiltonian systems with n degrees of freedom*, denoted by $\mathcal{IS}(n)$, has objects and isomorphisms as follows:

- Objects: *completely integrable Hamiltonian systems* (M, ω, Φ) where (M, ω) is a $2n$ -dimensional symplectic manifold and

$$\Phi := (H_1, \dots, H_n) : (M, \omega) \rightarrow \mathbb{R}^n$$

a smooth map satisfying

- $\{H_i, H_j\} = 0$ for all $i, j = 1, \dots, n$, where $\{\cdot, \cdot\}$ is the Poisson bracket induced by ω ;
- Φ is a submersion almost everywhere.

Sometimes, for brevity, Φ is referred to as a (completely) integrable (Hamiltonian) system. Its component H_i is called the i^{th} *integral (of motion)*.

- Morphisms: *isomorphisms of integrable systems* (Ψ, ψ) , where, for $i = 1, 2$, (M_i, ω_i, Φ_i) is a completely integrable Hamiltonian system, $\Psi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ is a symplectomorphism, $\psi : \Phi_1(M_1) \rightarrow \Phi_2(M_2)$ is a diffeomorphism, and the following diagram commutes:

$$\begin{array}{ccc} (M_1, \omega_1) & \xrightarrow{\Psi} & (M_2, \omega_2) \\ \Phi_1 \downarrow & & \downarrow \Phi_2 \\ \Phi_1(M_1) & \xrightarrow{\psi} & \Phi_2(M_2) \end{array}$$

Given $\Phi = (H_1, \dots, H_n) : (M, \omega) \rightarrow \mathbb{R}^n$ denote by X_i , for $1 \leq i \leq n$, the *Hamiltonian vector field* associated to H_i , defined implicitly by the equation $\omega(X_i, \cdot) = dH_i$. If the flows of the vector fields X_1, \dots, X_n are complete, then there is a Hamiltonian \mathbb{R}^n -action on (M, ω) , one of whose moment maps is precisely Φ .

Throughout this paper, integrable systems have compact fibers unless otherwise stated, so the above completeness assumption is satisfied and Φ is a moment map for a Hamiltonian \mathbb{R}^n -action (upon identifying

the Lie algebra of \mathbb{R}^n with \mathbb{R}^n). For this reason, it is referred to as the *moment map* of the system.

Many of the integrable systems considered in this paper arise from restricting a given system to a subset.

Definition 2.2. A *subsystem* of an integrable system (M, ω, Φ) is an integrable system $(V, \omega|_V, \Phi|_V)$ where V is an open subset of M . If $V = \Phi^{-1}(U)$ for some open subset U of $\Phi(M)$, the subsystem of (M, ω, Φ) relative to V is also referred to as the *subsystem of (M, ω, Φ) relative to U* .

Remark 2.3 Subsystems of integrable systems with compact fibers need not have compact fibers; moreover, subsystems of integrable systems supporting a Hamiltonian \mathbb{R}^n -action need not support a Hamiltonian \mathbb{R}^n -action.

The most accessible feature of an integrable system is the image of Φ , but, for arbitrary systems, there is no way to extract useful information from that image. However, the leaf space of the system reliably reflects some of the structure.

Definition 2.4. Given an integrable system (M, ω, Φ) ,

- a *leaf* is a connected component of a fiber of Φ ;
- its *leaf space* is the topological space $\mathcal{L} := M / \sim$, where $p \sim q$ if p and q belong to the same leaf, endowed with the quotient topology;
- a subsystem $(V, \omega|_V, \Phi|_V)$ of (M, ω, Φ) is *saturated* if any leaf of (M, ω, Φ) that intersects V is contained in V .

Remark 2.5 Given an integrable system (M, ω, Φ) and an open subset $U \subset \Phi(M)$, the leaf space of the subsystem relative to U is naturally included in the leaf space of (M, ω, Φ) .

Given an integrable system (M, ω, Φ) with leaf space \mathcal{L} , the moment map Φ factors through \mathcal{L} , giving rise to the quotient map $q : M \rightarrow \mathcal{L}$ and inducing a continuous map $\pi : \mathcal{L} \rightarrow \Phi(M)$ that makes the following diagram commute:

$$\begin{array}{ccc}
 & (M, \omega) & \\
 q \swarrow & & \searrow \Phi \\
 \mathcal{L} & \xrightarrow{\pi} & B := \Phi(M) \subset \mathbb{R}^n.
 \end{array}$$

Remark 2.6 Isomorphic integrable systems possess homeomorphic leaf spaces.

Remark 2.7 The sets of connected components of the leaf space of an integrable system and of connected components of its total space are in bijection.

A natural way to enhance the topological data encoded in the leaf space of an integrable system is to identify the singular leaves of the system.

Definition 2.8. Let (M, ω, Φ) be an integrable system with n degrees of freedom with leaf space \mathcal{L} .

- A point $p \in M$ is *singular* if $\text{rk } D_p\Phi < n$. Otherwise it is *regular*.
- A leaf of Φ is *singular* if it contains a singular point. Otherwise it is *regular*.
- The subset $\mathcal{L}_{\text{sing}} \subset \mathcal{L}$ consisting of the image of singular leaves of Φ is said to be the *singular part* of \mathcal{L} , while its complement \mathcal{L}_{reg} is said to be the *regular part*.

Definition 2.9. The pair $(\mathcal{L}, \mathcal{L}_{\text{reg}})$ associated to an integrable system (M, ω, Φ) is called the pair of *leaf and regular leaf spaces* of the system.

The above association descends to isomorphism classes of systems and behaves well with respect to saturated subsystems:

Remark 2.10.

- Isomorphic integrable systems possess homeomorphic pairs of leaf and regular leaf spaces.
- The pair of leaf and regular leaf spaces of saturated subsystems of (M, ω, Φ) naturally embed in the pair of leaf and regular leaf spaces of (M, ω, Φ) .

2.2. Faithful integrable systems. In light of Remark 2.6, it is helpful to distinguish those cases in which a moment map image at least carries the topological information of the leaf space.

Definition 2.11. An integrable system (M, ω, Φ) with leaf space \mathcal{L} is said to be *faithful* if the induced map $\pi : \mathcal{L} \rightarrow B = \Phi(M)$ is a homeomorphism. Here, $B \subset \mathbb{R}^n$ is equipped with the subset topology.

Faithful integrable systems form a full subcategory of the category of all integrable systems.

Remark 2.12. If an integrable system (M, ω, Φ) is faithful, then so is every integrable system isomorphic to it, and every saturated subsystem. In particular, by Remark 2.7, if $U \subset \Phi(M)$ is open, the set of connected components of U is in bijective correspondence with the set of connected components of the total space of the subsystem relative to U .

Remark 2.13. Because the leaf space of a faithful integrable system is homeomorphic to a subset of \mathbb{R}^n , such a leaf space is second countable and Hausdorff.

Example 2.14 illustrates two ways in which a failure of faithfulness can disrupt the relationship between the moment map image and the topology of the total space – first if the fibers of the moment map need not be connected, and second if the moment map need not be proper onto its image.

Example 2.14 Let $A \subset \mathbb{R}^2$ be the closed annulus centered at the origin with inner and outer radii 1 and e . Let (M, ω, Φ) be the toric system (cf. Definition 2.26) that underlies the compact symplectic toric manifold whose moment map image is the rectangle $R := [0, 1] \times [0, 2\pi] \subset \mathbb{R}^2$ and define $g_k : R \rightarrow A \subset \mathbb{R}^2$ by $g_k(x, y) = (e^x \cos(ky), e^x \sin(ky))$ with $k \in \mathbb{N}$.

- 1) For any $k \in \mathbb{N}$, the integrable system $(M, \omega, g_k \circ \Phi)$ has fibers that are the disjoint union of
 - k tori or
 - 2 circles and $k-1$ tori or
 - 2 points and $k-1$ circles.
- 2) Let (M', ω', Φ') be the subsystem of (M, ω, Φ) relative to the subset $[0, 1] \times [0, 2\pi[$. Then the integrable system $(M', \omega', g_1 \circ \Phi')$ has the same (connected) fibers as (M', ω', Φ') , but unlike (M', ω', Φ') – or any toric system defined on a compact manifold – has circle orbits in the preimage of some points on the interior of the moment map image A , cf. Figure 2.1. Notice however that, unlike Φ' , the map $g_1 \circ \Phi'$ is not proper onto its image: the preimage of the compact set A is not compact.

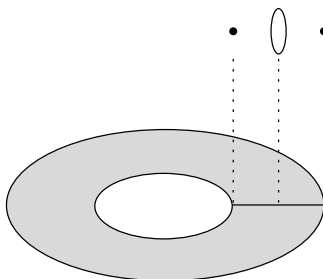


FIGURE 2.1. A circle as fiber over the interior of the moment map image in the segment $\{0\} \times]0, 1[$.

If the fibers of Φ are compact, necessary and sufficient conditions for faithfulness can be phrased without reference to the leaf space.

Lemma 2.15. *An integrable system (M, ω, Φ) with compact fibers is faithful if and only if Φ has connected fibers and is proper onto its image.*

Proof. Let \mathcal{L} and B be the leaf space and moment map image, respectively, of (M, ω, Φ) . Let $q : M \rightarrow \mathcal{L}$ be the quotient map and let $\pi : \mathcal{L} \rightarrow B$ be the induced map. Suppose first that Φ has connected fibers and is proper onto its image. By the definition of \mathcal{L} , the continuous map π is a bijection because the fibers of Φ are connected. It remains to show that π is a closed map. In general, a continuous proper map to a metrizable space is closed (cf. Palais [31]). To see that π is proper, consider an arbitrary compact set $K \subset B$. The preimage $\Phi^{-1}(K)$ is compact because Φ is proper onto its image. Furthermore, $\pi^{-1}(K) = q(\Phi^{-1}(K))$ because q is surjective, so $\pi^{-1}(K)$ is compact because q is continuous and $\Phi^{-1}(K)$ is compact. Therefore π is proper. Then, since $B \subset \mathbb{R}^2$ is metrizable, the map π is also closed. Consequently, π is a homeomorphism.

Conversely, suppose that π is a homeomorphism; in particular, it is a bijection, which implies that Φ has connected fibers. It remains to prove that Φ is proper onto its image; since π is a homeomorphism, it suffices to check that q is proper onto its image. Note that, since π is a homeomorphism, \mathcal{L} is second countable, locally compact and Hausdorff. Properness of q can be checked at every point of \mathcal{L} , using the pointwise version of properness of Duistermaat & Kolk [10, Definition 2.3.2]. Seeing as \mathcal{L} satisfies the above topological conditions, the criterion of del Hoyo [6, Proposition 2.1.3] can be used: since the fibers of Φ and hence of q are compact, it suffices to show that any open neighborhood of a fiber of q contains a q -saturated open neighborhood. To this end, observe that M is locally compact, the fibers of q are Hausdorff, and q has compact and connected fibers. Therefore the result of Mrčun [30, Theorem 3.3] can be applied: any open neighborhood of any given fiber of q contains an open neighborhood that is the union of compact connected components of fibers of q . Connectedness of the fibers of q implies that this neighborhood is q -saturated. \square

2.3. \mathbb{Z} -affine structure on the regular part of the leaf space.

When the fibers of an integrable system are compact, the regular leaf space inherits a *geometric* structure. This is a consequence of the Liouville-Arnol'd theorem, which provides a local normal form for a neighborhood of a regular leaf (cf. Cushman & Bates [5, Appendix

D], Duistermaat [9], Guillemin & Sternberg [16, Chapter 44], Sepe & Vũ Ngọc [38] for various versions of a proof). Let Ω be the canonical symplectic form on $\mathbb{T}^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ for which the projection $\text{pr}_2 : (\mathbb{T}^*\mathbb{T}^n, \Omega) \rightarrow \mathbb{R}^n$ defines an integrable system.

Theorem 2.16 (Liouville-Arnol'd). *Let (M, ω, Φ) be an integrable system with n degrees of freedom and let F be a regular, compact leaf. Then there exist open neighborhoods $V \subset (M, \omega)$ of F and $W \subset (\mathbb{T}^*\mathbb{T}^n, \Omega)$ of $\mathbb{T}^n \times \{0\}$, the latter saturated w.r.t. pr_2 , such that the subsystems of (M, ω, Φ) and of $(\mathbb{T}^*\mathbb{T}^n, \Omega, \text{pr}_2)$, relative to V and W , respectively, are isomorphic via a pair (Ψ, ψ) where $\psi(\Phi(F)) = 0$.*

Remark 2.17 Identify the Lie algebra of \mathbb{T}^n with \mathbb{R}^n . Using the notation of Theorem 2.16, the composition $\psi \circ \Phi|_V$ is the moment map of a free, effective Hamiltonian \mathbb{T}^n -action, *i.e.* the Hamiltonian vector fields of its components have 2π -periodic flows. Moreover, if $\Phi = (H_1, \dots, H_n)$ with H_1 being the moment map of an effective Hamiltonian S^1 -action, then the diffeomorphism ψ can be taken to be of the form

$$(2.1) \quad \begin{aligned} \psi(x_1, \dots, x_n) &= (\psi^{(1)}, \dots, \psi^{(n)})(x_1, \dots, x_n) \\ &= (x_1 + a, \psi^{(2)}(x_1, \dots, x_n), \dots, \psi^{(n)}(x_1, \dots, x_n)), \end{aligned}$$

where $a \in \mathbb{R}$ is a constant.

Corollary 2.18. *Let (M, ω, Φ) be an integrable system with compact fibers. The regular leaf space $\mathcal{L}_{\text{reg}} \subset \mathcal{L}$ is open and it inherits a structure of smooth, Hausdorff manifold uniquely defined by requiring that the restriction of the quotient map $q|_{q^{-1}(\mathcal{L}_{\text{reg}})}$ be a submersion onto \mathcal{L}_{reg} . In particular, the restriction $\pi|_{\mathcal{L}_{\text{reg}}}$ is smooth.*

Proof. Moerdijk and Mrčun proved, in [29, Section 2.4], that the leaf space of a submersion whose fibers are compact can be endowed with the structure of a smooth manifold uniquely defined by demanding that the quotient map be a submersion. The version of the Liouville-Arnol'd theorem given in Theorem 2.16 implies that, for any point $p \in \mathcal{L}_{\text{reg}}$, the corresponding regular leaf F_p has an open neighborhood that is saturated by regular leaves. Therefore, $\Phi|_{q^{-1}(\mathcal{L}_{\text{reg}})}$ is a submersion whose leaf space is naturally isomorphic to \mathcal{L}_{reg} , an open subset of \mathcal{L} . Since the fibers of Φ are compact by hypothesis, the result of Moerdijk and Mrčun implies the desired result. \square

A symplectomorphism φ of $(\mathbb{T}^n \times \mathbb{R}^n, \Omega)$ that preserves the fibers of pr_2 must have the form $\varphi = (\varphi^{(1)}, \varphi^{(2)})$ where $\varphi^{(1)}(t, x) = (A^{-1})^T t + f(x)$ and $\varphi^{(2)}(t, x) = Ax + c$ for some $A \in \text{GL}(n, \mathbb{Z})$, some $c \in \mathbb{R}^n$,

and a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the matrix $A^{-1} \frac{\partial f}{\partial x}$ is symmetric (cf. Symington [40, Lemma 2.5].)

This implies that maps of the form $\psi \circ \pi$, where ψ is as in Theorem 2.16, can be used to define a \mathbb{Z} -affine atlas on \mathcal{L}_{reg} .

Definition 2.19. For any $n \geq 1$, the category of n -dimensional \mathbb{Z} -affine manifolds, denoted by $\mathcal{A}ff_{\mathbb{Z}}(n)$, has objects and morphisms as follows:

- Objects: \mathbb{Z} -affine manifolds (with corners), as defined in Section 1.
- Morphisms: \mathbb{Z} -affine maps, i.e. maps $f : (N_1, \mathcal{A}_1) \rightarrow (N_2, \mathcal{A}_2)$ that are local diffeomorphisms such that $f^* \mathcal{A}_2$ and \mathcal{A}_1 define equivalent \mathbb{Z} -affine atlases (cf. Section 1).

If (N_2, \mathcal{A}_2) is a \mathbb{Z} -affine manifold and $f : N_1 \rightarrow N_2$ is locally a homeomorphism then there exists a unique (up to isomorphism) \mathbb{Z} -affine structure \mathcal{A}_1 on N_1 that makes f into a \mathbb{Z} -affine morphism. The structure \mathcal{A}_1 is henceforth referred to as being *induced* by f .

Example 2.20 For any $n \geq 1$, denote by \mathcal{A}_0 both the \mathbb{Z} -affine structure on \mathbb{R}^n and the \mathbb{Z} -affine structure with corners on $[0, \infty[^n$ obtained by declaring the standard coordinates x_1, \dots, x_n to be \mathbb{Z} -affine. Then the *standard \mathbb{Z} -affine structure* on an open subset of \mathbb{R}^n is the \mathbb{Z} -affine structure induced by inclusion of the subset in $(\mathbb{R}^n, \mathcal{A}_0)$. Likewise, an open subset of the subspace $[0, \infty[^n$ also inherits the standard \mathbb{Z} -affine structure from inclusion in $([0, +\infty[^n, \mathcal{A}_0)$.

For \mathbb{Z} -affine manifolds, it makes sense to consider (the sheaf of) \mathbb{Z} -affine functions, i.e. (locally defined) smooth functions that, in local \mathbb{Z} -affine coordinates (x_1, \dots, x_n) , are given by

$$\sum_{i=1}^n k_i x_i + c,$$

where $k_i \in \mathbb{Z}$ and $c \in \mathbb{R}$. The local normal form provided by the Liouville-Arnol'd Theorem (Theorem 2.16) implies that the regular leaf space of an integrable system with compact fibers can be characterized, as a \mathbb{Z} -affine manifold, by the sheaf of functions that generate 2π -periodic flows tangent to the fibers of the quotient map.

Corollary 2.21. *Let (M, ω, Φ) be an integrable system with compact fibers. Then the subset $\mathcal{L}_{\text{reg}} \subset \mathcal{L}$ inherits a \mathbb{Z} -affine structure \mathcal{A}_{reg} , uniquely defined by the property that locally defined \mathbb{Z} -affine functions from $(\mathcal{L}_{\text{reg}}, \mathcal{A}_{\mathcal{L}_{\text{reg}}})$ to $(\mathbb{R}, \mathcal{A}_0)$ correspond, by taking the pull-back along the restriction to $q^{-1}(\mathcal{L}_{\text{reg}})$ of the quotient map $q : M \rightarrow \mathcal{L}$, to functions*

on $q^{-1}(\mathcal{L}_{\text{reg}}) \subset M$ whose Hamiltonian vector fields are tangent to the fibers of q and have 2π -periodic flows.

Corollary 2.22. *For each $n \geq 1$, there is a functor $\mathcal{IS}(n) \rightarrow \text{Aff}_{\mathbb{Z}}(n)$ that, on objects, is precisely the map $(M, \omega, \Phi) \mapsto (\mathcal{L}_{\text{reg}}, \mathcal{A}_{\text{reg}})$ given by Corollary 2.21.*

Proof. The above functor is completely determined by the following property, which can be checked directly: an isomorphism of integrable systems induces a \mathbb{Z} -affine (iso)morphism between the associated \mathbb{Z} -affine manifolds. \square

Furthermore, the correspondence between integrable systems and \mathbb{Z} -affine manifolds given by Corollary 2.21 behaves well under restriction to saturated subsystems.

Corollary 2.23. *Given an integrable system (M, ω, Φ) , the natural inclusion of the leaf space of a saturated subsystem into the leaf space of (M, ω, Φ) corresponds to a \mathbb{Z} -affine embedding of one \mathbb{Z} -affine manifold into another.*

Finally, the above discussion allows further refinement of the set of invariants that can be associated to an integrable system. Given (M, ω, Φ) , associate the pair $(\mathcal{L}, (\mathcal{L}_{\text{reg}}, \mathcal{A}_{\text{reg}}))$ to it, where $(\mathcal{L}, \mathcal{L}_{\text{reg}})$ is the pair of leaf and regular leaf spaces of (M, ω, Φ) and \mathcal{A}_{reg} is the \mathbb{Z} -affine structure given by Corollary 2.21. This association descends to isomorphism classes of systems. In this case, isomorphisms of pairs are homeomorphisms of the underlying topological pairs that restrict to \mathbb{Z} -affine isomorphisms on the \mathbb{Z} -affine subspace.

Remark 2.24 For faithful integrable systems (M, ω, Φ) , the leaf space \mathcal{L} can be identified topologically with the moment map image $B = \Phi(M)$. Under this correspondence, \mathcal{L}_{reg} is identified with the subset of regular values $B_{\text{reg}} \subset B$. By Corollary 2.21, B_{reg} inherits a \mathbb{Z} -affine structure denoted by \mathcal{A}_{reg} which, in general, is *not isomorphic* to the standard one.

2.4. Developing maps. Given an n -dimensional \mathbb{Z} -affine manifold (with corners) (N, \mathcal{A}) , let \tilde{N} denote its universal cover. The universal covering map $\mathfrak{q} : \tilde{N} \rightarrow N$ induces a \mathbb{Z} -affine structure $\tilde{\mathcal{A}}$ on \tilde{N} , making $\mathfrak{q} : (\tilde{N}, \tilde{\mathcal{A}}) \rightarrow (N, \mathcal{A})$ into a \mathbb{Z} -affine morphism. Fix a basepoint $\mathbf{x}_0 \in N$, a point $\tilde{\mathbf{x}}_0 \in \tilde{N}$ with $\mathfrak{q}(\tilde{\mathbf{x}}_0) = \mathbf{x}_0$, and a \mathbb{Z} -affine coordinate chart $\phi_0 : U_0 \rightarrow \mathbb{R}^n$ defined near \mathbf{x}_0 . Identify \tilde{N} with the space of paths starting at $\tilde{\mathbf{x}}_0$, up to homotopy relative to endpoints. Then there is a \mathbb{Z} -affine map $\text{dev} : (\tilde{N}, \tilde{\mathcal{A}}) \rightarrow (\mathbb{R}^n, \mathcal{A}_0)$, uniquely determined by

the property that, near \tilde{x}_0 , it equals the restriction of ϕ_0 to a suitable neighborhood of x_0 , and a representation $\mathfrak{a} : \pi_1(N; x_0) \rightarrow \text{AGL}(n; \mathbb{Z})$, called *the affine holonomy of (N, \mathcal{A})* , which is intertwined with dev as follows: for all $\gamma \in \pi_1(N; x_0)$, the following diagram commutes

$$\begin{array}{ccc} (\tilde{N}, \tilde{\mathcal{A}}) & \xrightarrow{\text{dev}} & (\mathbb{R}^n, \mathcal{A}_0) \\ \gamma \downarrow & & \uparrow \mathfrak{a}(\gamma) \\ (\tilde{N}, \tilde{\mathcal{A}}) & \xrightarrow{\text{dev}} & (\mathbb{R}^n, \mathcal{A}_0), \end{array}$$

where $\cdot \gamma$ denotes the \mathbb{Z} -affine isomorphism of $(\tilde{N}, \tilde{\mathcal{A}})$ induced by the natural action of $\pi_1(N; x_0)$ on \tilde{N} .

The map $\text{dev} : \tilde{N} \rightarrow \mathbb{R}^n$ is called the *developing map* of (N, \mathcal{A}) (relative to the choices $(x_0, \tilde{x}_0, \phi_0)$), cf. for details Goldman & Hirsch [13] and references therein. Note that, using the fundamental groupoid of N , the information of a developing map can be packaged and conveyed independent of choices (cf. Crainic & Fernandes & Martínez-Torres [4]).

Remark 2.25

- If $\text{dev}, \text{dev}' : \tilde{N} \rightarrow \mathbb{R}^n$ are developing maps for (N, \mathcal{A}) constructed using different choices then there exists a unique element $h \in \text{AGL}(n; \mathbb{Z})$ such that $\text{dev}' = h \circ \text{dev}$.
- If (N, \mathcal{A}) is a \mathbb{Z} -affine manifold with corners, then the image of any codimension- k face of \tilde{N} ($0 < k \leq n$) under a developing map is the intersection of k linear hyperplanes of \mathbb{R}^n whose normals can be chosen to span a *unimodular* sublattice of \mathbb{Z}^n , *i.e.* this span is a direct summand of \mathbb{Z}^n .
- In general, developing maps need not be covering maps and their images can be rather complicated (cf. Sullivan & Thurston [39] for pathological examples).

2.5. Toric and Delzant systems. This section describes the connection between integrable toric actions and integrable systems.

Definition 2.26. For any $n \geq 1$, the category of *symplectic toric manifolds* of dimension $2n$, denoted by $\mathcal{TM}(2n)$, has objects and morphisms as follows:

- Objects: *symplectic toric manifolds*, *i.e.* $2n$ -dimensional symplectic manifolds (M, ω) endowed with an effective Hamiltonian \mathbb{T}^n -action with moment map $\mu : (M, \omega) \rightarrow \mathfrak{t}^*$, where \mathfrak{t}^* denotes the dual of the Lie algebra of \mathbb{T}^n . A symplectic toric manifold is henceforth

denoted by the triple (M, ω, μ) and, for brevity, referred to as a *toric manifold*.

- **Morphisms:** *isomorphisms of symplectic toric manifolds, i.e.* given (M_i, ω_i, μ_i) for $i = 1, 2$, a symplectomorphism $\Psi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ and an element $\xi \in \mathfrak{t}^*$ making the following diagram commute

$$\begin{array}{ccc} (M_1, \omega_1) & \xrightarrow{\Psi} & (M_2, \omega_2) \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ \mathfrak{t}^* & \xrightarrow{+\xi} & \mathfrak{t}^*, \end{array}$$

where $+\xi : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ denotes translation by ξ .

Henceforth, for each $n \geq 1$, fix an isomorphism $\mathfrak{t}^* \cong \mathbb{R}^n$ so that the standard lattice in \mathfrak{t}^* (dual to $\ker(\exp : \mathfrak{t} \rightarrow \mathbb{T}^n)$) is mapped to \mathbb{Z}^n .

Note that the individual components of the moment map $\mu : (M, \omega) \rightarrow \mathbb{R}^n$ Poisson commute and μ is a submersion almost everywhere, due to the Marle-Guillemin-Sternberg local normal form for Hamiltonian actions of compact Lie groups (cf. Guillemin & Sternberg, Marle [15, 26]). Therefore, taking $\Phi = \mu$, call (M, ω, Φ) the integrable system *underlying* (M, ω, μ) . Because isomorphisms of symplectic toric manifolds induce isomorphisms of underlying integrable systems, for each $n \geq 1$, the function from $\mathcal{TM}(2n)$ to $\mathcal{IS}(n)$ that maps a toric manifold to its underlying integrable system defines a ‘forgetful functor’ $\mathcal{F} : \mathcal{TM}(2n) \rightarrow \mathcal{IS}(n)$.

It is useful to identify the integrable systems that underlie symplectic toric manifolds and systems that are isomorphic to such.

Definition 2.27. An integrable system (M, ω, Φ) is *toric* if there exists a toric manifold (M, ω, μ) such that $(M, \omega, \Phi) = \mathcal{F}(M, \omega, \mu)$. An integrable system is *weakly toric* if it is isomorphic to a toric one.

Example 2.28. Let (M, ω, Φ) be an integrable system with compact fibers and $q : M \rightarrow \mathcal{L}$ the quotient map to its leaf space. A \mathbb{Z} -affine coordinate chart on $\mathcal{L}_{\text{reg}} \subset \mathcal{L}$ yields, pre-composing with the quotient map, the moment map of a locally defined effective (but not unique) Hamiltonian \mathbb{T}^n -action (cf. Corollary 2.21), and hence a toric system on the preimage by q of the domain of the coordinate chart.

Remark 2.29 The image of \mathcal{F} is a subcategory of $\mathcal{IS}(n)$, but it is not *full*. For instance, if $h \in \text{AGL}(n; \mathbb{Z})$ is an element different from the identity, the symplectic toric manifolds (M, ω, Φ) and $(M, \omega, h \circ \Phi)$ are not isomorphic, but the underlying integrable systems are. In fact, the integral affine group $\text{AGL}(n; \mathbb{Z})$ completely captures the failure of

the functor to be full: Suppose that (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2) are isomorphic toric systems with isomorphism denoted by (Ψ, ψ) . Then $\psi : \Phi_1(M_1) \rightarrow \Phi_2(M_2)$ is given, on each connected component of $\Phi_1(M_1)$, by the restriction of an element in $\text{AGL}(n; \mathbb{Z})$.

The classification by Delzant [7] of *compact* symplectic toric manifolds (M, ω, μ) is well-known. In particular, the manifold M , the symplectic form ω and the moment map μ up to isomorphism are determined by the image of the moment map, $\mu(M)$. Two properties of a compact symplectic toric manifold (M, ω, μ) that are important for the classification are that μ has connected fibers and that $\mu(M)$ is a convex polytope. These properties follow from work of Atiyah [1] and Guillemin & Sternberg [14] and hold for the more general family of effective Hamiltonian torus actions on compact symplectic manifolds.

If the underlying symplectic manifold is *not* compact, neither of the above properties need hold (cf. Karshon & Lerman [23]). Nevertheless, Karshon & Lerman [23] achieve a classification of these objects. Among others, they use the following result:

Orbits are tori that have a neighborhood that can be put in normal form (cf. Guillemin & Sternberg [15], Marle [26]): For each orbit \mathcal{O} of dimension k , there exist

- open neighborhoods $V \subset M$ of \mathcal{O} and $W \subset T^*\mathbb{T}^k \times \mathbb{R}^{2(n-k)} \cong \mathbb{T}^k \times \mathbb{R}^{2n-k}$ of $\mathbb{T}^k \times \{\mathbf{0}\}$;
- a symplectomorphism $\Psi : (V, \omega) \rightarrow (W, \omega_{\text{can}} \oplus \omega_0)$ sending \mathcal{O} to $\mathbb{T}^k \times \{\mathbf{0}\}$;
- an element $A \in \text{GL}(n-k; \mathbb{Z})$;
- a translation $-\mu(\mathcal{O}) : \mu(U) \rightarrow \mathbb{R}^n$

making the following diagram commute

$$\begin{array}{ccc} (V, \omega) & \xrightarrow{\Psi} & (W, \omega_{\text{can}} \oplus \omega_0) \\ \mu \downarrow & & \downarrow \text{pr}_2 \oplus A \circ \mathbf{q} \\ \mu(U) & \xrightarrow{-\mu(\mathcal{O})} & \mathbb{R}^k \times \mathbb{R}^{n-k}, \end{array}$$

where

- ω_{can} is the canonical symplectic form on $T^*\mathbb{T}^k \cong \mathbb{T}^k \times \mathbb{R}^k$;
- $\text{pr}_2 : \mathbb{T}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is projection onto the second factor;
- $\omega_0 = \sum_{i=1}^{n-k} dx_i \wedge dy_i$ with respect to standard symplectic coordinates on $\mathbb{R}^{2(n-k)}$;
- $\mathbf{q} = \left(\frac{x_1^2 + y_1^2}{2}, \dots, \frac{x_{n-k}^2 + y_{n-k}^2}{2} \right)$.

Observe that W is saturated by the fibers of $\text{pr}_2 \oplus A \circ \mathbf{q}$.

The existence of these local normal forms implies that even singular orbits of the action make up whole connected components of the fibers of the moment map: this sets toric systems apart from general integrable systems.

In analogy with Corollary 2.21, the above local normal forms imply the following:

Remark 2.30 Let (M, ω, μ) be a symplectic toric manifold with associated toric system (M, ω, Φ) . Then the above local normal forms imply the following.

- 1) The orbit space M/\mathbb{T}^n of (M, ω, μ) is a \mathbb{Z} -affine manifold with corners uniquely characterized as in Corollary 2.21.
- 2) M/\mathbb{T}^n is canonically homeomorphic to the leaf space \mathcal{L} of (M, ω, Φ) .
- 3) The image under π of a codimension- k face of \mathcal{L} , where $0 < k \leq n$, is the intersection of k hyperplanes of \mathbb{R}^n whose normals can be chosen to span a unimodular sublattice of \mathbb{Z}^n (cf. Remark 2.25).
- 4) With the above identification, the \mathbb{Z} -affine structure on \mathcal{L} , denoted by $\mathcal{A}_{\mathcal{L}}$, extends \mathcal{A}_{reg} , *i.e.* the inclusion $(\mathcal{L}_{\text{reg}}, \mathcal{A}_{\text{reg}}) \hookrightarrow (\mathcal{L}, \mathcal{A}_{\mathcal{L}})$ is a \mathbb{Z} -affine embedding.
- 5) The regular leaf space satisfies $\mathcal{L}_{\text{reg}} = \mathcal{L} \setminus \partial_{\infty} \mathcal{L}$.

Note that the developing map $\text{dev} : \tilde{\mathcal{L}} \rightarrow \mathbb{R}^n$ for $(\mathcal{L}, \mathcal{A}_{\mathcal{L}})$ makes the following diagram commute

$$\begin{array}{ccc} & \tilde{\mathcal{L}} & \\ \mathfrak{q} \swarrow & & \searrow \text{dev} \\ \mathcal{L} & \xrightarrow{\pi} & \mathbb{R}^n, \end{array}$$

where $\mathfrak{q} : \tilde{\mathcal{L}} \rightarrow \mathcal{L}$ is the universal covering map from Karshon & Lerman [23, Proposition 1.1, Remarks 1.4 and 1.5] and π the so-called *orbital moment map* of (M, ω, μ) therein.

The orbital moment map is an essential invariant of symplectic toric manifolds; this is the content of the following result stated below without proof.

Proposition 2.31. *Two toric manifolds are isomorphic only if their orbit/leaf spaces are homeomorphic and have, once the above homeomorphism is taken into consideration, orbital moment maps that are equal up to translation.*

The remaining ingredients in the classification up to isomorphism are topological invariants depending on $H^2(\mathcal{L}; \mathbb{Z})$; for a precise statement

see Karshon & Lerman [23, Theorem 1.3]. Motivated by the classification theorem of Karshon & Lerman [23, Theorem 1.3], we introduce a class of toric systems whose corresponding symplectic toric manifolds are determined up to isomorphism by their moment map images.

Definition 2.32. A toric system (M, ω, Φ) is said to be *Delzant* if it is faithful and $\Phi(M)$ is contractible.

Lemma 2.33. *The total space of a Delzant system is connected.*

Proof. This follows from the faithfulness of the moment map, connectedness of the moment map image, and the equivalence of the cardinality of the set of components of the leaf space and of the total space of an integrable system (Remark 2.7). \square

Proposition 2.34. *Two Delzant systems (M_i, ω_i, Φ_i) , $i = 1, 2$, are isomorphic if and only if there is an element $h \in \text{AGL}(n; \mathbb{Z})$ such that $\Phi_2(M_2) = h \circ \Phi_1(M_1)$. Furthermore, the two Delzant systems underlie isomorphic toric manifolds if and only if $\Phi_1(M_1)$ and $\Phi_2(M_2)$ agree up to translation.*

Proof. Because (M_i, ω_i, Φ_i) , $i = 1, 2$, are toric systems, there exist symplectic toric manifolds (M_i, ω_i, μ_i) such that $\mu_i = \Phi_i$, $i = 1, 2$. Under that equivalence, faithfulness of the moment maps Φ_i corresponds to the orbital moment maps of the symplectic toric manifolds being embeddings. Therefore, since $\mu_i(M_i) = \Phi_i(M_i)$, $i = 1, 2$, is contractible, Theorem 1.3 of Karshon & Lerman [23] implies the symplectic toric manifolds (M_i, ω_i, μ_i) are determined up to isomorphism by their moment map images. By the definition of $\mathcal{TM}(2n)$, the category of symplectic toric manifolds (Definition 2.26), (M_i, ω_i, μ_i) , $i = 1, 2$ belong to the same isomorphism class if and only if $\mu_1(M_1)$ and $\mu_2(M_2)$ differ by a translation. The criterion for isomorphism of the toric systems then follows from Remark 2.29 and connectedness of the total space (Lemma 2.33). \square

Remark 2.35 Given a Delzant system (M, ω, Φ) with n degrees of freedom, identify its leaf space with its moment map image B . The above discussion, together with the fact that B is simply connected, implies that the inclusion $B \hookrightarrow \mathbb{R}^n$ is a developing map for the induced \mathbb{Z} -affine structure on B . Thus the \mathbb{Z} -affine structure on B coming from (M, ω, Φ) is the standard one.

Unlike *compact* symplectic toric manifolds (and their associated toric systems), the moment map image of a Delzant system need not be convex; however, the above local normal form, together with the defining properties of Delzant systems, imply the following result.

Corollary 2.36. *The moment map image of a Delzant system (M, ω, Φ) is locally convex, i.e., for all $c \in \Phi(M)$, there exists an open neighborhood U of c in $\Phi(M)$ that is convex as a subset of Euclidean space.*

2.6. Locally weakly toric leaf spaces. Weakly toric provide examples of integrable systems whose leaf spaces are naturally endowed with the structure of a \mathbb{Z} -affine manifold with corners. This is not a phenomenon to be expected in general. However it is natural to ask, what is the largest subset of the leaf space of an integrable system that does inherit the structure of a \mathbb{Z} -affine manifold with corners?

That question motivates the following notions of a *weakly toric leaf* and the *weakly locally toric leaf space* of an integrable system.

Definition 2.37. Given an integrable system (M, ω, Φ) , a leaf $L \subset M$ is said to be *weakly toric* if there exists a connected open neighborhood $V \subset M$ such that the subsystem $(V, \omega|_V, \Phi|_V)$ is weakly toric.

Remark 2.38 A weakly toric leaf L is a smoothly embedded submanifold that must be compact even though the system (M, ω, Φ) may have non-compact fibers. Also, the open neighborhood V of Definition 2.37 is saturated with respect to the quotient map $q : M \rightarrow \mathcal{L}$ and all leaves contained in V are weakly toric. Finally, in analogy with Remark 2.17, if (Ψ, ψ) denotes the isomorphism between $(V, \omega|_V, \Phi|_V)$ and a toric system, and the first component of $\Phi|_V$ is the moment map of an effective Hamiltonian S^1 -action, then ψ can be taken to be of the form of equation (2.1).

Definition 2.39. Given an integrable system (M, ω, Φ) with leaf space \mathcal{L} , the subset $\mathcal{L}_{\text{lt}} \subset \mathcal{L}$ corresponding to weakly toric leaves is called the *locally weakly toric leaf space* associated to (M, ω, Φ) .

A priori, if the fibers of an integrable system are not necessarily compact, the associated locally weakly toric leaf space may be empty. In contrast, when the fibers are required to be compact, the Liouville-Arnol'd Theorem (Theorem 2.16) implies \mathcal{L}_{lt} must be dense in \mathcal{L} :

Corollary 2.40. *If (M, ω, Φ) has compact fibers, then $\mathcal{L}_{\text{reg}} \subset \mathcal{L}_{\text{lt}}$.*

In fact, if an integrable system has compact fibers, its locally weakly toric leaf space inherits a \mathbb{Z} -affine structure.

Proposition 2.41. *The locally weakly toric leaf space \mathcal{L}_{lt} of an integrable system, whenever non-empty, inherits a structure of \mathbb{Z} -affine manifold with corners denoted by \mathcal{A}_{lt} . This structure is uniquely defined by the property that locally defined \mathbb{Z} -affine functions from $(\mathcal{L}_{\text{lt}}, \mathcal{A}_{\mathcal{L}_{\text{lt}}})$*

correspond to functions on $q^{-1}(\mathcal{L}_{\text{lt}}) \subset M$ whose Hamiltonian vector fields are tangent to the fibers of q and have 2π -periodic flows.

Proof. Fix an integrable system (M, ω, Φ) with n degrees of freedom whose locally weakly toric leaf space \mathcal{L}_{lt} is not empty. To show that the locally weakly toric leaf space \mathcal{L}_{lt} is Hausdorff, it suffices to check that if L_1, L_2 are weakly toric leaves with $\Phi(L_1) = \Phi(L_2)$, then for $i = 1, 2$, there exists an open neighborhood V_i of L_i , saturated with respect to q and containing solely weakly toric leaves, with $V_1 \cap V_2 = \emptyset$. First, observe that $L_1, L_2 \subset M$ are closed; thus there exist $V'_1, V'_2 \subset M$ open, disjoint subsets with $L_i \subset V'_i$ for $i = 1, 2$. Since, for $i = 1, 2$, L_i is weakly toric, there exists an open neighborhood $V_i \subset V'_i$ of L_i saturated with respect to q containing solely weakly toric leaves; V_1 and V_2 are the desired separating open subsets. The space \mathcal{L}_{lt} is second countable because it is the image of a second countable space under an open map. Indeed $q^{-1}(\mathcal{L}_{\text{lt}}) \subset M$ is second countable as it is an open subset of a smooth manifold, and the topological quotient map $q|_{q^{-1}(\mathcal{L}_{\text{lt}})} : q^{-1}(\mathcal{L}_{\text{lt}}) \rightarrow \mathcal{L}_{\text{lt}}$ is open because the quotient maps in the local models for symplectic toric manifolds are open. Finally, since \mathcal{L}_{lt} is locally homeomorphic to a subset of Euclidean space, it is locally compact, thereby implying \mathcal{L}_{lt} is paracompact (by virtue of being a locally compact, Hausdorff, second countable space).

Next we define an open cover of \mathcal{L}_{lt} and coordinate charts whose codomain is $[0, +\infty]^n$. Let $L \subset M$ be a weakly toric leaf; by definition, there exists an open neighborhood $V \subset M$ of L whose corresponding subsystem is weakly toric. By restricting V if necessary, it may be assumed that the corresponding subsystem is isomorphic to a local model for symplectic toric manifolds. Since $q|_{q^{-1}(\mathcal{L}_{\text{lt}})}$ is open, $q(V)$ is an open subset of \mathcal{L}_{lt} ; moreover, the above isomorphism implies that there is a map $\chi : q(V) \rightarrow [0, +\infty]^n$ that is locally a homeomorphism. Since $L \subset M$ is arbitrary, the above reasoning defines an open cover of \mathcal{L}_{lt} , denoted by $\{U_i\}$, and, for each i , a map $\chi_i : U_i \rightarrow [0, +\infty]^n$. In fact, $\mathcal{A}_{\text{lt}} := \{(U_i, \chi_i)\}$ is an n -dimensional \mathbb{Z} -affine atlas with corners. To see this, fix i, j with $U_i \cap U_j \neq \emptyset$. Then, unraveling the above definitions, we obtain an isomorphism (Ψ_{ij}, ψ_{ij}) of saturated subsystems of local models of symplectic toric manifolds with $\psi_{ij} = \chi_j \circ \chi_i^{-1}$; observing that ψ_{ij} is necessarily the restriction of an element in $\text{AGL}(n; \mathbb{Z})$ shows the desired result. Finally, the defining property of \mathcal{A}_{lt} follows directly from its definition and the fact it holds for the (locally weakly toric) leaf spaces of symplectic toric manifolds. \square

Remark 2.42 Note that, by Proposition 2.41, the locally weakly toric leaf space of an integrable system is, tautologically, the largest subset

of the leaf space of an integrable system that inherits the structure of a \mathbb{Z} -affine manifold with corners. Moreover, if the system has compact fibers, the inclusion $(\mathcal{L}_{\text{reg}}, \mathcal{A}_{\text{reg}}) \hookrightarrow (\mathcal{L}_{\text{lt}}, \mathcal{A}_{\text{lt}})$ is a \mathbb{Z} -affine morphism.

As expected, the \mathbb{Z} -affine manifold with corners $(\mathcal{L}_{\text{lt}}, \mathcal{A}_{\text{lt}})$ associated to (M, ω, Φ) is an invariant of the isomorphism class of (M, ω, Φ) and behaves well with respect to restriction to saturated subsystems, *i.e.* statements analogous to Corollaries 2.22 and 2.23 hold for this (possibly larger) \mathbb{Z} -affine manifold with corners. This allows one to associate to an integrable system (M, ω, Φ) the pair $(\mathcal{L}, (\mathcal{L}_{\text{lt}}, \mathcal{A}_{\text{lt}}))$, the latter being an invariant of the isomorphism class of (M, ω, Φ) .

Remark 2.43 If (M, ω, Φ) is faithful and has compact fibers, the locally weakly toric leaf space corresponds to an open, dense subset denoted by $B_{\text{lt}} \subset B = \Phi(M)$ and its boundary as a manifold with corners (corresponding to singular weakly toric leaves) satisfies $\partial_{\infty} B_{\text{lt}} \subset B \cap \partial B$, where the inclusion may be strict.

2.7. Cartographic maps. The geometry of the locally weakly toric leaf space of a faithful integrable system (M, ω, Φ) with compact fibers is captured by the moment map image whenever the inclusion $B \hookrightarrow \mathbb{R}^n$ is a \mathbb{Z} -affine embedding when restricted to $B_{\text{lt}} \cong \mathcal{L}_{\text{lt}}$, as is the case for faithful toric (and, in particular, Delzant) systems. In this case, the subsystem relative to B_{lt} is toric; expanding the terminology of Pelayo & Ratiu & Vũ Ngọc [33], we say that the moment map Φ is *cartographic*.

When the moment map of a faithful integrable system (M, ω, Φ) with compact fibers is not cartographic, one could ask whether there is an isomorphic system whose moment map *is* cartographic. Existence of such a system is tantamount to finding a smooth embedding of the moment map image B into \mathbb{R}^n whose restriction to the locally weakly toric leaf space B_{lt} is a \mathbb{Z} -affine embedding into $(\mathbb{R}^n, \mathcal{A}_0)$. A necessary condition is that the affine holonomy of $(B_{\text{lt}}, \mathcal{A}_{\text{lt}})$ be trivial; however, that condition is not sufficient, as the following example illustrates.

Example 2.44 Let R be the open rectangle $]0, 1[\times]0, 2\pi[\subset \mathbb{R}^2$. Consider the unique symplectic toric manifold $(\mathbb{T}^2 \times R, \omega, \mu)$ defined by the orbital moment map $W : R \rightarrow \mathbb{R}^2$ given by $g_2(x, y) = (e^x \cos(2y), e^x \sin(2y))$. Observe that ω is not the standard symplectic structure induced by inclusion of $\mathbb{T}^2 \times R$ in $(\mathbb{T}^2 \times \mathbb{R}^2, \omega_{\text{can}})$. Let $\Phi : \mathbb{T}^2 \times R \rightarrow R \subset \mathbb{R}^2$ be the projection onto the second factor. Then $(\mathbb{T}^2 \times R, \omega, \Phi)$ defines a faithful integrable system. Note that for any open simply connected subset $U \subset \{(x_1, x_2) \mid 1 < x_1^2 + x_2^2 < e^2\}$, the subsystem of $(\mathbb{T}^2 \times R, \omega, \mu)$ relative to U is the union of at least two disjoint subsystems, each of

which is isomorphic to a subsystem of the toric system $(\mathbb{T}^2 \times R, \omega, \mu)$. However, the integrable system $(\mathbb{T}^2 \times R, \omega, \Phi)$ as a whole is not isomorphic to any system with a cartographic moment map as g_2 , which is not injective, is the unique map (up to composition on the right with an \mathbb{Z} -affine diffeomorphism of $(\mathbb{R}^2, \mathcal{A}_0)$) such that $g_2 \circ \Phi$ generates an effective Hamiltonian \mathbb{T}^2 -action on $(\mathbb{T}^2 \times \mathbb{R}, \omega)$.

The faithful integrable systems with compact fibers considered in Sections 3 and 4 allow for singular fibers (focus-focus fibers, cf. Section 3.1.2) that induce non-trivial affine holonomy on the locally weakly toric leaf space (cf. Theorem 3.27). Thus for such an integrable system there is not necessarily a system in its isomorphism class that has a cartographic moment map. However, following the insight of Symington [40] and Vũ Ngọc [46], for such systems it is reasonable to ask whether there is a homeomorphism of the moment map image that, when restricted to an *open, dense* subset of the locally weakly toric leaf space, is a \mathbb{Z} -affine embedding. That question motivates introducing the following notion.

Definition 2.45. Let (M, ω, Φ) be a faithful integrable system with n degrees of freedom and whose fibers are compact, and set $B = \Phi(M)$. A *cartographic pair* (f, S) for (M, ω, Φ) consists of a topological embedding $f : B \rightarrow f(B) \subset \mathbb{R}^n$, called *cartographic homeomorphism*, and an open, dense subset $S \subset B_{\text{lt}}$, with the property that $f|_S : (S, \mathcal{A}_{\text{lt}}|_S) \rightarrow (\mathbb{R}^n, \mathcal{A}_0)$ is a \mathbb{Z} -affine smooth embedding. If such a pair exists, (M, ω, Φ) is said to *admit a cartographic homeomorphism* whose image is said to be *cartographic*.

Remark 2.46 If (f, S) is a cartographic pair for (M, ω, Φ) , there is no guarantee that S is *maximal*, i.e. that it is the largest open, dense subset of B_{lt} on which f restricts to a \mathbb{Z} -affine embedding.

The following lemma provides a simple but useful way to adjust a given cartographic homeomorphism.

Lemma 2.47. *If (f, S) is a cartographic pair for (M, ω, Φ) , then, for any $h \in \text{AGL}(n; \mathbb{Z})$, $(h \circ f, S)$ is also a cartographic pair.*

Intuitively, cartographic homeomorphisms should be thought of as continuous extensions of restrictions of developing maps to suitable domains; for instance, if S is simply connected, S can be identified with a dense subset of a fundamental domain in the universal cover of B_{lt} .

If $S = B$ holds in Definition 2.45, then $f \circ \Phi$ is a cartographic moment map. If not, a cartographic homeomorphism at least provides a dense

subset of the total space on which the system is isomorphic to a toric, and possibly Delzant system. More precisely:

Corollary 2.48. *Let (M, ω, Φ) be a faithful integrable system with compact fibers. If (f, S) is a cartographic pair, then $(\Phi^{-1}(S), \omega|_{\Phi^{-1}(S)}, (f \circ \Phi)|_{\Phi^{-1}(S)})$ is toric. If, in addition, S is contractible, then the above system is Delzant.*

Cartographic homeomorphisms restrict appropriately when taking saturated subsystems.

Corollary 2.49. *Suppose that (M, ω, Φ) is a faithful integrable system with compact fibers with cartographic pair (f, S) . Let $U \subset B = \Phi(M)$ be an open subset. Then $(f|_U, S \cap U)$ is a cartographic pair for the subsystem relative to U .*

Combining Corollaries 2.48 and 2.49, we obtain the following simple description of cartographic homeomorphisms when restricted to open subsets of the moment map image whose corresponding subsystems are Delzant.

Corollary 2.50. *Let (M, ω, Φ) be a faithful integrable system with compact fibers with cartographic pair (f, S) . Suppose that $U \subset S$ is an open subset with the property that the subsystem relative to U is Delzant. Then $f|_U$ is the restriction of an element $h_U \in \text{AGL}(n; \mathbb{Z})$ and $(h_U^{-1} \circ f, S)$ is a cartographic pair for (M, ω, Φ) with $(h_U^{-1} \circ f)|_U = \text{id}|_U$.*

Proof. By assumption, $(\Phi^{-1}(U), \omega|_{\Phi^{-1}(U)}, \Phi|_{\Phi^{-1}(U)})$ is a Delzant system and, by Corollary 2.48, $(\Phi^{-1}(U), \omega|_{\Phi^{-1}(U)}, (f \circ \Phi)|_{\Phi^{-1}(U)})$ is also Delzant. In fact, the pair $(\text{id}, f|_U)$ defines an isomorphism between these two systems. Because U is connected (by virtue of being contractible), there exists an element $h_U \in \text{AGL}(n; \mathbb{Z})$ such that $f|_U = h_U$ (cf. Remark 2.29). Lemma 2.47 gives that $(h_U^{-1} \circ f, S)$ is a cartographic pair for (M, ω, Φ) that, by construction, satisfies $(h_U^{-1} \circ f)|_U = \text{id}|_U$. \square

Finally, it is important to notice that the property of admitting a cartographic homeomorphism is independent of the choice of representative of the isomorphism class of a faithful integrable system.

Corollary 2.51. *Let (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2) be faithful integrable systems with compact fibers isomorphic via (Ψ, ψ) and let (f_1, S_1) be a cartographic pair for (M_1, ω_1, Φ_1) . Then $(f_2 := f_1 \circ \psi^{-1}, S_2 := \psi(S_1))$ is a cartographic pair for (M_2, ω_2, Φ_2) .*

In fact, cartographic images of isomorphic systems are homeomorphic via maps that extend \mathbb{Z} -affine isomorphisms. More precisely, the following holds.

Corollary 2.52. *Consider, for $i = 1, 2$, a faithful integrable system (M_i, ω_i, Φ_i) with n degrees of freedom and compact fibers with a cartographic pair (f_i, S_i) . Assume that there exists an isomorphism (Ψ, ψ) between (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2) . Then the cartographic images $f_1(B_1)$ and $f_2(B_2)$ are homeomorphic by a map that, when restricted to each connected component of $f_1(S_1 \cap \psi^{-1}(S_2))$, is the restriction of an element of $\text{AGL}(n; \mathbb{Z})$.*

Proof. Fix an isomorphism $(\Psi, \psi) : (M_1, \omega_1, \Phi_1) \rightarrow (M_2, \omega_2, \Phi_2)$. The map $g := f_2 \circ \psi \circ f_1^{-1} : f_1(B_1) \rightarrow f_2(B_2)$ is a homeomorphism as it is the composition of homeomorphisms. In fact, we claim that it is the unique extension of a \mathbb{Z} -affine isomorphism $f_1(S_1 \cap \psi^{-1}(S_2)) \rightarrow f_2(\psi(S_1) \cap S_2)$. To see this, begin by observing that $S_1 \cap \psi^{-1}(S_2)$ and $\psi(S_1) \cap S_2$ are open and dense in B_1 and B_2 respectively. This implies that for each $i = 1, 2$, $f_i(S_1 \cap \psi^{-1}(S_2))$ is open and dense in $f_i(B_i)$ as f_i is a homeomorphism. Therefore, g is determined uniquely by its restriction to $f_1(S_1 \cap \psi^{-1}(S_2))$, which maps homeomorphically onto $f_2(\psi(S_1) \cap S_2)$. This restriction is a \mathbb{Z} -affine isomorphism, by the definition of cartographic homeomorphisms and Corollary 2.22. Since the \mathbb{Z} -affine structures on $f_1(S_1 \cap \psi^{-1}(S_2))$ and on $f_2(\psi(S_1) \cap S_2)$ are isomorphic to the ones induced by inclusion into $(\mathbb{R}^n, \mathcal{A}_0)$, the restriction of the above \mathbb{Z} -affine isomorphism to each connected component of $f_1(S_1 \cap \psi^{-1}(S_2))$ is the restriction of an element of $\text{AGL}(n; \mathbb{Z})$. \square

3. ALMOST-TORIC SYSTEMS

Motivated by Symington [40] and Vũ Ngọc [46], this section introduces and studies the fundamental properties of *almost-toric systems*, a category of integrable systems generalizing that of weakly toric systems (Definition 2.27) in dimension 4 by allowing for the presence of *focus-focus* fibers, which are the Lagrangian analog of nodal fibers in Lefschetz fibrations. To define this category formally, we first recall the notion of almost-toric singular orbits, which are a special class of non-degenerate singular orbits in dimension 4. This is achieved in Section 3.1. Section 3.2 defines almost-toric systems and establishes fundamental properties of leaves and their neighborhoods. Seeing as the vertical almost-toric systems of Section 4 are both faithful and almost-toric, Section 3.3 collects results about systems that satisfy both properties.

3.1. Singular orbits.

3.1.1. *Non-degenerate singular orbits in arbitrary dimension.* The types of singular orbits considered in Section 3.1.2 are a special case of *non-degenerate* singular orbits, a condition that is briefly recalled below and should be thought of as a ‘symplectic’ Morse-Bott condition. Throughout this subsection, let (M, ω, Φ) be an integrable system so that Φ is the moment map of an effective Hamiltonian \mathbb{R}^n -action; for any $\mathbf{t} \in \mathbb{R}^n$, denote by $\phi^{\mathbf{t}} : (M, \omega) \rightarrow (M, \omega)$ the symplectomorphism induced by acting via \mathbf{t} . Moreover, for any $p \in M$, denote by \mathcal{O}_p the \mathbb{R}^n -orbit through p . If p is singular, then every point in \mathcal{O}_p is singular; thus the notion of *singular orbit* is well-defined. Next, we introduce the following useful notion.

Definition 3.1. Given an integrable system (M, ω, Φ) , the *rank* of a point $p \in M$ is given by $\text{rk } D_p \Phi$.

Remark 3.2 With the above notation, if $p \in M$ is a point of rank $0 \leq k \leq n$, the existence of a Hamiltonian \mathbb{R}^n -action implies that the orbit \mathcal{O}_p is a k -dimensional immersed, isotropic submanifold of (M, ω) that is diffeomorphic to $\mathbb{R}^{k-c(p)} \times \mathbb{T}^{c(p)}$, where $0 \leq c(p) \leq k$ is called the *degree of closedness* of \mathcal{O}_p in Zung [49, Definition 3.4]. In particular, the rank and the degree of closedness of an orbit are well-defined notions.

Fix a singular orbit $\mathcal{O} \subset M$ of rank $0 \leq k < n$ and let $p \in \mathcal{O}$; since, for all $\mathbf{t} \in \mathbb{R}^n$, $\phi^{\mathbf{t}}$ is a symplectomorphism sending \mathcal{O} to itself, it follows that, for all $\mathbf{t} \in \mathbb{R}^n$, $D_p \phi^{\mathbf{t}}$ is a symplectomorphism of $((T_p \mathcal{O}_p)^\omega / T_p \mathcal{O}_p, \Omega)$, where $(T_p \mathcal{O}_p)^\omega$ is the symplectic orthogonal of $T_p \mathcal{O}_p$ and Ω is the symplectic form induced by performing linear reduction. Thus we obtain a Lie algebra homomorphism $\mathbb{R}^n \rightarrow \text{Sp}((T_p \mathcal{O}_p)^\omega / T_p \mathcal{O}_p, \Omega)$. In fact, this homomorphism only depends on the orbit and not on the choice of point; this is because the action is by an abelian Lie group. Choosing local Darboux coordinates, it is possible to identify $\text{Sp}((T_p \mathcal{O}_p)^\omega / T_p \mathcal{O}_p, \Omega)$ with $\text{Sp}(2(n-k); \mathbb{R})$; therefore, by taking derivative at the identity, we obtain a Lie algebra homomorphism $\mathbb{R}^n \rightarrow \mathfrak{sp}(2(n-k); \mathbb{R})$ whose image is denoted by $\mathfrak{h}_{\mathcal{O}}$.

Definition 3.3. A singular orbit \mathcal{O} of rank $0 \leq k < n$ is said to be *non-degenerate* if $\mathfrak{h}_{\mathcal{O}} \subset \mathfrak{sp}(2(n-k); \mathbb{R})$ is a Cartan subalgebra.

Remark 3.4 Since $\mathfrak{sp}(2n; \mathbb{R})$ is semisimple, its Cartan subalgebras are maximal Abelian and self-normalizing. A criterion to check that a fixed point in an integrable system with n -degrees of freedom is non-degenerate is as follows (cf. Bolsinov & Fomenko [2, Definitions 1.24 and 1.25]) for details. Let p be a singular point of rank 0 in (M, ω, Φ) , where $\Phi = (H_1, \dots, H_n)$. Then, for all $i = 1, \dots, n$, the Hamiltonian

vector field of H_i vanishes at p ; thus it makes sense to consider its *linearization* at p denoted by $X_i^{\text{Lin}}(p) \in \mathfrak{sp}(2n; \mathbb{R})$. The point p is non-degenerate if $X_1^{\text{Lin}}(p), \dots, X_n^{\text{Lin}}(p)$ are linearly independent and if there exists a linear combination $\lambda_1 X_1^{\text{Lin}}(p) + \dots + \lambda_n X_n^{\text{Lin}}(p)$ with $2n$ distinct, non-zero eigenvalues.

Cartan subalgebras of $\mathfrak{sp}(2(n-k); \mathbb{R})$ have been classified up to conjugacy in Williamson [48] using the standard isomorphism $\mathfrak{sp}(2(n-k); \mathbb{R}) \cong \text{Sym}(2(n-k); \mathbb{R})$, where the latter is the Lie algebra of symmetric bilinear forms on the linear symplectic vector space $\mathbb{R}^{2(n-k)}$ with Lie bracket given by the commutator, and the isomorphism sends a quadratic polynomial to its Hamiltonian vector field. The classification of Williamson [48] is recalled below without proof.

Theorem 3.5. *Fix a positive integer n and let $\mathfrak{h} \subset \text{Sym}(2n; \mathbb{R})$ be a Cartan subalgebra. Then there exist canonical coordinates x_i, y_i of the linear symplectic vector space \mathbb{R}^{2n} , a triple $(k_e, k_h, k_{\text{ff}}) \in \mathbb{Z}_{\geq 0}^3$ with $k_e + k_h + 2k_{\text{ff}} = n$, and a basis H_1, \dots, H_n of \mathfrak{h} such that*

$$H_i = \begin{cases} \frac{x_i^2 + y_i^2}{2} & \text{if } i = 1, \dots, k_e, \\ x_i y_i & \text{if } i = k_e + 1, \dots, k_e + k_h, \end{cases}$$

and, if $i = k_e + k_h + 1, k_e + k_h + 3, \dots, k_e + k_h + 2j - 1, \dots, k_e + k_h + 2k_{\text{ff}} - 1$, then

$$\begin{aligned} H_i &= x_i y_{i+1} - x_{i+1} y_i \\ H_{i+1} &= x_i y_i + x_{i+1} y_{i+1}. \end{aligned}$$

Moreover, the triple $(k_e, k_h, k_{\text{ff}})$ determines \mathfrak{h} up to conjugacy.

Definition 3.6. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}(2n; \mathbb{R})$, the triple $(k_e, k_h, k_{\text{ff}}) \in \mathbb{Z}_{\geq 0}^3$ with $k_e + k_h + 2k_{\text{ff}} = n$ classifying it up to conjugacy is called the *Williamson triple* of \mathfrak{h} , where k_e, k_h, k_{ff} are referred to as the number of *elliptic*, *hyperbolic* and *focus-focus* components.

Going back to non-degenerate singular orbits, adapting and following Zung [49, Definition 3.4], we introduce the following terminology.

Definition 3.7. Given an integrable system (M, ω, Φ) and a non-degenerate singular orbit $\mathcal{O} \subset M$, its *Williamson type* is the element $(k, c, k_e, k_h, k_{\text{ff}}) \in \mathbb{Z}_{\geq 0}^5$, where k is the rank of \mathcal{O} , c is its degree of closedness, and $(k_e, k_h, k_{\text{ff}}) \in \mathbb{Z}_{\geq 0}^3$ is the Williamson triple of $\mathfrak{h}_{\mathcal{O}}$.

Non-degenerate singular orbits can be *linearized*, i.e., there exist canonical coordinates which also put the moment map in standard form (cf. Dufour & Molino, Eliasson, Miranda & Zung [8, 12, 28] amongst others). While the various linearization results are beyond the scope

of this article, it is worthwhile observing that they are simpler in the absence of hyperbolic blocks, *i.e.* if $k_h = 0$, and when the orbits are compact, *i.e.* if $k = c$. This is part of the motivation for introducing the singular orbits studied in Section 3.1.2. To conclude this subsection, we state the following characterization of non-degenerate, compact singular orbits of *purely elliptic type*, *i.e.* whose Williamson types are given by elements of the form $(k, k, n - k, 0, 0)$, relating them to singular toric leaves (cf. Dufour & Molino, Eliasson [8, 12]). Such orbits are henceforth referred to as *elliptic tori*.

Theorem 3.8. *Let \mathcal{O} be an elliptic torus in an integrable system (M, ω, Φ) . Then there exists a (connected) open neighborhood $V \subset M$ of \mathcal{O} whose corresponding subsystem is isomorphic to a toric system. In particular, \mathcal{O} is a singular locally weakly toric leaf.*

Theorem 3.8 is the crucial ingredient in proving that elliptic tori can be linearized (cf. Dufour & Molino, Eliasson [8, 12]).

3.1.2. *Almost-toric orbits.* Motivated by the work of Symington, Vũ Ngọc [40, 46], we distinguish the following family of singular orbits.

Definition 3.9. An orbit \mathcal{O} in an integrable system (M, ω, Φ) is said to be *almost-toric* if it is compact and non-degenerate without hyperbolic blocks.

If \mathcal{O} is an almost-toric orbit, its Williamson type (cf. Definition 3.7) is constrained to be of one of three types, namely

- *elliptic-elliptic*, *i.e.* $(0, 0, 2, 0, 0)$,
- *elliptic-regular*, *i.e.* $(1, 1, 1, 0, 0)$,
- *focus-focus*, *i.e.* $(0, 0, 0, 0, 1)$.

The first two are elliptic tori of dimension 0 and 1 respectively. On the other hand, focus-focus points are completely characterized by the following local normal form (cf. Chaperon, Eliasson, Vũ Ngọc & Wacheux [3, 12, 47]):

Let p be such a point and consider (\mathbb{R}^4, ω_0) where $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. There exist open neighborhoods $V \subset M$ and $W \subset \mathbb{R}^4$ of p and the origin respectively, such that $(V, \omega|_V, \Phi|_V)$ is isomorphic to $(W, \omega_0|_W, \mathbf{q}|_W)$ via a pair (Ψ, ψ) with $\psi(p) = 0$, where $\mathbf{q} = (q_1, q_2)$, $q_1 = x_1 y_2 - x_2 y_1$ and $q_2 = x_1 y_1 + x_2 y_2$.

Observe that the flow of the Hamiltonian vector field X_{q_1} is periodic. In fact, W can be chosen to be saturated with respect to the effective Hamiltonian S^1 -action whose moment map is given by q_1 .

Remark 3.10 An immediate consequence of the above local normal form is that the set of focus-focus points is discrete.

Elliptic tori and focus-focus points differ significantly. An immediate topological difference is that the former are leaves of the system while for the latter are not. A crucial geometric difference lies in the fact that the former support a local effective Hamiltonian \mathbb{T}^2 -action whose moment map has components that Poisson commute with the integrals of the system (we say that the action is *system-preserving*), while the latter possesses a unique (up to sign) system-preserving effective Hamiltonian S^1 -action (cf. Zung [50, Proposition 4]).

Remark 3.11 Let p be a focus-focus point in an integrable system (M, ω, Φ) and let V be an open neighborhood of p that can be put in local normal form. Then

- the restriction $\Phi|_V$ is open;
- p is the only singular point of $\Phi|_V$;
- the fibers of $\Phi|_V$ are connected;
- a fiber of $\Phi|_V$ is either diffeomorphic to a cylinder (if it does not contain p) or given by the union of two Lagrangian planes intersecting transversally at p . In particular, if the latter fiber is denoted by L , then $(V \cap L) \setminus \{p\}$ consists of two connected components, each diffeomorphic to a cylinder;
- there exist smooth sections σ_1, σ_2 defined near $\Phi(p)$ whose image lies inside V with the property that $\sigma_1(\Phi(p)), \sigma_2(\Phi(p))$ lie in distinct connected components of $(V \cap L) \setminus \{p\}$, where L denotes the leaf through p .

3.2. Definition and fundamental properties. Following Symington [40] and Vũ Ngọc [46], we introduce a category of integrable systems of two degrees of freedom that generalize weakly toric systems on 4-dimensional manifolds while retaining significant similarities.

Definition 3.12. An integrable system (M, ω, Φ) on a 4-dimensional symplectic manifold is *almost-toric* if Φ is proper onto its image and all of its singular orbits are almost-toric.

The following result can be used to describe almost-toric systems in an equivalent fashion; seeing as it follows directly from Zung [49, Proposition 3.5], its proof is omitted.

Lemma 3.13. *Let (M, ω, Φ) be an integrable system on a 4-dimensional symplectic manifold with compact fibers, all of whose singular orbits are non-degenerate without hyperbolic blocks. Then the singular orbits are compact and, in particular, almost-toric.*

The above notion of almost-toric system differs slightly from that in Vũ Ngọc [46], for Definition 3.12 only requires that Φ be proper *onto its image*, as opposed to being proper.

Remark 3.14 Almost-toric systems form a full subcategory of $\mathcal{IS}(2)$, henceforth referred to as the *category of almost-toric systems* and denoted by \mathcal{AT} .

Remark 3.15 Note that saturated subsystems of almost-toric systems are almost-toric.

The restriction on the types of singular orbits in an almost-toric system (M, ω, Φ) implies the singular leaves of Φ are either elliptic tori or contain at least one focus-focus singular orbit. That dichotomy arises because the local normal form for elliptic singular orbits implies such orbits (which are elliptic tori) make up whole connected components of a fiber of Φ . Henceforth, leaves that contain focus-focus orbits are referred to as *focus-focus leaves*. Denote by \mathcal{L}_{ff} the set of points in the leaf space \mathcal{L} corresponding to leaves containing focus-focus singular orbits. Then $\mathcal{L}_{\text{sing}} = \mathcal{L}_{\text{e}} \cup \mathcal{L}_{\text{ff}}$, where \mathcal{L}_{e} is the elliptic part of \mathcal{L} . Elements of \mathcal{L}_{ff} are called *focus-focus values (in the leaf space)*.

Definition 3.16. Let $c \in \mathcal{L}_{\text{ff}}$ be a focus-focus value in the leaf space of an almost-toric system (M, ω, Φ) . The *multiplicity* of c , denoted $r_c \geq 1$, is the number of focus-focus singular orbits in the corresponding leaf of Φ .

Focus-focus values, counted with multiplicity, are an invariant of the isomorphism class of an almost-toric system. Henceforth, all focus-focus values are counted with multiplicity unless otherwise stated.

The topology of focus-focus leaves is well-known and is completely determined by the finite number $r \geq 1$ of focus-focus singular points contained in the corresponding leaf. In particular, each focus-focus leaf is homeomorphic to a torus with r homologous cycles, each collapsed to a point (cf. Bolsinov & Fomenko [2, Chapter 9.8]). Lying at the heart of this result is the existence of a vector field tangent to a focus-focus leaf, whose flow is periodic and whose fixed points are precisely the focus-focus points.

In fact, the S^1 -action on any focus-focus leaf can be extended to a suitable neighborhood of the leaf, one that is saturated with respect to the quotient map to the leaf space. To prove this, we start by establishing the existence of this suitable neighborhood.

Proposition 3.17. *Given an almost-toric system (M, ω, Φ) , any focus-focus leaf $L \subset M$ admits an open neighborhood V satisfying the following properties:*

- L is the only singular leaf in V ;
- V is saturated with respect to the quotient map $q : M \rightarrow \mathcal{L}$;
- V contains at most one leaf of any fiber of Φ .

Proof. Fix a focus-focus leaf L . First, we show that L admits an open neighborhood Z that is saturated with respect to q and in which L is the only focus-focus leaf. Suppose not; then there exists a sequence of focus-focus points $\{p_n\}$ with the property that $\Phi(p_n) \rightarrow \Phi(L)$; since Φ is proper onto its image, there exists a convergent subsequence $p_{n_j} \rightarrow p$. The limit point p is necessarily singular, but the local normal form for almost-toric orbits yields a contradiction.

Fix such a neighborhood Z and let $p_1, \dots, p_r \in L$ denote the focus-focus points in L . For $i = 1, \dots, r$, let $V_i \subset Z$ be an open neighborhood of p_i that can be put in local normal form (cf. Section 3.1.2). Consider the subset

$$\hat{V} := \bigcup_{i=1}^r \bigcup_{\mathbf{t} \in \mathbb{R}^2} \phi^{\mathbf{t}}(V_i)$$

where, as in Section 3.1.2, $\phi^{\mathbf{t}}$ denotes the Hamiltonian action by $\mathbf{t} \in \mathbb{R}^2$; this is the union of the orbits that intersect at least one V_i . Since, for $i = 1, \dots, r$, V_i is open, so is \hat{V} ; moreover, \hat{V} contains L because $L = \bigcup_{i=1}^r \bigcup_{\mathbf{t} \in \mathbb{R}^2} \phi^{\mathbf{t}}(V_i \cap L)$. Next we show that \hat{V} is also saturated with respect to q . To see this, observe that if $p \in V_i \setminus L$, then the leaf passing through p is contained in Z and is not equal to L , thus implying that it is not a focus-focus leaf. Since p is regular (by the local normal form for focus-focus points), the leaf through p is regular and is, therefore, an orbit of the Hamiltonian \mathbb{R}^2 -action, which is contained in \hat{V} by construction.

The above construction does not necessarily guarantee that \hat{V} contains at most one leaf of any fiber of Φ . However, the local normal form for a focus-focus point and the \mathbb{R}^2 action can be used to determine a possibly smaller neighborhood in which that property holds. Specifically, fix some $i \in \{1, \dots, r\}$. There exists a smooth section σ_i of Φ defined near $\Phi(p_i)$ whose image is contained in V_i ; this implies that $\sigma_i(\Phi(p_i)) \in (V_i \cap L) \setminus \{p_i\}$ (cf. Remark 3.11). The structure of the focus-focus leaf L implies that there exists $\mathbf{t}_0 \in \mathbb{R}^2$ with the following property

- if $r = 1$, then $\sigma_i(\Phi(p_i))$ and $\phi^{t_0}(\sigma_i(\Phi(p_i)))$ lie in different connected components of $(V_i \cap L) \setminus \{p_i\}$ (cf. Vũ Ngọc [44]);
- if $r > 1$, there exists $j \neq i$ with $\phi^{t_0}(\sigma_i(\Phi(p_i))) \in V_j$ (cf. Bolsinov & Fomenko [2, Chapter 9.8]).

In other words, the section σ_i flows out of V_i and into V_j (and if $i = j$, then it approaches V_i from the ‘opposite’ side). Let $c \in \mathbb{R}^2$ be sufficiently close to $\Phi(p_i)$ so that $\sigma_i(c)$ is defined; then the above properties show that

- if $r = 1$, $\Phi^{-1}(c) \cap \hat{V}$ is connected;
- if $r > 1$, the intersections $\Phi^{-1}(c) \cap V_i$ and $\Phi^{-1}(c) \cap V_j$ lie on the same leaf of $\Phi^{-1}(c)$.

In the latter case, using again the structure of the focus-focus leaf L (cf. Bolsinov & Fomenko [2, Chapter 9.8]), we can iterate the above argument finitely many times to ensure that, for all c sufficiently close to $\Phi(L)$ ($= \Phi(p_i)$ for all $i = 1, \dots, r$), $\Phi^{-1}(c) \cap \hat{V}$ is connected. This shows that \hat{V} can be shrunk as desired. \square

Corollary 3.18. *The set of focus-focus values in the leaf space of an almost-toric system is discrete.*

Any open neighborhood of a focus-focus leaf as in Proposition 3.17 is henceforth referred to as a (q) -saturated regular neighborhood of a focus-focus leaf. A saturated regular neighborhood has the necessary S^1 -symmetry.

Proposition 3.19. *Given a focus-focus leaf L of an almost-toric system (M, ω, Φ) , any saturated regular neighborhood of L admits a local system-preserving Hamiltonian S^1 -action.*

Sketch of proof. The ideas behind proving this result are known (cf. Bolsinov & Fomenko [2, Lemma 9.8] and Zung [50, Section 3]), but the key ideas are provided below for completeness. Let L be a focus-focus leaf and let V be a saturated regular neighborhood of L . Let p be a focus-focus point on L ; the local normal form for p implies that, near p , there exists a local system-preserving Hamiltonian S^1 -action. By construction of V , this action can be extended to the whole of V and is independent (up to sign) of the choice of focus-focus point $p \in L$. \square

In fact, a saturated regular neighborhood of a focus-focus leaf (of multiplicity one) is a singular Liouville foliation of (simple) focus-focus type in the sense of Vũ Ngọc [45, Definition 2.4]. Moreover, saturated regular neighborhoods of focus-focus leaves ought to be thought of as analogous to the neighborhoods of elliptic tori that can be put in local normal form. For instance, the following result holds.

Proposition 3.20. *Let V be a saturated regular neighborhood of a focus-focus leaf L in an almost-toric system (M, ω, Φ) . The subsystem $(V, \omega|_V, \Phi|_V)$ is faithful and almost-toric.*

Proof. By construction, the fibers of $\Phi|_V$ are connected and all singular orbits in the subsystem are almost-toric. To prove the result, it suffices to show that $\Phi|_V$ is proper onto its image, for then the subsystem is almost-toric by Definition 3.12 and faithful by Lemma 2.15. Set $B_V := \Phi|_V(V)$; to prove that $\Phi|_V$ is proper onto its image, it suffices to check that it is proper at every point $c \in B_V$. Observe that, by definition of V , B_V contains only one singular value, which equals $\Phi(L)$. If $c \neq \Phi(L)$, then $\Phi|_V$ is proper at c as $\Phi|_{V \setminus L}$ is a submersion with compact and connected fibers. If $c = \Phi(L)$, then arguing as in the second half of the proof of Lemma 2.15, it can be shown that $\Phi|_V$ is also proper at c . \square

To summarize the above results and motivate subsequent sections, we state the following description of regular neighborhoods of leaves in almost-toric systems.

Corollary 3.21. *Given an almost-toric system (M, ω, Φ) , any leaf L admits an open neighborhood V satisfying the following properties:*

- V is saturated with respect to the quotient map $q : M \rightarrow \mathcal{L}$;
- the subsystem $(V, \omega|_V, \Phi|_V)$ is faithful and admits a system-preserving Hamiltonian S^1 -action.

3.3. Faithful almost-toric systems. By Corollary 3.21, any leaf in an almost-toric system admits an open neighborhood whose corresponding subsystem is faithful almost-toric. Therefore it makes sense to view faithful almost-toric systems as building blocks of almost-toric systems. Throughout this subsection, let (M, ω, Φ) be a faithful almost-toric system. Moreover, fix the identification between \mathcal{L} and $B = \Phi(M)$ and denote by B_{ff} the image of \mathcal{L}_{ff} under this identification.

Remark 3.22 The local normal form for focus-focus points implies that $B_{\text{ff}} \subset \text{Int}(B)$. Thus, by Corollary 3.18, B_{ff} is discrete in $\text{Int}(B)$.

The arguments in Vũ Ngọc [46, Proposition 3.9] can be used to prove the following result, stated without proof.

Lemma 3.23. *Given a faithful almost-toric system on a connected symplectic manifold, the subsets B_{reg} and $\text{Int}(B)$ of B are path-connected.*

The restriction on the types of singular orbits, as well as faithfulness, imply the following useful fact for faithful almost-toric systems.

Lemma 3.24. *Given a faithful almost-toric system (M, ω, Φ) and a continuous path $\gamma : [0, 1] \rightarrow B$, the subset $\Phi^{-1}(\gamma([0, 1]))$ is path-connected.*

Proof. Fix a path γ as in the statement. By the local normal form for almost-toric singular points, given any point $c \in B$, there exists an open, path-connected neighborhood $U \subset B$ of c and a continuous section $\sigma : U \rightarrow M$. The image of γ is contained in the union of finitely many such neighborhoods. Connectedness of the fibers of Φ then implies the desired result. \square

Remark 3.25 Given a faithful almost-toric system (M, ω, Φ) ,

- the smooth manifold with corners structure on B_{lt} , the locally weakly toric part of the leaf space, extends to all of B ;
- the image of elliptic tori is precisely $\partial_{\infty} B = \partial_{\infty} B_{\text{lt}}$, where corners and facets (or *curved edges*) of B are the images of elliptic-elliptic and elliptic-regular points, respectively (cf. Remark 2.43).
- The set of focus-focus values $B_{\text{ff}} \subset \text{Int}(B)$ is at most countable, and the set of its limit points in \mathbb{R}^2 is contained in $\partial B \setminus \partial_{\infty} B$.

Example 3.26 Suppose (M, ω) is symplectomorphic to a K3 surface (for example, a smooth quartic hypersurface in $\mathbb{C}P^3$). Such a symplectic manifold admits singular Lagrangian fibrations over S^2 in which each singular fiber has a neighborhood that, with respect to an appropriate coordinate chart on S^2 , defines a faithful almost-toric system that is a regular saturated neighborhood of a focus-focus fiber with one singular orbit. Suppose $\Pi : (M, \omega) \rightarrow S^2$ is such a fibration and let $p \in S^2$ be the image of a regular fiber. Let $N = M \setminus \Pi^{-1}(p)$ and let $\phi : S^2 \setminus p \rightarrow \mathbb{R}^2$ be an embedding. Then $(N, \omega|_N, \phi \circ \Pi|_N)$ defines a faithful almost-toric system with 24 focus-focus leaves.

A natural question arising from Remark 3.25 is whether the \mathbb{Z} -affine structure \mathcal{A}_{lt} on B_{lt} can be extended to B . The presence of focus-focus fibers prevents this from happening, as the \mathbb{Z} -affine structure on any neighborhood of a focus-focus value has non-trivial affine holonomy.

Theorem 3.27 (Zung [50], Prop. 3 and Cor. 1). *Let (M, ω, Φ) be faithful almost-toric system and let $U \subset B = \Phi(M)$ be an open neighborhood of a focus-focus value c , sufficiently small such that U contains*

no other focus-focus value. The affine holonomy of the \mathbb{Z} -affine structure on $U \setminus \{c\} \subset B_{\text{lt}}$ is given, in a suitable basis, by

$$(3.1) \quad \begin{aligned} \pi_1(U) &\cong \mathbb{Z} \rightarrow \text{AGL}(2; \mathbb{Z}) \\ k &\mapsto \left(\begin{pmatrix} 1 & 0 \\ kr_c & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \end{aligned}$$

where $r_c \geq 1$ is the multiplicity of c .

Remark 3.28 The eigenspace associated to the above representation reflects the uniqueness (up to sign) of the local effective system-preserving Hamiltonian S^1 -action in a neighborhood of a singular fiber containing focus-focus points. With respect to the local choice of basis in Theorem 3.27, this action is induced by the first integral of the moment map.

Another natural question is which almost-toric systems are isomorphic to toric ones? Certainly, the system must not contain focus-focus points. If the system is faithful, it suffices that there be a \mathbb{Z} -affine immersion of the moment map image into $(\mathbb{R}^2, \mathcal{A}_0)$. However, as the next example illustrates, an absence of focus-focus points does not suffice.

Example 3.29 Let (M, ω, Φ) be the Delzant system defined (up to isomorphism) by the moment map image $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$. Then, up to isomorphism, $M = S^2 \times \mathbb{R} \times S^1$ with $\omega = \omega_{S^2} \oplus da \wedge d\theta$, where ω_{S^2} is a suitable symplectic form on S^2 , and $\Phi(p, a, \theta) = (h(p), a)$, where $h : S^2 \rightarrow \mathbb{R}$ is a suitable height function. Let $(M, \omega, \hat{\Phi})$ be the almost-toric system defined by $\hat{\Phi} = g_1 \circ \mu$ where $g_1(x, y) = (e^x \cos y, e^x \sin y)$ and consider the symplectic \mathbb{Z} -action on M given by $k \cdot (p, a, \theta) = (p, a + 2\pi k, \theta)$. The quotient is $M' = S^2 \times T^2$. Observe that $\hat{\Phi}$ is invariant under this \mathbb{Z} -action. Therefore, taking the quotient yields an almost-toric system (M', ω', Φ') . This system has no focus-focus points but is not isomorphic to a toric system: it is faithful, and yet has a moment map image that is not simply connected and hence not homeomorphic to a polygon.

Remark 3.30 Consider a faithful almost-toric system (M, ω, Φ) with focus-focus points and suppose that (f, S) is a cartographic pair. The definition of a cartographic pair (Definition 2.45) and the local normal form for singular orbits of elliptic type together imply that the cartographic image of curved edges in S are line segments whose tangent vectors can be chosen to have coprime integer coefficients, and then those tangent vectors span the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ whenever the edges are incident to a corner of S .

An important question is thus, when does a faithful almost-toric system admit a cartographic homeomorphism? Addressing this problem in full generality is beyond the scope of this paper. However, in light of Corollary 3.21, a natural family of faithful almost-toric systems to consider arises: namely, those admitting a system-preserving S^1 -action.

4. VERTICAL ALMOST-TORIC SYSTEMS

This section studies vertical almost-toric systems and proves the main results of the paper. As pointed out in the introduction, vertical almost-toric systems are closely related to (generalized) semi-toric systems (cf. Pelayo & Ratiu & Vũ Ngọc, Vũ Ngọc [33, 46]). Many of the ideas and proofs that appear in this section are inspired by the work in *op. cit.* Section 4.1 introduces vertical almost-toric systems and establishes their basic properties, as well as proving that such systems can be viewed as ‘building blocks’ of almost-toric systems (see Proposition 4.9 for a formal statement). However, unlike their almost-toric counterparts, vertical almost-toric systems possess a global Hamiltonian S^1 -action, a fact that has some important geometric consequences which are investigated in Section 4.2. Section 4.3 is devoted to proving that vertical almost-toric systems admit cartographic homeomorphisms constructed by choosing suitable vertical cuts of their moment map images (Theorem 4.24). The set of all cartographic homeomorphisms of a given vertical almost-toric system is described in Section 4.4, which generalizes Vũ Ngọc [46, Section 4]. Finally, Section 4.5 shows that cartographic homeomorphisms can be made smooth by modifying them on arbitrarily small neighborhoods of the corresponding cuts (Theorem 4.48). This result provides representatives (which we call $\boldsymbol{\eta}$ -cartographic) in the isomorphism class of any vertical almost-toric system, representatives that are particularly useful when defining surgeries on vertical almost-toric systems (cf. the forthcoming [20]).

4.1. Definition and basic properties. To the best of our knowledge, there are no general results regarding the existence of cartographic homeomorphisms for faithful almost-toric systems, even if the total space is closed (cf. Leung & Symington [25] and Symington [40]). However, the existence results of Pelayo & Ratiu & Vũ Ngọc [33] and Vũ Ngọc [46] hint at the fact that if the first integral in an almost-toric system $(M, \omega, \Phi = (J, H))$ is the moment map for an effective Hamiltonian S^1 -action, then some control on J suffices.

Definition 4.1. The category of *vertical almost-toric systems*, denoted by \mathcal{VAT} , has objects and morphisms as follows.

- Objects: *vertical almost-toric systems*, i.e., 4-dimensional faithful almost-toric systems $(M, \omega, \Phi = (J, H))$ satisfying:
 - (V1) the total space M is connected;
 - (V2) the first integral J is the moment map of an effective Hamiltonian S^1 -action;
 - (V3) the set of critical values of J does not contain any limit points in $J(M)$;
 - (V4) any fiber of J contains at most finitely many isolated fixed points of the S^1 -action;
 - (V5) any fiber of J is connected.
- Morphisms: *isomorphisms of vertical almost-toric systems*, i.e. isomorphisms (Ψ, ψ) of integrable systems between vertical almost-toric systems $(M_i, \omega_i, \Phi_i = (J_i, H_i))$, $i = 1, 2$, where $\psi : B_1 \rightarrow B_2$ is of the form $\psi = (\psi^{(1)}, \psi^{(2)})$ with $\psi^{(1)}(x, y) = x$.

Note that the leaves of a vertical almost-toric system are compact because the system is almost-toric and each fiber consists of a single leaf (and hence is connected) because the system is faithful. Moreover, property (V5) is equivalent to path-connectedness of the fibers of J by the local normal form for Hamiltonian S^1 -action (cf. Guillemin & Sternberg, Marle [15, 26]).

Remark 4.2 Vertical almost-toric systems do not form a full subcategory of \mathcal{AT} , as the above notions are not invariant under general isomorphisms of integrable systems. The more restrictive notion of isomorphisms of vertical almost-toric systems places special emphasis on the S^1 -action and suggests that the geometry of a vertical almost-toric system $(M, \omega, \Phi = (J, H))$ is closely related to the geometry of the triple (M, ω, J) obtained by ‘forgetting’ the second integral (cf. Hohloch & Sabatini & Sepe [19] and the forthcoming [21]).

Henceforth, let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system. The important property of connectedness of the fibers of J can be checked directly on the moment map image.

Lemma 4.3. *Let $(M, \omega, \Phi = (J, H))$ be a faithful almost-toric system. Then the fibers of J are path-connected if and only if, for all $x_0 \in \mathbb{R}$, the intersection $B \cap \{(x, y) \mid x = x_0\}$ is either empty or path-connected.*

Proof. If the fibers of J are path-connected, then the claimed result holds. Conversely, suppose that $B \cap \{(x, y) \mid x = x_0\} \neq \emptyset$ is path-connected; the aim is to prove that $J^{-1}(x_0) = \Phi^{-1}(B \cap \{(x, y) \mid x = x_0\})$ is path-connected. Let $p_1, p_2 \in J^{-1}(x_0)$ and set $c_i = \Phi(p_i)$, for $i = 1, 2$.

Then the vertical segment joining c_1 to c_2 is contained in B by assumption; Lemma 3.24 then implies that there exists a path joining p_1 to p_2 contained in the preimage under Φ of that vertical segment. Since that preimage is contained in $J^{-1}(x_0)$, this shows the desired result. \square

In fact, the path-connectedness of the intersection of B with any vertical line in turn implies that intersections of vertical lines with the interior of B are path-connected.

Lemma 4.4. *The set $\text{Int}(B) \cap \{(x, y) \mid x = x_0\}$ is path-connected for any $x_0 \in \text{pr}_1(\text{Int}(B))$.*

Proof. Suppose not, then there exists $x_0 \in \text{pr}_1(\text{Int}(B))$ such that $\text{Int}(B) \cap \{(x, y) \mid x = x_0\}$ is not path-connected. Since $B \cap \{(x, y) \mid x = x_0\}$ is nonempty and path-connected, it follows that there exist $y_1 < y_0 < y_2$ such that $(x_0, y_0) \in \partial_\infty B \subset \partial B$ and $(x_0, y_i) \in \text{Int}(B)$, for $i = 1, 2$. However this is impossible because it forces a disconnection of the intersection of B and a vertical line as follows.

For $i = 1, 2$, because the point (x_0, y_i) is in $\text{Int}(B)$, there is open disk of radius r_i centered at (x_0, y_i) that is a subset of $\text{Int}(B)$. And because $(x_0, y_0) \in \partial_\infty B \subset \partial B$, there exists a sequence of points $\{(x'_n, y'_n)\}$ such that each (x'_n, y'_n) is in the open ball of radius $\frac{1}{n}$ centered at (x_0, y_0) but $(x'_n, y'_n) \notin B$. Consequently, for each n such that $\frac{1}{n} < \min(r_1, r_2)$ the intersection $B \cap \{(x, y) \mid x = x_n\}$ is disconnected because $y_1 < y_0 - \frac{1}{n} < y'_n < y_0 + \frac{1}{n} < y_2$, with $(x'_n, y_1), (x'_n, y_2) \in B$ and $(x'_n, y'_n) \notin B$. \square

Corollary 4.5. *The moment map image of a vertical almost-toric system is contractible.*

Proof. Fix a vertical almost-toric system with moment map image B . As noted in Remark 3.25, B is a manifold with corners. The inclusion $\text{Int}(B) = B \setminus \partial_\infty B \hookrightarrow B$ is a homotopy equivalence because B is homeomorphic to a smooth manifold with boundary, which is homotopy equivalent to the complement of its boundary. Thus it suffices to prove that $\text{Int}(B)$ is contractible. By Lemma 3.23, $\text{Int}(B)$ is path-connected and so is $\text{pr}_1(\text{Int}(B))$. Moreover, $\text{Int}(B) \subset \mathbb{R}^2$ is open and pr_1 is an open map. Therefore $\text{pr}_1 : \text{Int}(B) \rightarrow \text{pr}_1(\text{Int}(B))$ is a surjective submersion whose fibers are diffeomorphic to \mathbb{R} by Lemma 4.4. Thus it is a fiber bundle (cf. Meigniez [27, Page 3778]). Since $\text{pr}_1(\text{Int}(B))$ is an interval, the bundle is trivial. Since both the base and the fiber of this trivial bundle are contractible, so is the total space. \square

Corollary 4.6. *A vertical almost-toric system is Delzant if and only if it is toric.*

Vertical almost-toric systems behave well with respect to taking certain ‘vertical subsystems’:

Proposition 4.7. *Let (M, ω, Φ) be a vertical almost-toric system and let $U \subset B$ be open and path-connected. Then the subsystem of (M, ω, Φ) relative to U is a faithful almost-toric system that satisfies properties (V1) – (V4) of a vertical almost-toric system. Moreover, if for all $x_0 \in \mathbb{R}$, $\{(x, y) \mid x = x_0\} \cap U$ is either empty or path-connected, then the subsystem relative to U is, in fact, a vertical almost-toric system.*

Proof. The subsystem relative to U is faithful by Remark 2.12 and almost-toric by Remark 3.15. The total space is connected because U is connected and the subsystem is faithful. Property (V2) is satisfied because $\Phi^{-1}(U)$ is a union of orbits of its first integral. Finally, Properties (V3) and (V4) are preserved under taking subsystems. This proves the first assertion.

Assume that the intersection of U with any vertical line is either empty or path-connected. Because the subsystem relative to U is faithful almost-toric, Lemma 4.3 implies the fibers of $J|_U$ are path-connected, as desired. \square

Remark 4.8 Vertical almost-toric systems are very closely related to the generalized semi-toric systems introduced in Pelayo & Ratiu & Vũ Ngọc [33]. Both are special cases of integrable systems on 4-manifolds in which all singular orbits are almost-toric and in which J is assumed to generate a Hamiltonian S^1 action and have connected fibers. But there are some significant differences:

- The definition of generalized semi-toric systems assumes properness of Φ (cf. Pelayo & Ratiu & Vũ Ngọc [33, Definition 1.3]) which, together with the other properties, can be proved to imply connectedness of the fibers of Φ (cf. Pelayo & Ratiu & Vũ Ngọc [32]).
- On the other hand, as a result of being faithful, the moment map of a vertical almost-toric system is assumed to be proper onto its image and to have connected fibers.

Accordingly, vertical almost-toric systems can be thought of as subsystems of generalized semi-toric systems. However, the notion of isomorphism of vertical almost-toric systems is strictly weaker than that of generalized semi-toric systems, as it allows for diffeomorphisms of the moment map images whose derivative has negative determinant (cf. Pelayo & Ratiu & Vũ Ngọc [33, Definition 2.4]).

Almost-toric systems (cf. Definition 3.12) are a source of examples of vertical almost-toric systems.

Proposition 4.9. *Given an almost-toric system (M, ω, Φ) , any leaf L admits an open neighborhood V such that $(V, \omega|_V, \Phi|_V)$ is isomorphic to a vertical almost-toric system.*

Proof. A leaf of an almost-toric system is either weakly toric or is a focus-focus leaf. In the former case, the local normal forms for elliptic tori yield the result. Thus, suppose that L is a focus-focus leaf and let V be a saturated regular neighborhood of L . By construction, V is connected; moreover, Proposition 3.20 gives that $(V, \omega|_V, \Phi|_V)$ is faithful almost-toric. Proposition 3.19 gives that there exists a system-preserving effective Hamiltonian S^1 -action on $(V, \omega|_V, \Phi|_V)$; in other words, that subsystem is isomorphic to an almost-toric system whose first integral is the moment map of an effective Hamiltonian S^1 -action. Thus, without loss of generality, it may be assumed that the first component of $\Phi|_V$ is the moment map of an effective Hamiltonian S^1 -action. By construction of V , properties (V3) and (V4) of a vertical almost-toric system hold for $(V, \omega|_V, \Phi|_V)$. Set $c = \Phi(L)$. Let $U \subset \Phi|_V(V)$ be an open neighborhood of c with the property that its intersection with any vertical line is either empty or path-connected. Using Proposition 4.7, the subsystem relative to U of $(V, \omega|_V, \Phi|_V)$ is vertical almost-toric as desired. \square

Remark 4.10 The vertical almost-toric systems of Proposition 4.9 are not necessarily generalized semi-toric because their moment map images are not necessarily closed in \mathbb{R}^2 , while those of generalized semi-toric systems are.

4.2. Geometric implications of the S^1 -action. Fix a vertical almost-toric system $(M, \omega, \Phi = (J, H))$ and denote the set of fixed points of the Hamiltonian S^1 -action, one of whose moment maps is J , by M^{S^1} . Its connected components are either isolated fixed points or *symplectic fixed surfaces*, i.e., symplectic submanifolds of dimension 2 that are fixed under the S^1 -action. This is a consequence of the Marle-Guillemin-Sternberg local normal form (cf. Guillemin & Sternberg [15], Marle [26]).

Proposition 4.11. *Let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system such that M^{S^1} contains a fixed surface Σ . Then $J(\Sigma)$ is a global extremum of J .*

Proof. To show that $J(\Sigma)$ is a global extremum of J it suffices to show that it cannot lie in the interior of the interval $J(M)$. Assume the contrary: then $J^{-1}(J(\Sigma)) = \Sigma$ as the fibers of J are connected by property (V5). Since $J(\Sigma) \in \text{Int}(J(M))$, it follows that $J(M) \setminus J(\Sigma)$

is disconnected, thus implying that $M \setminus \Sigma = J^{-1}(J(M) \setminus J(\Sigma))$ is disconnected. However, this is absurd, since $\Sigma \subset M$ is a submanifold of codimension 2. \square

In fact, given a vertical almost-toric system $(M, \omega, \Phi = (J, H))$ for which M^{S^1} contains a fixed surface Σ , each point of Σ belongs to some singular orbit of elliptic type, so $\Phi(\Sigma) \subset \partial_\infty B$. Moreover $\Phi(\Sigma) \subset \{(x, y) \mid x = J(\Sigma)\}$. The image $\Phi(\Sigma) \subset B$ is henceforth referred to as a *vertical edge* of B . The following result provides a characterization of vertical edges.

Proposition 4.12. *Let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system and suppose that there exists $x_0 \in J(M)$ and distinct points $c_1^\infty, c_2^\infty, c_3^\infty \in \partial_\infty B \cap \{(x, y) \mid x = x_0\}$. Then $\partial_\infty B \cap \{(x, y) \mid x = x_0\}$ is a vertical edge.*

Proof. For $i = 1, 2, 3$, set $c_i^\infty = (x_0, y_i)$ and assume, without loss of generality, that $y_1 < y_2 < y_3$. Since the fibers of J are connected by property (V5), it follows that

$$\{(x, y) \mid x = x_0, y_1 \leq y \leq y_3\} \subset B.$$

The local normal forms for almost-toric singular orbits in the presence of a system-preserving Hamiltonian S^1 -action (cf. Remark 2.38), together with faithfulness of $(M, \omega, \Phi = (J, H))$, force $\Phi^{-1}(c_2^\infty)$ to be a singular orbit of elliptic-regular type all of whose points are critical for J . Then $\Phi^{-1}(c_2^\infty)$ lies on a fixed surface and Proposition 4.11, together with connectedness of the fibers of J , completes the proof. \square

Combining Propositions 4.11 and 4.12, we obtain the following result.

Corollary 4.13. *Let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system and consider a point $x_0 \in \text{Int}(J(M))$. Then the intersection $\partial_\infty B \cap \{(x, y) \mid x = x_0\}$ consists of at most two points.*

To conclude this section, we state the following result, which is entirely analogous to Hohloch & Sabatini & Sepe [19, Lemma 3.3].

Proposition 4.14. *Let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system. Then the isolated fixed points in M^{S^1} are either*

- *focus-focus singular orbits, or*
- *elliptic-elliptic singular orbits whose image is not a corner adjacent to a vertical edge.*

4.3. Cuts and cartographic homeomorphisms. An important property of vertical almost-toric systems is that, as proved below, assuming the following mild restriction, they admit cartographic homeomorphisms (cf. Theorem 4.24).

Definition 4.15. A vertical almost-toric system $(M, \omega, \Phi = (J, H))$ is said to be *simple* if the following property holds:

(V6) Any focus-focus value has multiplicity 1.

The above condition is generic according to Zung [49]. Moreover, it is invariant under taking isomorphisms of vertical almost-toric systems and descends to saturated subsystems satisfying all the hypotheses of Proposition 4.7.

While the existence of cartographic homeomorphisms is expected to hold without imposing property (V6), proofs would require a more detailed understanding of neighborhoods of focus-focus fibers with more than one focus-focus point, which is beyond the scope of this paper (cf. Vũ Ngọc [45, Section 7] for some sketched proofs in this direction). To the best of our knowledge, all existing proofs of the existence of cartographic homeomorphisms assume, either tacitly or explicitly, simplicity of the system (cf. Vũ Ngọc [46, Step 4 of the Proof of Theorem 3.8] and Pelayo & Ratiu & Vũ Ngọc [33, Step 4 of the Proof of Theorem B]).

Remark 4.16 Note that the notion of *simple* used in the literature on semi-toric systems differs from ours: There it means that there exists at most one focus-focus point on any fiber of J .

The aim of this section is to prove that any simple vertical almost-toric system admits a cartographic homeomorphism that, loosely speaking, encodes the affine holonomy of the \mathbb{Z} -affine structure on the locally weakly toric part of the leaf space (cf. Theorem 4.24 for a precise statement). It is important to remark that there are proofs of this result for special families of vertical almost-toric systems (cf. Vũ Ngọc [46, Theorem 3.8] and Pelayo & Ratiu & Vũ Ngọc [33, Theorem B] for semi-toric and generalized semi-toric systems respectively). Those proofs are utilized and adjusted as needed in what follows.

Lemma 4.17. *Let $(M, \omega, \Phi = (J, H))$ be a vertical almost-toric system without focus-focus points. Then there exists a cartographic pair (f, B) , where f is of the form*

$$f(x, y) = (f^{(1)}, f^{(2)})(x, y) = (x, f^{(2)}(x, y)).$$

In particular, $(M, \omega, f \circ \Phi)$ is a Delzant system and $f(B) \subset \mathbb{R}^2$ is locally convex.

Proof. The proof is analogous to Pelayo & Ratiu & Vũ Ngọc [33, Step 2 of Theorem B], but is included in this paper for completeness.

The lack of focus-focus points implies $B = B_{\text{lt}}$, thus B inherits a \mathbb{Z} -affine structure. By Corollary 4.5, B is contractible, so there exists a developing map $f : B \cong \tilde{B} \rightarrow \mathbb{R}^2$. By definition of the \mathbb{Z} -affine structure on B , since the first integral J of $(M, \omega, \Phi = (J, H))$ is the moment map of an effective Hamiltonian S^1 -action, one can choose the above developing map to be of the form $f(x, y) = (x, f^{(2)}(x, y))$ for some smooth function $f^{(2)} : B \rightarrow \mathbb{R}$. Fix such a choice.

To show that f is the required cartographic homeomorphism, it suffices to show that f is injective. If $f(x_0, y_0) = f(x_1, y_1)$, one gets immediately $x_0 = x_1$. The map $f^{(2)}(x_0, \cdot) : B \cap \{(x, y) \mid x = x_0\} \rightarrow \mathbb{R}$ is strictly monotone as $\frac{\partial f^{(2)}}{\partial y}$ does not vanish on B , because f is locally a diffeomorphism. This implies that $y_0 = y_1$ as required.

Since $(M, \omega, \Phi = (J, H))$ is vertical almost-toric and because of the form of f , $(M, \omega, f \circ \Phi)$ is vertical almost-toric. In fact, it is toric because $f \circ \Phi : (M, \omega) \rightarrow \mathbb{R}^2$ is the moment map of an effective Hamiltonian \mathbb{T}^2 -action. By Corollary 4.6, $(M, \omega, f \circ \Phi)$ is Delzant and by Corollary 2.36, $f(B)$ is locally convex. \square

Remark 4.18 There is freedom in choosing the cartographic homeomorphism as in Lemma 4.17. If $f, \hat{f} : B \rightarrow \mathbb{R}^2$ are two such choices, then by Remark 2.25, there exists an element $h \in \text{AGL}(2; \mathbb{Z})$ such that $\hat{f} = h \circ f$. However, since f and \hat{f} are chosen to have the restricted form given by Lemma 4.17, it follows that the element h belongs, in fact, to the subgroup $\text{Vert}(2; \mathbb{Z})$ consisting of \mathbb{Z} -affine transformations that fix all vertical lines in \mathbb{R}^2 . Using the standard basis for \mathbb{R}^2 , elements of $\text{Vert}(2; \mathbb{Z})$ are of the form

$$\left(\begin{pmatrix} 1 & 0 \\ k & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right),$$

where $k \in \mathbb{Z}$ and $a \in \mathbb{R}$. In fact, if f is a cartographic homeomorphism as in Lemma 4.17 and $h \in \text{Vert}(2; \mathbb{Z})$, then $h \circ f$ is also a cartographic homeomorphism satisfying the conditions of Lemma 4.17.

Let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system with $B = \Phi(M)$ that contains at least one focus-focus point. An example is shown in Figure 4.1. We introduce vertical ‘cuts’ at the focus-focus values. This terminology is motivated by Vũ Ngọc[46]. Symington [40]

refers to these as ‘eigenrays’ in the more general context of (faithful) almost-toric systems.

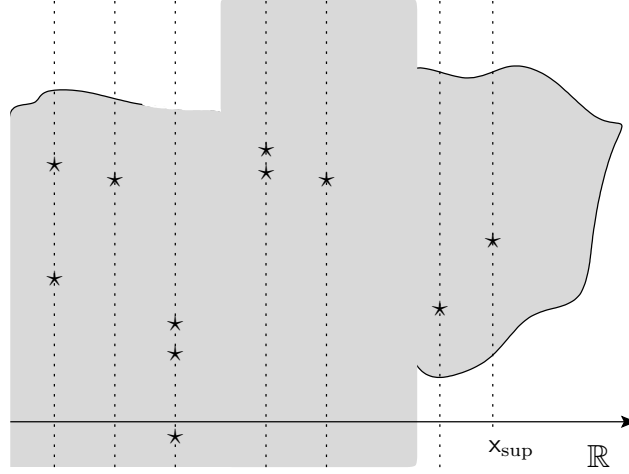


FIGURE 4.1. The image of the moment map (gray) with the focus-focus values (marked by ★) and their projection onto the first component.

Let $B_{\text{ff}} \subset \text{Int}(B)$ denote the set of focus-focus values. By Remark 3.25, it is a countable subset. To order the elements of B_{ff} we fix the following convention for the indexing set of B_{ff} . Set

$$(4.1) \quad \begin{aligned} x_{\text{sup}} &:= \sup\{\text{pr}_1(c) \mid c \in B_{\text{ff}}\}, \\ x_{\text{inf}} &:= \inf\{\text{pr}_1(c) \mid c \in B_{\text{ff}}\}, \end{aligned}$$

where $\text{pr}_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is projection onto the first component. By property (V3), this supremum x_{sup} (respectively infimum x_{inf}) is either attained as a maximum (respectively as a minimum) or does not lie in $J(M)$. Set

$$(4.2) \quad I := \begin{cases} \{1, 2, \dots, |B_{\text{ff}}|\} & \text{if } |B_{\text{ff}}| < \infty; \\ \{1, 2, \dots\} & \text{if } |B_{\text{ff}}| = |\mathbb{N}| \text{ and } x_{\text{inf}} \in J(M); \\ \{0, -1, -2, \dots\} & \text{if } |B_{\text{ff}}| = |\mathbb{N}| \text{ and } x_{\text{sup}} \in J(M); \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

By construction, the cardinality of I equals that of B_{ff} and thus we think of the elements of the latter as being indexed by I . Order the elements of B_{ff} as follows. For $i \in I$, set $c_i = (x_i, y_i)$. Then require that $i < j$ implies either $x_i < x_j$, or $x_i = x_j$ and $y_i < y_j$; moreover, if $0, 1 \in I$, require that $x_0 < x_1$. (If $I = \mathbb{Z}$, the above ordering is unique up to the choice of which focus-focus value is labeled with 0.)

For each $i \in I$ choose a sign $\varepsilon_i \in \{+1, -1\}$, and denote the associated vertical *cut* in B at c_i by

$$l^{\varepsilon_i} := \{(x, y) \in \mathbb{R}^2 \mid x = x_i, \varepsilon_i y \geq \varepsilon_i y_i\} \cap B.$$

When $\varepsilon_i = +1$ (respectively -1), the cut l^{ε_i} is simply the intersection of B with the vertical half-line starting at c_i going ‘up’ (respectively ‘down’), cf. Figure 4.2. Therefore the former is referred to as being *upward*, while the latter as being *downward*. For a fixed $\varepsilon \in \{+1, -1\}^I$, denote the union of the cuts by l^ε and set $S_\varepsilon := B \setminus l^\varepsilon$. Moreover, to each element $(x, y) \in B_{\text{lt}} = B \setminus B_{\text{ff}}$, associate the integer

$$j_\varepsilon(x, y) := \sum_{\{i \in I \mid (x, y) \in l^{\varepsilon_i}\}} \varepsilon_i,$$

with the convention that $j_\varepsilon(x, y) = 0$ for $(x, y) \in S_\varepsilon$. Finiteness of $j_\varepsilon(x, y)$ follows from property (V4) and Proposition 4.14.

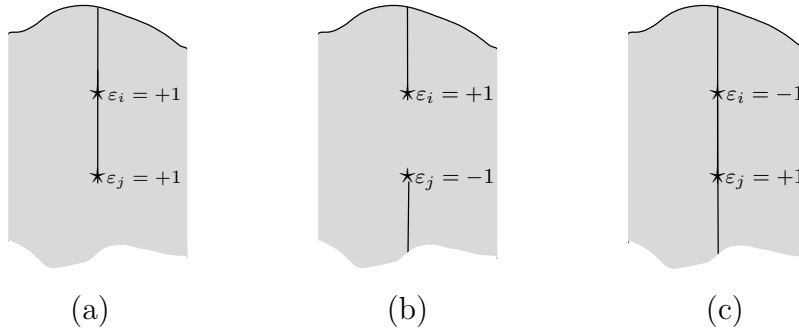


FIGURE 4.2. Image of the moment map (gray) with cuts emanating from the focus-focus values (marked by \star). The choice of cuts in (a) and (b) leads to a simply connected set whereas the choice in (c) yields two connected components.

Corollary 4.19. *The subset S_ε is open and dense in B .*

Proof. Density of S_ε in B is trivial, so it remains to prove its openness in B and, to this end, it suffices to prove that l^ε is closed in B . Let $\{(x_n, y_n)\} \subset l^\varepsilon$ be a sequence which converges to $(x_0, y_0) \in B$. This implies that the sequence $\{x_n\} = \text{pr}_1(\{(x_n, y_n)\}) \subset J(M)$ converges to $x_0 \in J(M)$. By construction and by Proposition 4.14, $\{x_n\}$ is contained in the subset of critical values of J , which does not contain any limit points in $J(M)$ by property (V3). Therefore, for all but finitely many n , $x_n = x_0$, which, in turn, implies that $(x_n, y_n) \in \{(x, y) \mid x = x_0\}$ for all but finitely many n . By property (V4) and Proposition 4.14, the

vertical line $\{(x, y) \mid x = x_0\}$ contains finitely many focus-focus values and, therefore, finitely many cuts. Seeing as each cut is a closed subset, then the union of all cuts contained on $\{(x, y) \mid x = x_0\}$ is closed. Therefore, $(x_0, y_0) \in l^\varepsilon$ as required. \square

Remark 4.20 In general, it is not true that l^ε is closed in \mathbb{R}^2 , for $x_{\text{sup}}, x_{\text{inf}}$ may belong to $\mathbb{R} \setminus J(M)$.

The notation S_ε is suggestive of the fact that there exists a cartographic homeomorphism $f_\varepsilon : B \rightarrow \mathbb{R}^2$ such that $(f_\varepsilon, S_\varepsilon)$ is a cartographic pair for $(M, \omega, \Phi = (J, H))$. Before stating and proving the precise existence statement, we prove some further properties of S_ε (cf. Figure 4.2).

Lemma 4.21. *The subset S_ε is path-connected if and only if $\varepsilon_i \geq \varepsilon_j$ for all $i > j$ with $x_i = x_j$.*

Proof. Suppose first that S_ε is path-connected and let $i > j \in I$ be such that $x_i = x_j$. Let $(x_1, y_1), (x_2, y_2) \in S_\varepsilon$ be points with $x_1 < x_i < x_2$. Such points exist because focus-focus values are contained in $\text{Int}(B)$. Since S_ε is path-connected, there exists a path in S_ε starting at (x_1, y_1) and ending at (x_2, y_2) . Therefore, there exists a point $(x_i, y') \in S_\varepsilon$ and thus $\varepsilon_i \geq \varepsilon_j$.

Conversely, suppose that ε satisfies the condition that $\varepsilon_i \geq \varepsilon_j$ for all $i > j$ with $x_i = x_j$. First we show that, for all $x_1 \in J(M)$, $\{(x, y) \mid x = x_1\} \cap S_\varepsilon \neq \emptyset$ and that the set is path-connected. If $x_1 \notin \text{pr}_1(B_{\text{ff}})$, one obtains

$$\{(x, y) \mid x = x_1\} \cap S_\varepsilon = \{(x, y) \mid x = x_1\} \cap B$$

and the result follows from the fact that $(M, \omega, \Phi = (J, H))$ is a vertical almost-toric system. Suppose, therefore, that $x_1 \in \text{pr}_1(B_{\text{ff}})$. By property (V4) and Proposition 4.14, there are finitely many focus-focus values $(x_1, y_{i_1}), \dots, (x_1, y_{i_N})$ lying on the vertical line $\{(x, y) \mid x = x_1\}$. Set

$$\begin{aligned} y_+ &:= \inf\{y_{i_k} \mid \varepsilon_{i_k} = +1\}, \\ y_- &:= \sup\{y_{i_k} \mid \varepsilon_{i_k} = -1\}. \end{aligned}$$

Since ε satisfies the condition in the statement, it follows that $y_+ > y_-$ and therefore,

$$\{(x, y) \mid x = x_1\} \cap S_\varepsilon = \{(x, y) \mid x = x_1, y_+ > y > y_-\},$$

which shows that $\{(x, y) \mid x = x_1\} \cap S_\varepsilon$ is path-connected. By Corollary 4.19, S_ε is open in B . Thus S_ε satisfies all the hypotheses of Proposition 4.7 and, therefore, the subsystem of $(M, \omega, \Phi = (J, H))$ relative to S_ε

is vertical almost-toric. By Corollary 4.5, S_ε is contractible and, in particular, path-connected. \square

Corollary 4.22. *The subset S_ε is path-connected if and only if it is contractible.*

Corollary 4.23. *There exists a choice of ε making S_ε path-connected.*

Proof. The choice of $\varepsilon_i = +1$ for all $i \in I$ satisfies the condition of Lemma 4.21. \square

Having established the above preliminary results, we can state and prove existence of cartographic homeomorphisms for simple vertical almost-toric systems.

Theorem 4.24. *Let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system with $B_{\text{ff}} = \{c_i\}_{i \in I} \neq \emptyset$. For any $\varepsilon \in \{+1, -1\}^I$, there exists a cartographic pair $(f_\varepsilon, S_\varepsilon)$, where $f_\varepsilon(x, y) = \left(x, f_\varepsilon^{(2)}(x, y)\right)$, satisfying the following properties*

(C1) *the quantity $\text{sgn}\left(\frac{\partial f_\varepsilon^{(2)}}{\partial y}(x, y)\right) =: \text{sgn}(f_\varepsilon)$ is constant for all $(x, y) \in S_\varepsilon$;*

(C2) *for all $(x, y) \in B_{\text{lt}}$,*

$$(4.3) \quad \lim_{\substack{(x, y) \rightarrow (x, y) \\ x < x}} Df_\varepsilon(x, y) = \begin{pmatrix} 1 & 0 \\ \text{sgn}(f_\varepsilon)j_\varepsilon(x, y) & 1 \end{pmatrix} \lim_{\substack{(x, y) \rightarrow (x, y) \\ x > x}} Df_\varepsilon(x, y).$$

In particular, $f_\varepsilon(B)$ is locally convex.

Any cartographic homeomorphism $f_\varepsilon : B \rightarrow \mathbb{R}^2$ satisfying the properties in Theorem 4.24 is said to be *associated to ε* .

The proof of Theorem 4.24 is split into two cases: when S_ε is path-connected and when it is not.

Proof of Theorem 4.24 if S_ε is path-connected. Suppose that S_ε is path-connected; by Corollary 4.23, a choice of ε whose associated set S_ε is path-connected exists. The idea is to argue as in Pelayo & Ratiu & Vũ Ngọc [33, Steps 2 – 4 in the proof of Theorem B] recalling and adjusting it as much as necessary for our purposes.

By construction $S_\varepsilon \subset B_{\text{lt}}$. Now let $\mathfrak{q} : \tilde{B}_{\text{lt}} \rightarrow B_{\text{lt}}$ denote the universal covering. By Corollary 4.22, S_ε is contractible and, in particular, simply connected. Therefore, there exists a smooth section $\sigma : S_\varepsilon \rightarrow \tilde{B}_{\text{lt}}$ of \mathfrak{q} . Consider a developing map $\text{dev} : \tilde{B}_{\text{lt}} \rightarrow \mathbb{R}^2$ constructed by fixing basepoints $\mathbf{x}_0 \in S_\varepsilon$ and $\tilde{\mathbf{x}}_0 \in \sigma(S_\varepsilon)$. Set $f_\varepsilon := \text{dev} \circ \sigma : S_\varepsilon \rightarrow \mathbb{R}^2$. Arguing as in the proof of Lemma 4.17, it is possible to choose dev so that $f_\varepsilon(x, y) = \left(x, f_\varepsilon^{(2)}(x, y)\right)$ for any $(x, y) \in S_\varepsilon$. Fix such a

choice. Following the arguments in Pelayo & Ratiu & Vũ Ngọc [33, Step 4 of the proof of Theorem B], f_ε can be extended to an embedding $B \rightarrow \mathbb{R}^2$ which, by abuse of notation, is also denoted by f_ε . By construction and by density of $S_\varepsilon \subset B$, $(f_\varepsilon, S_\varepsilon)$ is a cartographic pair with $f_\varepsilon(x, y) = \left(x, f_\varepsilon^{(2)}(x, y)\right)$. Thus for all $(x, y) \in S_\varepsilon$, $\frac{\partial f_\varepsilon^{(2)}}{\partial y}(x, y) \neq 0$. Since S_ε is path-connected, property (C1) follows.

To complete the proof, there are two cases to consider, depending on whether $\text{sgn}(f_\varepsilon) = +1$ or $\text{sgn}(f_\varepsilon) = -1$. In the first case, property (C2) and local convexity of $f_\varepsilon(B)$ can be proved as in Vũ Ngọc [46, Steps 5 and 6 of the proof of Theorem 3.8]. Thus suppose that $\text{sgn}(f_\varepsilon) = -1$. Setting $\hat{f}_\varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ f_\varepsilon$, $(\hat{f}_\varepsilon, S_\varepsilon)$ is a cartographic pair which can be constructed as above satisfying $\text{sgn}(\hat{f}_\varepsilon) = +1$. (This corresponds to adjusting the above choice of developing map by composing on the left with the map $(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}) \in \text{Vert}(2; \mathbb{Z})$, cf. Remark 4.18.) Fix $(\mathbf{x}, \mathbf{y}) \in B_{\text{lt}}$. Then, using property (C2) for \hat{f}_ε and the fact that $\text{sgn}(f_\varepsilon) = -1$,

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (\mathbf{x}, \mathbf{y}) \\ x < \mathbf{x}}} Df_\varepsilon(x, y) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lim_{\substack{(x,y) \rightarrow (\mathbf{x}, \mathbf{y}) \\ x < \mathbf{x}}} D\hat{f}_\varepsilon(x, y) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ j_\varepsilon(\mathbf{x}, \mathbf{y}) & 1 \end{pmatrix} \lim_{\substack{(x,y) \rightarrow (\mathbf{x}, \mathbf{y}) \\ x > \mathbf{x}}} D\hat{f}_\varepsilon(x, y) \\ &= \begin{pmatrix} 1 & 0 \\ \text{sgn}(f_\varepsilon)j_\varepsilon(\mathbf{x}, \mathbf{y}) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lim_{\substack{(x,y) \rightarrow (\mathbf{x}, \mathbf{y}) \\ x > \mathbf{x}}} D\hat{f}_\varepsilon(x, y) \\ &= \begin{pmatrix} 1 & 0 \\ \text{sgn}(f_\varepsilon)j_\varepsilon(\mathbf{x}, \mathbf{y}) & 1 \end{pmatrix} \lim_{\substack{(x,y) \rightarrow (\mathbf{x}, \mathbf{y}) \\ x > \mathbf{x}}} Df_\varepsilon(x, y), \end{aligned}$$

This proves property (C2) in general.

Finally, observe that $\hat{f}_\varepsilon(B)$ is locally convex and that \mathbb{Z} -affine maps preserve this property. Thus $f_\varepsilon(B)$ is locally convex as required. \square

Now we turn to the case of S_ε not being path-connected. There exist proofs for such cases in the literature (cf. for instance Pelayo & Ratiu & Vũ Ngọc [33, Step 5 of the proof of Theorem B]). The argument presented below, however, uses different techniques.

Before delving into the proof, we introduce some useful notions and notation. For any $\mathbf{x} \in J(M)$, set

$$(4.4) \quad N_{\mathbf{x}} := |\{i \in I \mid x_i = \mathbf{x}\}|.$$

By property (V4) and Proposition 4.14, N_x is finite for any $x \in J(M)$. Moreover, for a fixed $\varepsilon \in \{+1, -1\}^I$, set

$$(4.5) \quad N_x^\pm(\varepsilon) := \pm |\{i \in I \mid x_i = x \text{ and } \varepsilon_i = \pm 1\}|.$$

Observe that, for any $x \in J(M)$ and any $\varepsilon \in \{+1, -1\}^I$,

$$N_x = N_x^+(\varepsilon) - N_x^-(\varepsilon).$$

Moreover, for any $(x, y) \in B_{\text{lt}}$ and any $\varepsilon \in \{+1, -1\}^I$,

$$j_\varepsilon(x, y) = N_x^+(\varepsilon) + N_x^-(\varepsilon).$$

Fix $x \in J(M)$ with $N_x \neq 0$. Then, by property (V4) and Proposition 4.14, there exist finitely many indices $i_1 < i_2 < \dots < i_{N_x}$ in I with $x_{i_j} = x$. Observe that, by definition of the ordering on B_{ff} ,

$$\begin{aligned} \{(x, y) \mid x = x\} \cap B_{\text{lt}} &\subset \{(x, y) \mid x = x \text{ and } y < y_{i_1}\} \\ &\cup \bigcup_{j=1}^{N_x-1} \{(x, y) \mid x = x \text{ and } y_{i_j} < y < y_{i_{j+1}}\} \\ &\cup \{(x, y) \mid x = x \text{ and } y > y_{i_{N_x}}\}. \end{aligned}$$

Lemma 4.25. *For any $\varepsilon \in \{+1, -1\}^I$ and for all $x \in J(M)$ with $N_x \neq 0$, the function $j_\varepsilon(x, \cdot) : \{(x, y) \mid x = x\} \cap B_{\text{lt}} \rightarrow \mathbb{Z}$ satisfies*

- $j_\varepsilon(x, y) = N_x^-(\varepsilon)$ for all $(x, y) \in \{(x, y) \mid x = x \text{ and } y < y_{i_1}\}$;
- $j_\varepsilon(x, y) = N_x^-(\varepsilon) + k = N_x^+(\varepsilon) - N_x + j$ for all $k = 1, \dots, N_x - 1$ and for all $(x, y) \in \{(x, y) \mid x = x \text{ and } y_{i_k} < y < y_{i_{k+1}}\}$;
- $j_\varepsilon(x, y) = N_x^+(\varepsilon)$ for all $(x, y) \in \{(x, y) \mid x = x \text{ and } y > y_{i_{N_x}}\}$.

Hereby, $i_1 < i_2 < \dots < i_{N_x}$ are the elements of I such that $x_{i_j} = x$. In particular, the function $j_\varepsilon(x, \cdot)$ only depends on $N_x^\pm(\varepsilon)$.

Proof. Fix $\varepsilon \in \{+1, -1\}^I$ and consider $(x, y) \in B_{\text{lt}}$ such that $N_x \neq 0$. Suppose first that $y < y_{i_1}$. This means that (x, y) is ‘below’ all focus-focus values on the vertical line $\{(x, y) \mid x = x\}$. By definition of the ordering on B_{ff} and of the cuts associated to ε , for all $k \in \{1, \dots, N_x\}$, if $\varepsilon_{i_k} = -1$ then $(x, y) \in l^{\varepsilon_{i_k}}$, while if $\varepsilon_{i_k} = +1$, then $(x, y) \notin l^{\varepsilon_{i_k}}$. Thus

$$j_\varepsilon(x, y) = -|\{i \in I \mid x_i = x \text{ and } \varepsilon_i = -1\}| = N_x^-(\varepsilon)$$

as required. Similarly, if $y > y_{i_{N_x}}$, then $j_\varepsilon(x, y) = N_x^+(\varepsilon)$, for (x, y) is ‘above’ all focus-focus values on the vertical line $\{(x, y) \mid x = x\}$.

It remains to prove the intermediate cases for which we proceed by induction on k . The base case is $y < y_{i_1}$, which has already been proved. Suppose that the required statement holds for all $m < k$ and

let $y_{i_k} < y < y_{i_{k+1}}$. Set, for any $(\bar{x}, \bar{y}) \in B_{\text{lt}}$,

$$j_{\varepsilon}^{\pm}(\bar{x}, \bar{y}) := \sum_{\{i \in I | (\bar{x}, \bar{y}) \in l^{\varepsilon_i} \text{ and } \varepsilon_i = \pm 1\}} \varepsilon_i.$$

Clearly, for any $(\bar{x}, \bar{y}) \in B_{\text{lt}}$, $j_{\varepsilon}(\bar{x}, \bar{y}) = j_{\varepsilon}^+(\bar{x}, \bar{y}) + j_{\varepsilon}^-(\bar{x}, \bar{y})$. Fix some $(x, y') \in B_{\text{lt}}$ with $y_{i_{k-1}} < y' < y_{i_k}$. The inductive hypothesis implies $j_{\varepsilon}(x, y') = N_x^-(\varepsilon) + k - 1$. There are two cases to consider, depending on whether $\varepsilon_{i_k} = +1$ or $\varepsilon_{i_k} = -1$. In the former case, observe that $j_{\varepsilon}^+(x, y) = j_{\varepsilon}^+(x, y') + 1$, while $j_{\varepsilon}^-(x, y) = j_{\varepsilon}^-(x, y')$. Thus, $j_{\varepsilon}(x, y) = j_{\varepsilon}(x, y') + 1 = N_x^-(\varepsilon) + k$ as required. The latter case is proved analogously swapping the roles of $j_{\varepsilon}^+(x, y)$ and $j_{\varepsilon}^-(x, y)$. \square

With the above results at hand, we finish the proof of Theorem 4.24.

Proof of Theorem 4.24 if S_{ε} is not path-connected. Suppose that S_{ε} is not path-connected. The idea is to reduce this situation to the path-connected case by appealing to the following result.

Lemma 4.26. *There exists a unique $\hat{\varepsilon} \in \{+1, -1\}^I$ such that*

- $S_{\varepsilon} \subset S_{\hat{\varepsilon}}$;
- for all $(x, y) \in B_{\text{lt}}$, $j_{\varepsilon}(x, y) = j_{\hat{\varepsilon}}(x, y)$;
- $S_{\hat{\varepsilon}}$ is path-connected.

Assume Lemma 4.26, whose proof is below. Let $\hat{\varepsilon}$ be as in Lemma 4.26 and let $f_{\hat{\varepsilon}} : B \rightarrow \mathbb{R}^2$ be a cartographic homeomorphism associated to $\hat{\varepsilon}$. Set $f_{\varepsilon} := f_{\hat{\varepsilon}}$. Since $S_{\varepsilon} \subset S_{\hat{\varepsilon}}$, Corollary 4.19 implies that $(f_{\varepsilon}, S_{\varepsilon})$ is a cartographic pair; property (C1) holds by construction. Property (C2) holds because Lemma 4.26 implies that $j_{\varepsilon}(x, y) = j_{\hat{\varepsilon}}(x, y)$ for all $(x, y) \in B_{\text{lt}}$. Moreover, $\text{sgn}(f_{\varepsilon}) = \text{sgn}(f_{\hat{\varepsilon}})$ holds by definition. Local convexity of $f_{\varepsilon}(B) = f_{\hat{\varepsilon}}(B)$ is also true as $f_{\hat{\varepsilon}}$ is associated to $\hat{\varepsilon}$ in the sense of Proposition 4.24. This finishes the proof of Theorem 4.24 for the case that S_{ε} is not path-connected. \square

Proof of Lemma 4.26. Since S_{ε} is not path-connected, Lemma 4.21 implies that there exists $\mathbf{x} \in \text{pr}_1(B_{\text{ff}})$ with $\{(x, y) \mid x = \mathbf{x}\} \cap B \subset l^{\varepsilon}$. Denote the set of such \mathbf{x} by $\text{Disc}(\varepsilon) \subset \text{pr}_1(B_{\text{ff}})$; since B_{ff} is countable, so is $\text{Disc}(\varepsilon)$. The idea is to define $\hat{\varepsilon} \in \{+1, -1\}^I$ as follows:

- If $i \in I$ is such that $x_i \notin \text{Disc}(\varepsilon)$ then $\hat{\varepsilon}_i := \varepsilon_i$.
- If $i \in I$ is such that $x_i \in \text{Disc}(\varepsilon)$ set $\mathbf{x} = x_i$ and let $i_1 < i_2 < \dots < i_{N_{\mathbf{x}}}$ be the elements in I such that $x_{i_j} = \mathbf{x}$. Then
 - if $i = i_j$ for some $j = 1, \dots, |N_{\mathbf{x}}^-(\varepsilon)|$, set $\hat{\varepsilon}_i := -1$;
 - if $i = i_j$ for some $j = |N_{\mathbf{x}}^-(\varepsilon)| + 1, \dots, N_{\mathbf{x}}$, set $\hat{\varepsilon}_i := +1$.

First, it is shown that $\hat{\varepsilon}$ satisfies the properties itemized in Lemma 4.26. To see that $S_{\hat{\varepsilon}}$ is path-connected, we use the criterion of Lemma 4.21.

Suppose that there exist $i > j$ with $x_i = x_j$. There are two cases to consider, depending on whether $x_i \notin \text{Disc}(\varepsilon)$ or $x_i \in \text{Disc}(\varepsilon)$. In the former, observe that $\hat{\varepsilon}_i = \varepsilon_i$ and $\hat{\varepsilon}_j = \varepsilon_j$. Since $x_i \notin \text{Disc}(\varepsilon)$, it follows that $\hat{\varepsilon}_i = \varepsilon_i \geq \varepsilon_j = \hat{\varepsilon}_j$. On the other hand, if $x_i \in \text{Disc}(\varepsilon)$, then the definition of $\hat{\varepsilon}$ implies that $\hat{\varepsilon}_i \geq \hat{\varepsilon}_j$. Thus S_ε is path-connected.

Now, to show that $S_\varepsilon \subset S_{\hat{\varepsilon}}$, consider $(x, y) \in S_\varepsilon$. By definition of the cuts, if $x \neq x_i$ for any $i \in I$, then $(x, y) \in S_{\hat{\varepsilon}}$. Therefore, suppose that there exists $i \in I$ with $x = x_i$. By property (V4) and Proposition 4.14, there exist finitely many indices $i_0, \dots, i_r \in I$ such that $x_{i_j} = x$ for all $j = 0, \dots, r$. Then $(x, y) \in S_\varepsilon$, implies that, for all $j = 0, \dots, r$, $x_{i_j} \notin \text{Disc}(\varepsilon)$. Thus $\hat{\varepsilon}_{i_j} = \varepsilon_{i_j}$ for all $j = 0, \dots, r$ by definition, which implies $(x, y) \in S_{\hat{\varepsilon}}$.

Finally, to show that $j_\varepsilon(\mathbf{x}, \mathbf{y}) = j_{\hat{\varepsilon}}(\mathbf{x}, \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in B_{\text{lt}}$, consider some $(\mathbf{x}, \mathbf{y}) \in B_{\text{lt}}$. If $\mathbf{x} \neq x_i$ for all $i \in I$, the above equality is trivially true. Suppose, therefore, that there exists $i \in I$ with $\mathbf{x} = x_i$. By Lemma 4.25, $j_\varepsilon(\mathbf{x}, \mathbf{y}) = j_{\hat{\varepsilon}}(\mathbf{x}, \mathbf{y})$ is equivalent to $N_{\mathbf{x}}^\pm(\varepsilon) = N_{\mathbf{x}}^\pm(\hat{\varepsilon})$, but the latter equality follows from the way in which $\hat{\varepsilon}$ has been defined.

It remains to check that $\hat{\varepsilon}$ is the unique choice of signs satisfying the above properties. Suppose that $\hat{\varepsilon}'$ is another such choice. The inclusion $S_\varepsilon \subset S_{\hat{\varepsilon}'}$ implies $\hat{\varepsilon}'_i = \varepsilon_i = \hat{\varepsilon}_i$ for all $i \in I$ such that $x_i \notin \text{Disc}(\varepsilon)$. On the other hand, the condition that $j_\varepsilon(\mathbf{x}, \mathbf{y}) = j_{\hat{\varepsilon}'}(\mathbf{x}, \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in B_{\text{lt}}$ implies, using the above argument, that $\hat{\varepsilon}'_i = \hat{\varepsilon}_i$ for all $i \in I$ such that $x_i \in \text{Disc}(\varepsilon)$. This shows that $\hat{\varepsilon}' = \hat{\varepsilon}$ as required. \square

Remark 4.27 The above argument for the case of S_ε not being path-connected only works because the system $(M, \omega, \Phi = (J, H))$ is assumed to be simple. If focus-focus values of higher multiplicity are allowed, then there may be no analogue of $\hat{\varepsilon}$ as in Lemma 4.26.

Remark 4.28 The argument proving the case of Theorem 4.24 when S_ε is not path-connected shows that, for a given vertical almost toric system, the set of all cartographic homeomorphisms that satisfy the hypotheses of Theorem 4.24 equals that of the cartographic homeomorphisms associated to those ε for which S_ε is path-connected (cf. Vũ Ngọc [46, Proposition 4.1]).

4.4. The set of cartographic pairs. Given a simple vertical almost-toric system $(M, \omega, \Phi = (J, H))$, it is natural to ask for a description of the set of all cartographic pairs of $(M, \omega, \Phi = (J, H))$ that satisfy the properties in Theorem 4.24. Providing such a description is the aim of this section, which generalizes, while being heavily inspired by, work of Vũ Ngọc [46, Section 4]. Henceforth, fix a vertical almost-toric system $(M, \omega, \Phi = (J, H))$. If $(M, \omega, \Phi = (J, H))$ does not have any

focus-focus points, then Remark 4.18 describes all cartographic pairs as in Lemma 4.17: In that case, any cartographic homeomorphism can be obtained from a fixed one by composing on the left by an element of $\text{Vert}(2; \mathbb{Z})$.

Thus, assume that $(M, \omega, \Phi = (J, H))$ contains at least one focus-focus point. The idea is to show that any cartographic homeomorphism of $(M, \omega, \Phi = (J, H))$ as in Theorem 4.24 can be constructed from a fixed one by means of composing on the left with a suitable homeomorphism (see Corollary 4.29 and Theorem 4.36 for precise statements). It is convenient to consider two separate cases:

- Determine all cartographic homeomorphisms associated to a given choice of signs (Corollary 4.29).
- Determine how cartographic homeomorphisms associated to possibly distinct choices of signs are related (Theorem 4.36).

First, we consider the set of all cartographic homeomorphisms associated to a given choice of signs; this is described in the following result, which is analogous to Remark 4.18.

Corollary 4.29. *Fix a vertical almost-toric system $(M, \omega, \Phi = (J, H))$ and a choice of signs ε . If $f_\varepsilon, \hat{f}_\varepsilon$ are cartographic homeomorphisms associated to ε , then there exists an element $h \in \text{Vert}(2; \mathbb{Z})$ with $\hat{f}_\varepsilon = h \circ f_\varepsilon$. Conversely, for any $h \in \text{Vert}(2; \mathbb{Z})$, the map $\hat{f}_\varepsilon := h \circ f_\varepsilon$ is a cartographic homeomorphism associated to ε .*

Proof. It may be assumed without loss of generality that S_ε is path-connected since the not path-connected case can be reduced to the path-connected one as in the proof of Theorem 4.24. Let $f_\varepsilon, \hat{f}_\varepsilon : B \rightarrow \mathbb{R}^2$ be cartographic homeomorphisms associated to ε . Then their restrictions to S_ε are developing maps for the induced \mathbb{Z} -affine structure on S_ε . Therefore, arguing as in Remark 4.18, there exists an element $h \in \text{Vert}(2; \mathbb{Z})$ with $\hat{f}_\varepsilon|_{S_\varepsilon} = h \circ f_\varepsilon|_{S_\varepsilon}$. Since $S_\varepsilon \subset B$ is dense, this implies that $\hat{f}_\varepsilon = h \circ f_\varepsilon$ as required.

Conversely, the proof of Theorem 4.24 gives that composing a cartographic homeomorphism associated to ε on the left with an element of $\text{Vert}(2; \mathbb{Z})$ yields another cartographic homeomorphism associated to ε . \square

Having established Corollary 4.29, we study the problem of relating cartographic homeomorphisms whose associated signs are not necessarily equal. Before stating the main result of this section we introduce some tools akin to those needed in Vũ Ngọc [46, Section 4], but slightly more involved as vertical almost-toric systems allow for the presence

of infinitely many focus-focus points (see Remarks 4.30, 4.31 and 4.37 below).

As in Section 4.3, let I be the indexing set of the set of focus-focus values B_{ff} defined in equation (4.2). Also, fix the ordering on B_{ff} as in the paragraph following equation (4.2), so elements of B_{ff} are denoted by $c_i = (x_i, y_i)$ for $i \in I$. Henceforth, fix elements $\varepsilon, \hat{\varepsilon} \in \{+1, -1\}^I$ and associated cartographic homeomorphisms $f_\varepsilon, f_{\hat{\varepsilon}} : B \rightarrow \mathbb{R}^2$. Furthermore, fix a basepoint $(\mathbf{x}, \mathbf{y}) \in B_{\text{lt}}$ with the property that $x_0 < \mathbf{x} < x_1$. (If x_0 or x_1 is not defined, then only the other inequality is required.)

Remark 4.30 The above choice of basepoint agrees with that made in Vũ Ngọc [46, Proof of Proposition 4.1]. However, there is an important difference that arises because of the possibility of having infinitely many focus-focus points for vertical almost-toric systems. In what follows we must allow for the case in which there are focus-focus values ‘to the left’ of the basepoint, i.e., with notation as above, for the case in which there exists $i \in I$ with $x_i < \mathbf{x}$. (If this is the case, then by the choices of indexing set I of equation (4.2) and of basepoint, then there are infinitely many such indices.)

Throughout this section, set $T := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. For any $i \in I$, set

$$(4.6) \quad k_i(\varepsilon, \hat{\varepsilon}) := \text{sgn}(f_{\hat{\varepsilon}}) \left(\frac{\varepsilon_i - \hat{\varepsilon}_i}{2} \right).$$

Moreover, for any $i \in I$ define $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows:

- If $i \leq 0$, let $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ be the identity.
- If $i > 0$, let $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ be the piece-wise \mathbb{Z} -affine transformation that acts as the identity on the half-space $x < x_i$ and as the shear $T^{k_i(\varepsilon,\hat{\varepsilon})}$ on the half-space $x \geq x_i$.

Analogously, for any $i \in I$ define $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows:

- If $i \leq 0$, let $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$ be the piece-wise \mathbb{Z} -affine transformation that acts as the shear $T^{-k_i(\varepsilon,\hat{\varepsilon})}$ on the half-space $x < x_i$ and as the identity on the half-space $x \geq x_i$.
- If $i > 0$ let $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$ be the identity.

Remark 4.31 While the maps $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ are those used in Vũ Ngọc [46, Section 4], the maps $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$ are needed in the following precisely because of the possibility of focus-focus values existing ‘to the left’ of the basepoint (cf. Remark 4.30).

Explicitly, we have that if $i > 0$ then

$$(4.7) \quad \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}} := \begin{cases} \text{id} & \text{if } x < x_i, \\ \left(\left(\begin{pmatrix} 1 & 0 \\ k_i(\varepsilon, \hat{\varepsilon}) & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -k_i(\varepsilon, \hat{\varepsilon})x_i \end{pmatrix} \right) \right) & \text{if } x \geq x_i, \end{cases}$$

and if $i \leq 0$ then

$$(4.8) \quad \mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}} = \begin{cases} \left(\left(\begin{pmatrix} 1 & 0 \\ -k_i(\varepsilon, \hat{\varepsilon}) & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ k_i(\varepsilon, \hat{\varepsilon})x_i \end{pmatrix} \right) \right) & \text{if } x \leq x_i, \\ \text{id} & \text{if } x > x_i. \end{cases}$$

The maps $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ and $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$ are well-defined and satisfy the following properties, the proofs of which are left to the reader.

Corollary 4.32. *For each positive (respectively non-positive) $i \in I$, the map $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ (respectively $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$) is a homeomorphism that fixes the vertical line $\{(x, y) \mid x = x_i\}$ pointwise and is a \mathbb{Z} -affine isomorphism of $\mathbb{R}^2 \setminus \{(x, y) \mid x = x_i\}$.*

Corollary 4.33. *For any $i, j \in I$,*

$$\begin{aligned} \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{j,\varepsilon,\hat{\varepsilon}} &= \mathfrak{l}_{j,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}, \\ \mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{r}_{j,\varepsilon,\hat{\varepsilon}} &= \mathfrak{r}_{j,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}, \\ \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{r}_{j,\varepsilon,\hat{\varepsilon}} &= \mathfrak{r}_{j,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}. \end{aligned}$$

One ingredient in the construction of the homeomorphism that relates f_ε and $f_{\hat{\varepsilon}}$ is the composition of the maps $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$ or $\mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}}$ as i range over all indices in I . Seeing as this may involve the composition of infinitely many maps different from the identity (as I may be infinite), some care is needed. To this end, we first introduce notation for the domains of these possibly infinite compositions. Set

$$D_{\text{sup}} := \begin{cases} \mathbb{R}^2 & \text{if } x_{\text{sup}} \in J(M), \\ \{(x, y) \in \mathbb{R}^2 \mid x < x_{\text{sup}}\} & \text{otherwise,} \end{cases}$$

$$D_{\text{inf}} := \begin{cases} \mathbb{R}^2 & \text{if } x_{\text{inf}} \in J(M), \\ \{(x, y) \in \mathbb{R}^2 \mid x > x_{\text{inf}}\} & \text{otherwise,} \end{cases}$$

where $x_{\text{sup}}, x_{\text{inf}} \in \mathbb{R}$ are defined as in equation (4.1). Observe that $B \subset D_{\text{sup}} \cap D_{\text{inf}}$.

Next, we define the desired compositions. If the cardinality of I is finite, the situation is entirely analogous to the one considered in Vũ Ngọc [46, Section 4]. For, with the above conventions, I being finite implies that $x_{\text{sup}}, x_{\text{inf}} \in J(M)$ and that I only contains positive elements. In this case, set $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}} : D_{\text{sup}} = \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}} : D_{\text{inf}} = \mathbb{R}^2 \rightarrow$

\mathbb{R}^2 to be equal to the finite compositions $\mathfrak{l}_{|I|,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{|I|-1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1,\varepsilon,\hat{\varepsilon}}$ and $\mathfrak{r}_{|I|,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{r}_{|I|-1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{r}_{1,\varepsilon,\hat{\varepsilon}}$ respectively. Observe that the latter is, by definition, equal to the identity. It remains to consider the case in which I is infinite. If I has infinitely many positive elements, set, for any (x, y) with $x \leq x_i$,

$$\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}(x, y) := \mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{i-1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1,\varepsilon,\hat{\varepsilon}}(x, y).$$

Analogously, if I has infinitely many non-positive elements, set, for any (x, y) with $x \geq x_i$,

$$\mathfrak{r}_{\varepsilon,\hat{\varepsilon}}(x, y) := \mathfrak{r}_{i,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{r}_{i+1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{r}_{0,\varepsilon,\hat{\varepsilon}}(x, y).$$

In the remaining cases, set $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}} = \text{id} = \mathfrak{r}_{\varepsilon,\hat{\varepsilon}}$.

Lemma 4.34. *The maps $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}} : D_{\text{sup}} \rightarrow D_{\text{sup}}$ and $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}} : D_{\text{inf}} \rightarrow D_{\text{inf}}$ are well-defined.*

Proof. The result follows immediately if I is finite, and for $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ (respectively $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}}$) if I does not contain positive (respectively non-negative) elements. Suppose that I contains infinitely many positive elements. By definition of the maps $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$, for all $i_1 \geq i_2 \geq 1$, the restrictions of $\mathfrak{l}_{i_1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{i_2,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{i_2-1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1,\varepsilon,\hat{\varepsilon}}$ and of $\mathfrak{l}_{i_2,\varepsilon,\hat{\varepsilon}} \circ \mathfrak{l}_{i_2-1,\varepsilon,\hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1,\varepsilon,\hat{\varepsilon}}$ to $\{(x, y) \mid x \leq x_{i_2}\}$ are equal. This implies that $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ is well-defined. The domain of $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ is given by the subset of \mathbb{R}^2 consisting of points $(x, y) \in \mathbb{R}^2$ for which there exists a positive $i \in I$ with $x \leq x_i$. By definition of x_{sup} (see equation (4.1)), this is precisely D_{sup} . Moreover, $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ sends the first coordinate to itself, as do all $\mathfrak{l}_{i,\varepsilon,\hat{\varepsilon}}$. This means that the codomain of $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ can be taken to be D_{sup} . An analogous argument works for $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}}$ in the case of I containing infinitely many non-positive elements by reversing the above inequalities. \square

In fact, more is true.

Lemma 4.35. *The maps $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}} : D_{\text{sup}} \rightarrow D_{\text{sup}}$ and $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}} : D_{\text{inf}} \rightarrow D_{\text{inf}}$ are homeomorphisms that are \mathbb{Z} -affine isomorphisms away from the set $\bigcup_{i \in I} \{(x, y) \mid x = x_i\}$.*

Proof. As in the proof of Lemma 4.34, the only non-trivial cases to consider are those of $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ if I contains infinitely many positive elements and of $\mathfrak{r}_{\varepsilon,\hat{\varepsilon}}$ if I contains infinitely many non-positive elements. Consider the former case (the latter is entirely analogous). To see that $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}$ is a homeomorphism, observe that the proof of Lemma 4.34 implies that, for all $(x, y) \in D_{\text{sup}}$, $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}(x, y) = \left(x, \mathfrak{l}_{\varepsilon,\hat{\varepsilon}}^{(2)}(x, y)\right)$, for some continuous function $\mathfrak{l}_{\varepsilon,\hat{\varepsilon}}^{(2)} : D_{\text{sup}} \rightarrow \mathbb{R}$. Moreover, it can be checked directly that,

for any $x < x_{\text{sup}}$, the function $\mathfrak{l}_{\varepsilon, \hat{\varepsilon}}^{(2)}(x, \cdot)$ is strictly increasing. Therefore, $\mathfrak{l}_{\varepsilon, \hat{\varepsilon}}$ is a homeomorphism onto its image. Since for any $i \geq 1$ the map $\mathfrak{l}_{i, \varepsilon, \hat{\varepsilon}} \circ \mathfrak{l}_{i-1, \varepsilon, \hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1, \varepsilon, \hat{\varepsilon}}$ sends $\{(x, y) \mid x \leq x_i\}$ onto itself, it follows that $\mathfrak{l}_{\varepsilon, \hat{\varepsilon}}(D_{\text{sup}}) = D_{\text{sup}}$. The fact that it is a \mathbb{Z} -affine isomorphism away from $\bigcup_{i \in I} \{(x, y) \mid x = x_i\}$ follows from the fact that, for any $i \geq 1$, $\mathfrak{l}_{i, \varepsilon, \hat{\varepsilon}} \circ \mathfrak{l}_{i-1, \varepsilon, \hat{\varepsilon}} \circ \dots \circ \mathfrak{l}_{1, \varepsilon, \hat{\varepsilon}}$ also satisfies this property (see Corollary 4.32). \square

With the maps $\mathfrak{l}_{\varepsilon, \hat{\varepsilon}}$ and $\mathfrak{r}_{\varepsilon, \hat{\varepsilon}}$ at hand, we can state the main result of this section.

Theorem 4.36. *Let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system. Given any $\varepsilon, \hat{\varepsilon} \in \{+1, -1\}^I$ and any two cartographic homeomorphisms $f_\varepsilon, f_{\hat{\varepsilon}} : B \rightarrow \mathbb{R}^2$ associated to $\varepsilon, \hat{\varepsilon}$ respectively, there exists a transformation $h_{\varepsilon, \hat{\varepsilon}} \in \text{Vert}(2; \mathbb{Z})$ such that*

$$(4.9) \quad f_{\hat{\varepsilon}} = \mathfrak{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathfrak{l}_{\varepsilon, \hat{\varepsilon}} \circ h_{\varepsilon, \hat{\varepsilon}} \circ f_\varepsilon.$$

The main idea behind Theorem 4.36 is not new; it first appears in Vũ Ngọc [46, Proposition 4.1] in the context of semi-toric systems. However, the more general context of vertical almost-toric systems, where there may be infinitely many focus-focus points, deserves to be dealt with carefully. For instance, the transformation $\mathfrak{r}_{\varepsilon, \hat{\varepsilon}}$, which is not needed in the study of semi-toric systems, is necessary in this context (cf. Remarks 4.30 and 4.31).

Remark 4.37 In fact, the statement and proof of Theorem 4.36 may be of use in the study of semi-toric systems as well. The main references for these systems make the underlying (tacit) assumption that the signs of the cartographic homeomorphisms, as in Theorem 4.24, are positive (cf. Pelayo & Vũ Ngọc, Vũ Ngọc [34, 35, 37, 46]).

Proof of Theorem 4.36. The proof is split into three steps:

- Construct the map $h_{\varepsilon, \hat{\varepsilon}} \in \text{Vert}(2; \mathbb{Z})$.
- Reduce to the simpler case in which ε and $\hat{\varepsilon}$ differ in precisely one component.
- Prove the simpler case.

Step 1: constructing the map $h_{\varepsilon, \hat{\varepsilon}} \in \text{Vert}(2; \mathbb{Z})$. As above, fix a base-point $(x, y) \in B_{\text{lt}}$ with the property that $x_0 < x < x_1$. (If x_0 or x_1 is not defined, then only the other inequality is required.) Denote by \mathfrak{S} the connected component of $S_\varepsilon \cap S_{\hat{\varepsilon}}$ containing (x, y) . Since both S_ε and $S_{\hat{\varepsilon}}$ are open, so is \mathfrak{S} . Moreover it is path-connected by definition. Furthermore, it can be checked that \mathfrak{S} intersects any vertical

line either in an empty or in a connected set. Therefore, by Proposition 4.7, the subsystem of $(M, \omega, \Phi = (J, H))$ relative to \mathfrak{S} is vertical almost-toric and contains (x, y) . Moreover, by construction, it contains no focus-focus value. By Corollary 2.49, the maps $f_\varepsilon|_{\mathfrak{S}}$ and $f_{\hat{\varepsilon}}|_{\mathfrak{S}}$ are cartographic homeomorphisms for the subsystem of $(M, \omega, \Phi = (J, H))$ relative to \mathfrak{S} . Therefore, by Remark 4.18, there exists $h_{\varepsilon, \hat{\varepsilon}} \in \text{Vert}(2; \mathbb{Z})$ such that $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = h_{\varepsilon, \hat{\varepsilon}}|_{f_\varepsilon(\mathfrak{S})} \circ f_\varepsilon|_{\mathfrak{S}}$. The map $h_{\varepsilon, \hat{\varepsilon}}$ is the desired one.

Step 2: reducing to a simpler case. Observe that, by Corollary 4.29, the map $h_{\varepsilon, \hat{\varepsilon}} \circ f_\varepsilon$ is a cartographic homeomorphism associated to ε . Moreover, the above argument shows that $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = (h_{\varepsilon, \hat{\varepsilon}} \circ f_\varepsilon)|_{\mathfrak{S}}$. Thus, in order to prove the result in the statement of the theorem, it suffices to prove that, if $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = f_\varepsilon|_{\mathfrak{S}}$, then $f_{\hat{\varepsilon}} = \mathbf{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathbf{l}_{\varepsilon, \hat{\varepsilon}} \circ f_\varepsilon$. Henceforth, assume that $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = f_\varepsilon|_{\mathfrak{S}}$, which implies $\text{sgn}(f_{\hat{\varepsilon}}) = \text{sgn}(f_\varepsilon)$. In fact, we can simplify the argument further: it suffices to prove the claimed result under the assumption that all but one of the components of $\varepsilon, \hat{\varepsilon}$ are equal. For, if the latter holds, we can argue as follows. Consider a sequence of choices of signs ε_s , for $s \in \{1, 2, \dots\} \cup \{\infty\}$, such that $\varepsilon_1 = \varepsilon$, $\varepsilon_\infty = \hat{\varepsilon}$, and, for any $s \geq 1$, all but one component of ε_s and ε_{s+1} are equal. Moreover, if $\varepsilon, \hat{\varepsilon}$ differ in finitely components, say in r components, choose the above sequence so that for all $s \geq r+1$, $\varepsilon_s = \hat{\varepsilon}$. For each $s \geq 1$, fix a choice of cartographic homeomorphism f_{ε_s} with the property that $f_{\varepsilon_s}|_{\mathfrak{S}} = f_\varepsilon|_{\mathfrak{S}}$. Moreover, require that $f_{\varepsilon_1} = f_\varepsilon$, that $f_{\varepsilon_\infty} = f_{\hat{\varepsilon}}$, and that, if $\varepsilon, \hat{\varepsilon}$ differ in precisely r components, then for all $s \geq r+1$, $f_{\varepsilon_s} = f_{\hat{\varepsilon}}$. Using the above sequence of signs and associated cartographic homeomorphisms and the fact that the claimed result holds when all but one component of the signs are equal, we obtain, for all $s \geq 1$, maps $\mathbf{l}_{\varepsilon_s, \varepsilon_{s+1}}, \mathbf{r}_{\varepsilon_s, \varepsilon_{s+1}}$ satisfying

$$(4.10) \quad f_{\varepsilon_{s+1}} = \mathbf{r}_{\varepsilon_s, \varepsilon_{s+1}} \circ \mathbf{l}_{\varepsilon_s, \varepsilon_{s+1}} \circ f_{\varepsilon_s}.$$

Therefore, iterating equation (4.10), for all $s \geq 1$,

$$(4.11) \quad f_{\varepsilon_{s+1}} = \mathbf{r}_{\varepsilon_s, \varepsilon_{s+1}} \circ \mathbf{l}_{\varepsilon_s, \varepsilon_{s+1}} \circ \mathbf{r}_{\varepsilon_{s-1}, \varepsilon_s} \circ \mathbf{l}_{\varepsilon_{s-1}, \varepsilon_s} \circ \dots \circ \mathbf{r}_{\varepsilon_1, \varepsilon_2} \circ \mathbf{l}_{\varepsilon_1, \varepsilon_2} \circ f_\varepsilon,$$

where we use the fact that $f_\varepsilon = f_{\varepsilon_1}$. If ε and $\hat{\varepsilon}$ differ in precisely r components, then, by construction, for all $s \geq r+1$,

$$\mathbf{r}_{\varepsilon_s, \varepsilon_{s+1}} = \text{id} = \mathbf{l}_{\varepsilon_s, \varepsilon_{s+1}}.$$

Therefore, in this case, equation (4.11) yields that

$$f_{\hat{\varepsilon}} = \mathbf{r}_{\varepsilon_r, \varepsilon_{r+1}} \circ \mathbf{l}_{\varepsilon_r, \varepsilon_{r+1}} \circ \mathbf{r}_{\varepsilon_{r-1}, \varepsilon_r} \circ \mathbf{l}_{\varepsilon_{r-1}, \varepsilon_r} \circ \dots \circ \mathbf{r}_{\varepsilon_1, \varepsilon_2} \circ \mathbf{l}_{\varepsilon_1, \varepsilon_2} \circ f_\varepsilon.$$

Because the homeomorphisms in the above composition commute (cf. Corollary 4.33), by definition of $\mathbf{l}_{\varepsilon, \hat{\varepsilon}}$ and $\mathbf{r}_{\varepsilon, \hat{\varepsilon}}$,

$$(4.12) \quad f_{\hat{\varepsilon}} = \mathbf{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathbf{l}_{\varepsilon, \hat{\varepsilon}}.$$

Thus the result is proved if ε and $\hat{\varepsilon}$ differ in finitely many components.

The case in which they differ by infinitely many components is entirely analogous, as we can consider the composite of infinitely many maps of the above form on, say, any compact subset of B (cf. the proof of Lemma 4.34) and use a compact exhaustion of B . Thus, assuming that the result holds when the choices of signs differ in precisely one component, the result holds in general.

Step 3: proving the simple case. Assume that $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = f_{\varepsilon}|_{\mathfrak{S}}$ and that ε and $\hat{\varepsilon}$ differ in precisely one component. Under these assumptions the result can be proved exactly as in Vũ Ngọc [46, Proposition 4.1], whose key ideas are explained below. Suppose that ε and $\hat{\varepsilon}$ differ precisely in the i th component. By Corollary 2.52, $f_{\hat{\varepsilon}} \circ f_{\varepsilon}^{-1}$ is a homeomorphism that is piecewise \mathbb{Z} -affine. Using the proof of Theorem 4.24, it may be assumed without loss of generality that S_{ε} and $S_{\hat{\varepsilon}}$ are path-connected. As both sets are dense, it suffices to check the desired equality on $S_{\varepsilon} \cap S_{\hat{\varepsilon}}$. Since ε and $\hat{\varepsilon}$ differ in precisely one component, it follows that $S_{\varepsilon} \cap S_{\hat{\varepsilon}}$ has two connected components, \mathfrak{S} and \mathfrak{S}' , which are open and satisfy the assumptions of Proposition 4.7. Thus the subsystems of $(M, \omega, \Phi = (J, H))$ relative to \mathfrak{S} and \mathfrak{S}' are vertical almost-toric. By Corollary 2.49, the restrictions of $f_{\varepsilon}, f_{\hat{\varepsilon}}$ to \mathfrak{S} and \mathfrak{S}' are cartographic homeomorphisms for the respective subsystems of $(M, \omega, \Phi = (J, H))$. By construction, these subsystems contain no focus-focus points. Thus, Remark 4.18 implies that there exist $h_{\mathfrak{S}}, h_{\mathfrak{S}'} \in \text{Vert}(2; \mathbb{Z})$ with $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = h_{\mathfrak{S}} \circ f_{\varepsilon}|_{\mathfrak{S}}$ and $f_{\hat{\varepsilon}}|_{\mathfrak{S}'} = h_{\mathfrak{S}'} \circ f_{\varepsilon}|_{\mathfrak{S}'}$.

By assumption, $f_{\hat{\varepsilon}}|_{\mathfrak{S}} = f_{\varepsilon}|_{\mathfrak{S}}$, so $h_{\mathfrak{S}} = \text{id}$; on the other hand, the above assumptions imply that $(\mathbf{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathbf{l}_{\varepsilon, \hat{\varepsilon}})|_{\mathfrak{S}} = \text{id}$. Thus the desired equality holds on \mathfrak{S} . It remains to check that it does on \mathfrak{S}' . Using property (C2) for f_{ε} and $f_{\hat{\varepsilon}}$ and the fact that $\text{sgn}(f_{\varepsilon}) = \text{sgn}(f_{\hat{\varepsilon}})$, it can be shown that the linear parts of $h_{\mathfrak{S}'}$ and of $(\mathbf{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathbf{l}_{\varepsilon, \hat{\varepsilon}})|_{\mathfrak{S}'}$ are equal. To see that their translational components are equal, observe that the piecewise \mathbb{Z} -affine transformation given on $f_{\varepsilon}(\mathfrak{S})$ and on $f_{\varepsilon}(\mathfrak{S}')$ by $h_{\mathfrak{S}}$ and $h_{\mathfrak{S}'}$, respectively, extends uniquely to a topological embedding of $f_{\varepsilon}(B)$ onto $f_{\hat{\varepsilon}}(B)$ (which equals $f_{\hat{\varepsilon}} \circ f_{\varepsilon}^{-1}$). In particular, it acts as the identity on the vertical line containing the i th focus-focus value. This implies that the translational component of $h_{\mathfrak{S}'}$ equals that of $(\mathbf{r}_{\varepsilon, \hat{\varepsilon}} \circ \mathbf{l}_{\varepsilon, \hat{\varepsilon}})|_{\mathfrak{S}'}$. \square

4.5. η -cartographic vertical almost-toric systems. Let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system. The presence of a focus-focus point implies that no vertical almost-toric system isomorphic to $(M, \omega, \Phi = (J, H))$ has a cartographic moment map (cf. Section 2.7).

On the other hand, Theorem 4.24 provides cartographic homeomorphisms associated to choices of vertical cuts. Fix any such cartographic homeomorphism f_ε ; while it is tempting to think of $(M, \omega, f_\varepsilon \circ \Phi)$ as an integrable system, the lack of smoothness of f_ε prevents it from being one. (If we were to adopt the non-standard convention of Harada & Kaveh [17, Definition 2.1], $(M, \omega, f_\varepsilon \circ \Phi)$ would be an integrable system.) The aim of this section is to show that, in some sense, the next best scenario holds: Given a choice of signs ε with S_ε connected, any cartographic homeomorphism associated to ε can be modified in an arbitrarily small neighborhood of the cuts associated to ε so that it becomes everywhere smooth (see Theorem 4.48 for a precise statement). This smoothing of cartographic homeomorphisms generates representatives in the isomorphism class of a simple vertical almost-toric system, which we call η -*cartographic*, that are particularly useful when defining surgeries on (isomorphism classes of) simple vertical almost-toric systems (cf. the forthcoming [20]). Moreover, we show that cartographic homeomorphisms are, in some sense, limits of what we call η -*cartographic embeddings* (see Proposition 4.52).

Throughout the rest of this section, a simple vertical almost-toric system $(M, \omega, \Phi = (J, H))$ containing at least one focus-focus singular point is fixed. As above, set $B = \Phi(M)$, let $\{c_i\} \subset \text{Int}(B)$ denote the set of focus-focus values of (M, ω, Φ) , while l^ε denotes the union of the vertical cuts in B associated to a choice of signs $\varepsilon \in \{+1, -1\}^I$. Also let $S_\varepsilon = B \setminus l^\varepsilon$ denote the complement of those cuts. Finally, assume a cartographic homeomorphism has positive sign (cf. Theorem 4.24) unless otherwise stated.

4.5.1. *Admissible half-strips for simple vertical almost-toric systems.* First, we define the (closed) neighborhoods of vertical half-lines that we use to construct the smoothing of a given cartographic homeomorphism.

Definition 4.38. Fix $\varepsilon \in \{+1, -1\}$, $(x_0, y_0) \in \mathbb{R}^2$, $\eta > 0$, and a continuous map $\gamma : [x_0 - \frac{\eta}{2}, x_0 + \frac{\eta}{2}] \rightarrow \mathbb{R}$ satisfying $\varepsilon y_0 > \varepsilon \gamma(x)$ for all $x \in [x_0 - \frac{\eta}{2}, x_0 + \frac{\eta}{2}]$. A *half-strip centered at (x_0, y_0) of sign ε and width η with bounding curve γ* is the following closed subset of \mathbb{R}^2 :

$$\sigma_{\eta, \gamma}^\varepsilon(x_0, y_0) := \left\{ (x, y) \mid x_0 - \frac{\eta}{2} \leq x \leq x_0 + \frac{\eta}{2} \text{ and } \varepsilon y \geq \varepsilon \gamma(x) \right\}$$

(see Figure 4.3). The vertical line $\{(x, y) \mid x = x_0\}$ is called the *center line* of the half-strip. When the center point (x_0, y_0) and the bounding curve γ are not of particular concern, the half-strip is denoted by σ_η^ε .

The *base* of a half-strip $\sigma_{\eta,\gamma}^\varepsilon(x_0, y_0)$ is the subset

$$\sigma_{\eta,\gamma}^\varepsilon(x_0, y_0) \cap \left\{ (x, y) \mid \varepsilon y < \varepsilon y_0 + \frac{\eta}{2} \right\}.$$

Consider a choice of (countably many) points $\{(x_i, y_i)\}_{i \in I}$, of signs $\varepsilon \in \{+1, -1\}^I$, of positive numbers $\eta \in \{\eta_i\}_{i \in I}$, and of continuous curves $\gamma = \{\gamma_i\}_{i \in I}$. Let $\sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i)$ be the half-strip centered at (x_i, y_i) of sign ε_i and width $\eta_i > 0$ with bounding curve γ_i . Moreover, set

$$\sigma_{\eta,\gamma}^\varepsilon := \bigcup_i \sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i),$$

and denote the above choices of signs, widths and curves by the triple $(\varepsilon, \eta, \gamma)$.

Definition 4.39. Suppose $B \subset \mathbb{R}^2$ has the property that its intersection with any vertical line is either empty or path-connected, and consider a countable set of points $\{(x_i, y_i)\}_{i \in I}$ therein. A triple $(\varepsilon, \eta, \gamma)$ as above is *admissible* for the subset B relative to the points $\{(x_i, y_i)\}_{i \in I}$ if it satisfies the following conditions:

- For all i , the base of the half-strip $\sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i)$ is contained in $\text{Int}(B)$.
- If $(x_i, y_i) \in \sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$ for $i \neq j$, then $x_i = x_j$.
- Whenever the half-strips $\sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i)$ and $\sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$ share the same center line, $\eta_i = \eta_j$.
- The intersection of any two distinct half-strips is either empty or equal to one of the half-strips.

In this case, the corresponding half-strips are called *admissible* for B relative to the points $\{(x_i, y_i)\}_{i \in I}$.

Examples of admissible half-strips are sketched in Figure 4.3 (a) and (b).

Definition 4.40. Consider a simple vertical almost-toric system $(M, \omega, \Phi = (J, H))$ whose set of focus-focus values is indexed by I as in equation (4.2). A triple $(\varepsilon, \eta, \gamma)$ as in Definition 4.39 and their corresponding half-strips are *admissible* for $(M, \omega, \Phi = (J, H))$ if they are admissible for B relative to the set of focus-focus values $B_{\text{ff}} = \{c_i\}_{i \in I}$.

Before establishing the existence of admissible half-strips for any simple vertical almost-toric system (Proposition 4.42), we derive the following necessary condition. Recall that the focus-focus values of a simple vertical almost-toric system are ordered as in Section 4.3.

Proposition 4.41. *If $(\varepsilon, \eta, \gamma)$ is an admissible triple for $(M, \omega, \Phi = (J, H))$, then $S_\varepsilon = B \setminus l^\varepsilon$ is contractible.*

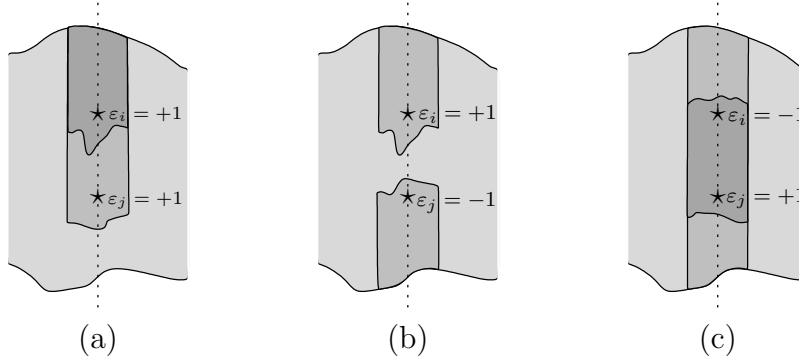


FIGURE 4.3. The symbol \star indicates the points at which the half-strips are centered. Figures (a) and (b) show admissible half-strips with the same center line. Figure (c) shows half-strips that are not admissible.

Proof. Fix an admissible triple $(\varepsilon, \eta, \gamma)$. By Lemma 4.21 and Corollary 4.22, it suffices to check that, if $i > j$ and there are focus-focus points $c_i = (x_i, y_i)$, $c_j = (x_j, y_j)$ with $x_i = x_j$, then $\varepsilon_i \geq \varepsilon_j$. Suppose not, then the half-strips $\sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i)$, $\sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$ intersect, but neither is contained in the other (cf. Figure 4.3 (c)), thus contradicting admissibility of the given triple. \square

Recall that, by Corollary 4.22, S_ε is path-connected if and only if it is contractible. The next result establishes the converse to Proposition 4.41.

Proposition 4.42. *Given a simple vertical almost-toric system $(M, \omega, \Phi = (J, H))$ with at least one focus-focus value and any choice $\varepsilon \in \{+1, -1\}^I$ making S_ε path-connected, there exist a choice of positive numbers $\eta = \{\eta_i\}_{i \in I}$ and of continuous curves $\gamma = \{\gamma_i\}_{i \in I}$ such that the triple $(\varepsilon, \eta, \gamma)$ is admissible for $(M, \omega, \Phi = (J, H))$.*

Proof. Fix a choice of $\varepsilon \in \{+1, -1\}^I$ as above and let $\{c_i = (x_i, y_i)\}_{i \in I}$ denote the set of focus-focus values of $(M, \omega, \Phi = (J, H))$. By Proposition 4.14, the set of first coordinates of focus-focus values is a subset of the set of critical values of J , which, by property (V3), does not contain any limit point in $J(M)$. Moreover, focus-focus values are discrete in B and, by property (V4), there are finitely many of them on a given vertical line. The above facts imply that there exists a choice of positive numbers $\eta = \{\eta_i\}_{i \in I}$ such that

- if $x_i = x_j$, $\eta_i = \eta_j$,
- if, for $i \neq j$, $x_j \in [x_i - \frac{\eta_i}{2}, x_i + \frac{\eta_i}{2}]$, then $x_i = x_j$, and

• for all i , $[x_i - \frac{\eta_i}{2}, x_i + \frac{\eta_i}{2}] \times [\varepsilon_i y_i - \frac{\eta_i}{2}, \varepsilon_i y_i + \frac{\eta_i}{2}]$ is contained in $\text{Int}(B)$.
 For each $i \in I$, define $\gamma_i : [x_i - \frac{\eta_i}{2}, x_i + \frac{\eta_i}{2}] \rightarrow \mathbb{R}$ to be $\gamma_i(x) := y_i - \varepsilon_i \frac{\eta_i}{2}$, and set $\gamma = \{\gamma_i\}_{i \in I}$. It can be checked that the triple $(\varepsilon, \eta, \gamma)$ is admissible for $(M, \omega, \Phi = (J, H))$. \square

Remark 4.43 The choice of widths η in the proof of Proposition 4.42 can be made so that the following property also holds: If a half-strip $\sigma_{\eta_i}^{\varepsilon_i}$ contains a corner of B , then it contains precisely one, and that corner lies on the center line of the half-strip. Henceforth, any admissible triple for a simple vertical almost-toric system is assumed to satisfy this property unless otherwise stated.

Fix an admissible triple $(\varepsilon, \eta, \gamma)$ for $(M, \omega, \Phi = (J, H))$. The next results can be interpreted as showing that the complement of the corresponding half-strips in B behaves like S_ε .

Lemma 4.44. *If $(\varepsilon, \eta, \gamma)$ is admissible for a simple vertical almost-toric system (M, ω, Φ) , then $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ is open in B .*

Proof. As in the proof of Corollary 4.19, it suffices to show that $\sigma_{\eta, \gamma}^\varepsilon$ is closed in B . Let $\{(x_n, y_n)\} \subset \sigma_{\eta, \gamma}^\varepsilon$ be a sequence that converges to $(x_0, y_0) \in B$ and consider the sequence $\{x_n\} = \text{pr}_1(\{(x_n, y_n)\})$ which converges to $x_0 \in J(M)$. Since $J(M)$ is locally compact, there exists a compact neighborhood $K \subset J(M)$ of x_0 . Since $x_n \rightarrow x_0$, it follows that all but finitely many of the x_n are contained in K . Since K is compact and the critical values of J are discrete in $J(M)$ by property (V3), K contains at most finitely many critical values of J . Therefore, by property (V4) and Proposition 4.14, there are at most finitely many focus-focus values contained in $\text{pr}_1^{-1}(K) \cap B$. Hence, all but finitely many of the (x_n, y_n) are contained in the union of finitely many admissible half-strips, each of which is a closed subset of \mathbb{R}^2 and, hence, of B . Thus (x_0, y_0) belongs to this union of finitely many admissible half-strips and so to $\sigma_{\eta, \gamma}^\varepsilon$. \square

Corollary 4.45. *If $(\varepsilon, \eta, \gamma)$ is admissible for a simple vertical almost-toric system (M, ω, Φ) , then $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ is contractible.*

Proof. By Lemma 4.44 the subset $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ is open in B . If the intersection of $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ with every vertical line were either empty or connected, then Proposition 4.7 would imply that the subsystem relative to $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ would be vertical almost-toric, after which Corollary 4.5 would ensure that $B \setminus \sigma_{\eta, \gamma}^\varepsilon$, the moment map image of that vertical almost-toric subsystem, would be contractible. Thus it suffices to show that the intersection of $B \setminus \sigma_{\eta, \gamma}^\varepsilon$ with every vertical line is either empty

or connected. Fix $x_0 \in \text{pr}_1(B)$; if $x_0 \notin \text{pr}_1(\sigma_{\eta,\gamma}^\varepsilon)$,

$$(B \setminus \sigma_{\eta,\gamma}^\varepsilon) \cap \{(x, y) \mid x = x_0\} = B \cap \{(x, y) \mid x = x_0\},$$

and the result follows from the fact that (M, ω, Φ) is vertical almost-toric. Suppose that $x_0 \in \text{pr}_1(\sigma_{\eta,\gamma}^\varepsilon)$ and call a half-strip $\sigma_{\eta_k, \gamma_k}^{\varepsilon_k}(x_k, y_k)$ *maximal* if it is not a proper subset of $\sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$ for any $j \neq k$. Because the triple $(\varepsilon, \eta, \gamma)$ is admissible for (M, ω, Φ) , there are at most two maximal half-strips $\sigma_{\eta_i, \gamma_i}^{\varepsilon_i}(x_i, y_i)$, $\sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$ with the property that for $s = i, j$, $x_0 \in [x_s - \frac{\eta_s}{2}, x_s + \frac{\eta_s}{2}]$. These half-strips are, by definition, disjoint and their bases are contained in $\text{Int}(B)$. That property is sufficient to ensure that $(B \setminus \sigma_{\eta,\gamma}^\varepsilon) \cap \{(x, y) \mid x = x_0\}$ is connected, as desired. \square

Finally, we note that admissible triples behave well under isomorphisms and taking saturated subsystems.

Corollary 4.46. *Let $(\varepsilon, \eta, \gamma)$ be an admissible triple for the vertical almost toric system (M, ω, Φ) . Then any vertical almost-toric system isomorphic to (M, ω, Φ) inherits an admissible triple, as does any subsystem whose image contains every half-strip of $\sigma_{\eta,\gamma}^\varepsilon$ that it intersects. Moreover, for any cartographic homeomorphism f_ε associated to ε , $(\varepsilon, \eta, \text{the triple } \gamma)$ induces an admissible triple for $f_\varepsilon(B)$ relative the image of the focus-focus values $\{f_\varepsilon(c_i)\}_{i \in I}$.*

Proof. Let (M', ω', Φ') be a vertical almost-toric system isomorphic to (M, ω, Φ) via the isomorphism (Ψ, ψ) . The choices $\varepsilon' = \text{sgn}(\det D\psi) \varepsilon$, $\eta' = \eta$ and $\gamma' = \psi \circ \gamma := \{\psi \circ \gamma_i\}_{i \in I}$ define an admissible triple for (M', ω', Φ') because of the special form of ψ (cf. Definition 4.1) and the connectedness of B , ensuring that $\text{sgn}(\det D\psi)$ is constant. Any vertical almost system whose image contains the half-strips of $\sigma_{\eta,\gamma}^\varepsilon$ that it intersects inherits an admissible triple simply by restriction. Finally, given a cartographic homeomorphism f_ε , the signs $\tilde{\varepsilon} = \text{sgn}(f_\varepsilon) \varepsilon$, widths $\tilde{\eta} = \eta$ and continuous curves $\tilde{\gamma} = f_\varepsilon \circ \gamma := \{f_\varepsilon \circ \gamma_i\}_{i \in I}$ define an admissible triple for $f_\varepsilon(B)$ relative to $\{f_\varepsilon(c_i)\}_{i \in I}$. \square

Remark 4.47 In fact, an admissible triple on the cartographic moment map image induces an admissible triple for the system by reversing the above construction.

4.5.2. *Smoothing.* With admissible half-strips at hand, we can state and prove the main result of this section.

Theorem 4.48. *Let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system and let ε be a choice of signs such that $B \setminus l^\varepsilon$ is connected.*

Given any cartographic homeomorphism $f_\varepsilon : B \rightarrow \mathbb{R}^2$, there exists a smooth embedding $F_\varepsilon : B \rightarrow \mathbb{R}^2$ of the form

$$F_\varepsilon(x, y) = (F_\varepsilon^{(1)}, F_\varepsilon^{(2)})(x, y) = (x, F_\varepsilon^{(2)}(x, y))$$

agreeing with f_ε on the complement of an arbitrarily small neighborhood of l^ε .

Proof. Let ε be as in the statement and fix an admissible triple $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$ for (M, ω, Φ) . The map f_ε is smooth on the complement of the cuts. As in the proof of Corollary 4.45, say that a half-strip is maximal if it is not a proper subset of any other half-strip. It is sufficient to modify f_ε in the interior of maximal admissible half-strips, and since maximal half-strips are pairwise disjoint, it suffices to construct the modified map in the interior of each one separately.

Consider a maximal admissible half-strip, say $\sigma_{\eta_j, \gamma_j}^{\varepsilon_j}(x_j, y_j)$. Without loss of generality assume that $\varepsilon_j = +1$ so as to drop the notational dependence of the half-strip on ε_j . Moreover, fix an admissible triple $(\varepsilon, \boldsymbol{\eta}', \boldsymbol{\gamma}')$ for $(M, \omega, \Phi = (J, H))$, where, if $i \neq j$, $\eta'_i = \eta_i$ and $\gamma'_i = \gamma_i$, and if $i = j$ and $\eta'_j < \eta_j$ and $\gamma'_j(x) > \gamma_j(x)$ whenever both make sense.

There are two cases to consider, namely if η'_j can be chosen so that $\partial_\infty B \cap \sigma_{\eta'_j, \gamma'_j} = \emptyset$ or not. Suppose that the former holds; then the set $W_j = B \cap \text{Int}(\sigma_{\eta'_j, \gamma'_j})$ is open in \mathbb{R}^2 . The situation is sketched in Figure 4.4 (a). Let Γ_j be an embedded curve in W_j of the form $\Gamma_j(x) = (x, h_j(x))$, where h_j is a smooth function, that is disjoint from the cut l^{ε_j} and is such that $W_j \setminus \Gamma_j$ has two components. Let K_j, L_j be the closures in W_j of the two components of $W_j \setminus \Gamma_j$, so $K_j \cap L_j = \Gamma_j$, and assume without loss of generality that the cut l^{ε_j} lies in K_j .

Recall that the cartographic homeomorphism f_ε is orientation-preserving and of the special form

$$f_\varepsilon(x, y) = (f_\varepsilon^{(1)}(x, y), f_\varepsilon^{(2)}(x, y)) = (x, f_\varepsilon^{(2)}(x, y)).$$

Therefore, $f \circ \Gamma_j(x) = (x, f_\varepsilon^{(2)}(x, h_j(x)))$ and, since $\varepsilon_j = +1$, if $(x, y) \in K_j$ then $y \geq h_j(x)$ and $f_\varepsilon^{(2)}(x, y) \geq f_\varepsilon^{(2)}(x, h_j(x))$, as f_ε is orientation-preserving. Define $g_j : K_j \rightarrow \mathbb{R}^2$ by

$$g_j(x, y) := (x, y + f_\varepsilon^{(2)}(x, h_j(x)) - h_j(x)),$$

which is an orientation-preserving diffeomorphism of K_j onto its image that satisfies

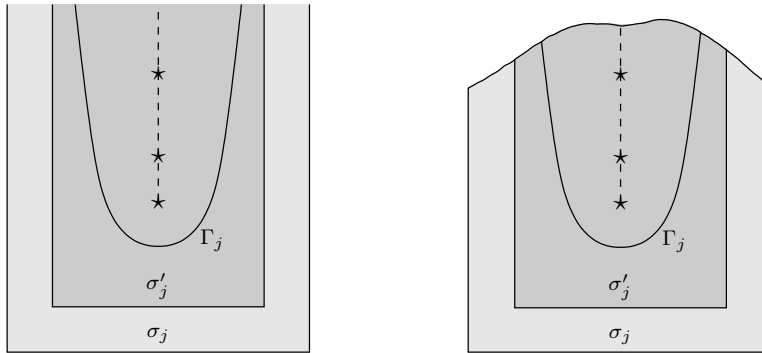
$$g_j(x, h_j(x)) = f_\varepsilon(x, h_j(x)).$$

Now consider the map

$$F'_{\varepsilon_j} : \text{Int}(\sigma_{\eta'_j, \gamma'_j}) \cap B \rightarrow \mathbb{R}^2, \quad \begin{cases} f_\varepsilon(x, y) & \text{if } (x, y) \in L_j \\ g_j(x, y) & \text{if } (x, y) \in K_j. \end{cases}$$

which is a homeomorphism onto its image. Furthermore, because F'_{ε_j} is a diffeomorphism on the complement of Γ_j , which is a closed submanifold of W_j , F'_{ε_j} can be isotoped to be a diffeomorphism onto the image $F'_{\varepsilon_j}(W_j)$ via an isotopy that is supported in an arbitrarily small neighborhood of Γ_j and is the identity in L_j (cf. Hirsch [18, Chapter 8]). By construction, F'_{ε_j} extends to all of $\sigma_{\eta_j, \gamma_j}$ as a diffeomorphism, say F_{ε_j} , on $\sigma_{\eta_j, \gamma_j}$ that agrees with f_{ε} on $\sigma_{\eta_j, \gamma_j} \setminus K_j$. The map F_{ε_j} is the desired smoothing.

It remains to consider the case in which an admissible triple $(\varepsilon, \eta', \gamma')$ as above does not exist, *i.e.*, for any choices of η'_j and γ'_j as above, the corresponding half-strip also intersects $\partial_{\infty} B$ (cf. Figure 4.4 (b)). In this case, modify the argument as follows. Let $\Gamma_j \subset B \cap \text{Int}(\sigma_{\eta_j, \gamma_j})$ be chosen as above, and so that any boundary point p of Γ_j also lies in path-connected component of $\partial_{\infty} B_0 \cap \text{Int}(\sigma_{\eta_j, \gamma_j})$. Because f_{ε} is, by definition, smooth at p , the map f_{ε} and the smooth curve Γ_j can be extended to a neighborhood of $p \in \mathbb{R}^2$. Make such an extension near the one or two boundary points of Γ , and let W_j be an open tubular neighborhood of the extended curve Γ_j . Let K_j and L_j be defined as in the first case (enlarged as per the extension just described), with the map F'_{ε_j} defined as above. But to apply the smoothing argument, restrict attention to $K_j \cap W_j$ and $L_j \cap W_j$ so that Γ_j is a closed submanifold of an open manifold, in this case the tubular neighborhood W_j . \square



(a)

(b)

FIGURE 4.4. Consider the curve Γ_j as ‘seam’ and glue the diffeomorphisms smoothly along Γ_j . (a) sketches the case $\partial_{\infty} B_0 \cap \sigma'_j = \emptyset$ and (b) sketches $\partial_{\infty} B_0 \cap \sigma'_j \neq \emptyset$.

Remark 4.49. Given an admissible triple $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$, a map F_ε as constructed in Theorem 4.48 is referred to as an $\boldsymbol{\eta}$ -cartographic embedding. Moreover, if the dependence on $\boldsymbol{\eta}$ is to be remembered, an $\boldsymbol{\eta}$ -cartographic embedding is denoted by $F_{\varepsilon, \boldsymbol{\eta}}$.

Theorem 4.48 motivates introducing the following notion.

Definition 4.50. A simple vertical almost-toric system is $\boldsymbol{\eta}$ -cartographic if it admits an admissible triple $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$ and a cartographic homeomorphism f_ε whose restriction to the complement of the union of the corresponding admissible half-strips is the identity. If the choice of $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$ is to be remembered, the system is said to be $\boldsymbol{\eta}$ -cartographic with respect to $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$.

The first application of Theorem 4.51 is the following result.

Theorem 4.51. *Any simple vertical almost-toric system (M, ω, Φ) is isomorphic to an $\boldsymbol{\eta}$ -cartographic one.*

Proof. Fix an admissible triple $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$ for (M, ω, Φ) and let $f_\varepsilon : B \rightarrow \mathbb{R}^2$ be the cartographic homeomorphism associated to ε . Let $F_\varepsilon : B \rightarrow \mathbb{R}^2$ be the associated smooth $\boldsymbol{\eta}$ -cartographic embedding constructed in the proof of Theorem 4.48. The form of F_ε implies that, by construction, $(M, \omega, F_\varepsilon \circ \Phi)$ is a vertical almost-toric system isomorphic to (M, ω, Φ) . Moreover, since F_ε is orientation-preserving, $(M, \omega, F_\varepsilon \circ \Phi)$ inherits an admissible triple $(\varepsilon, \boldsymbol{\eta}, F_\varepsilon \circ \boldsymbol{\gamma})$ by Corollary 4.46. The map $f_\varepsilon \circ F_\varepsilon^{-1} : F_\varepsilon(B) \rightarrow \mathbb{R}^2$ is a cartographic homeomorphism for $(M, \omega, F_\varepsilon \circ \Phi)$, which, by definition of F_ε , is the identity on the complement of the admissible half-strips for $(M, \omega, F_\varepsilon \circ \Phi)$ corresponding to $(\varepsilon, \boldsymbol{\eta}, F_\varepsilon \circ \boldsymbol{\gamma})$. Therefore, $(M, \omega, F_\varepsilon \circ \Phi)$ is $\boldsymbol{\eta}$ -cartographic as required. \square

To conclude this section, we show that the image of a cartographic homeomorphism can be seen as a ‘limit’ of the moment map images of $\boldsymbol{\eta}$ -cartographic systems. To make the above precise, let $(M, \omega, \Phi = (J, H))$ be a simple vertical almost-toric system, fix a choice of signs ε for which S_ε is path-connected, and fix a cartographic homeomorphism f_ε associated to ε . Consider the set consisting of quadruples $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma}, F_{\varepsilon, \boldsymbol{\eta}})$, where $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma})$ is an admissible triple for $(M, \omega, \Phi = (J, H))$ and $F_{\varepsilon, \boldsymbol{\eta}}$ is an $\boldsymbol{\eta}$ -cartographic embedding constructed starting from the cartographic homeomorphism f_ε . On this set, we define a partial order \succeq by setting

$$(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma}, F_{\varepsilon, \boldsymbol{\eta}}) \succeq (\varepsilon, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\gamma}}, F_{\varepsilon, \tilde{\boldsymbol{\eta}}}),$$

if and only if $\eta_j \geq \tilde{\eta}_j$ for all j , and $F_{\varepsilon, \boldsymbol{\eta}}(\boldsymbol{\sigma}_{\boldsymbol{\eta}, \boldsymbol{\gamma}}^\varepsilon) \supseteq F_{\varepsilon, \tilde{\boldsymbol{\eta}}}(\boldsymbol{\sigma}_{\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\gamma}}}^\varepsilon)$. Note that \succeq is reflexive and transitive. Moreover, for any two elements $(\varepsilon, \boldsymbol{\eta}, \boldsymbol{\gamma}, F_{\varepsilon, \boldsymbol{\eta}})$

and $(\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}})$, there exists an element $(\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}})$ such that $(\varepsilon, \eta, \gamma, F_{\varepsilon, \eta}) \succeq (\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}})$ and $(\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}}) \succeq (\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}})$, as the construction of $F_{\varepsilon, \eta}$ and $F_{\varepsilon, \tilde{\eta}}$ shows. Thus \succeq turns the set of quadruples $(\varepsilon, \eta, \gamma, F_{\varepsilon, \eta})$ into a directed set.

Proposition 4.52. *A cartographic moment map image of a simple vertical almost-toric system associated to a choice of signs whose corresponding cuts do not disconnect the moment map image is the direct limit of η -cartographic moment map images.*

Proof. As above, fix a vertical almost-toric system $(M, \omega, \Phi = (J, H))$, a choice of signs ε making S_ε path-connected, and a cartographic homeomorphism f_ε associated to ε . The set of quadruples $(\varepsilon, \eta, \gamma, F_{\varepsilon, \eta})$ is a directed set with the above partial order \succeq . The condition $(\varepsilon, \eta, \gamma, F_{\varepsilon, \eta}) \succeq (\varepsilon, \tilde{\eta}, \tilde{\gamma}, F_{\varepsilon, \tilde{\eta}})$ implies that

$$(4.13) \quad F_{\varepsilon, \eta}(B \setminus \sigma_{\eta, \gamma}^\varepsilon) \subseteq F_{\varepsilon, \tilde{\eta}}(B \setminus \sigma_{\tilde{\eta}, \tilde{\gamma}}^\varepsilon)$$

such that $f_\varepsilon(B \setminus l^\varepsilon)$ coincides with the direct limit in the category of topological spaces given by

$$\lim_{\rightarrow} F_{\varepsilon, \eta}(B \setminus \sigma_{\eta, \gamma}^\varepsilon) = \left(\bigsqcup_{\eta} F_{\varepsilon, \eta}(B \setminus \sigma_{\eta, \gamma}^\varepsilon) \right) / \sim,$$

where $z \in F_{\varepsilon, \eta}(B \setminus \sigma_{\eta, \gamma}^\varepsilon) \sim \tilde{z} \in F_{\varepsilon, \tilde{\eta}}(B \setminus \sigma_{\tilde{\eta}, \tilde{\gamma}}^\varepsilon)$ if z and \tilde{z} get mapped under the corresponding inclusions in (4.13) to the same point. \square

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