CHARACTERIZATION OF TORIC SYSTEMS VIA TRANSPORT COSTS

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Abstract. We characterize completely integrable Hamiltonian systems inducing an effective Hamiltonian torus action as systems with zero transport costs w.r.t. the time-T map where $T \in \mathbb{R}^n$ is the period of the acting n-torus.

1. Introduction

Intuitively, integrable Hamiltonian systems in the sense of Liouville can be seen as Hamiltonian systems with 'sufficiently many conserved quantities' — for the precise definition we refer the reader to Section 2.2. Standard examples are the spherical pendulum, coupled spin oscillators, coupled angular momenta, the spinning top etc. Integrable systems are of interest for many reasons, among others, since they display very diverse dynamical behaviour although their solutions are confined to lower dimensional sets and since not all 'nice properties' disappear under small perturbations (cf. KAM theory).

During the last 3-4 decades, there have been several breakthroughs in terms of achieving local or global, topological or symplectic classifications of certain types of integrable systems. Being far from exhaustive, let us just recall some (symplectic) classifications of interest for this short note.

There is Delzant's [12] symplectic classification of completely integrable systems on compact symplectic manifolds that induce effective Hamiltonian torus actions, briefly '(compact) toric systems'. The classifying invariant of a toric system is the image of its momentum map. It is a convex polytope, more precisely a so-called *Delzant polytope*. This class of polytopes in fact represents all possible compact toric systems up to equivariant symplectomorphism (for more details consult e.g. the lecture notes by Audin & Cannas da Silva & Lerman [8] and Cannas da Silva [11]). A toric system is thus in fact determined by the *finite* set of data given by its 'momentum polytope'. This makes the class of toric systems very special within the class of all integrable systems. The singular points of toric or toric type systems only admit elliptic and regular components, but no focus-focus or hyperbolic components — which may occur at nondegenerate singularities of general integrable systems (see the local normal form based on the works by Eliasson [15, 16], Vũ Ngọc & Wacheux [37], Miranda & Zung [25], and others).

Semitoric systems are completely integrable Hamiltonian systems on 4-dimensional manifolds such that (a) one of the integrals is proper and in addition the momentum map of an effective Hamiltonian $\mathbb{S}^1$-action and (b) the whole system admits, in addition to the singularities occuring in toric systems, also focus-focus singularities but no hyperbolic components. Thus semitoric systems induce an $\mathbb{S}^1 \times \mathbb{R}$-action and are thus 'sandwiched between' toric systems (inducing an $\mathbb{S}^1 \times \mathbb{S}^1$-action) and arbitrary completely integrable

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systems (inducing an $\mathbb{R}^2$-action). An overview over the most prominent features and developments can be found in the surveys by Pelayo & Vũ Ngọc [30] and, more recently, by Alonso & Hohloch [3]. Under certain assumptions, semitoric systems have been classified by Pelayo & Vũ Ngọc [28, 29]. Palmer & Pelayo & Tang [27] generalized this classification by removing some of these assumptions. Moreover, there have been recent efforts towards dealing with hyperbolic components, see Dullin & Pelayo [14] and Le Floch & Palmer [22], which seems to go hand in hand with the appearance of certain degenerate singularities. The class of semitoric systems is much richer than the one of toric systems since the invariants in the (constructive) semitoric classification correspond to infinitely many data needed to determine a given semitoric system. Nevertheless the class of semitoric systems is 'still manageable' since the invariants are computable (see e.g. Pelayo & Vũ Ngọc [31], Babelon & Douçot [9], Le Floch & Pelayo [23] for the linear term of the so-called *Taylor series*, and Alonso & Hohloch & Dullin [1, 2] for parameter dependence and higher order terms and the *twisting index*). Furthermore, the classification of semitoric systems fits well together with existing classifications like Delzant's [12] and Karshon's [21] of effective Hamiltonian $\mathbb{S}^1$-actions on compact symplectic 4-dimensional manifolds as shown by Hohloch & Sabatini & Sepe [18]).

In the present note, we give a characterization of toric systems by means of 'transport costs' which measure 'how toric or nontoric' a system is: toric systems on $2n$-dimensional manifolds are precisely those systems having zero transport costs w.r.t. the time-T map where $T \in \mathbb{R}^n$ is the period of the acting n-torus. To be more precise, we introduce the notion of *periodicity costs* (see Definition 4.2) and characterize toric systems as having zero periodicity costs (see Proposition 4.3 and Corollary 4.4).

This approach is inspired by the techniques developed around the transport problems of Monge [26] and Kantorovich [19, 20] where, for given cost functions, optimal transport functions resp. transport measures are looked for. Transport problems have been vividly studied during the last 3-4 decades. For an introduction and overview, see e.g. the lecture notes by Ambrosio & Gigli [6] and Thorpe [35] and the monographs by Rachev & Rüschendorf [32], Santambrogio [33], and Villani [36].

Since transport problems 'live naturally' within the calculus of variation, but integrable systems, due to their rigidity, usually do not at all 'fit well together' with variational methods, it is quite astonishing that 'periodicity costs' in the sense of transport costs provide a meaningful notion within the class of integrable systems, even singling out the subclass of toric systems.

In future projects, we hope to find an answer to e.g. the following questions:

1) Can one use the notion of periodicity costs to investigate the behaviour around focus-focus points in semitoric systems in particular and integrable systems in general, for instance by restricting to a neighbourhood of the focus-focus point or of the whole focus-focus fiber?
2) Do periodicity costs relate to the symplectic invariants classifying semitoric systems, in particular to the Taylor series invariant when restricting to a neighbourhood of the focus-focus fiber?
3) Are there applications of periodicity costs to integrable systems with singularities having hyperbolic components?

4) Are there more methods and ideas from transport theory that have an application within integrable systems?

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2. Notions and conventions

Within this section, let (M, ω) be a $2n$-dimensional compact symplectic manifold.

2.1. **Basic definitions and conventions.** The *Hamiltonian vector field* X^f of a smooth function $f : M \to \mathbb{R}$ is defined by $\omega(X^f, \cdot) = -df$. The flow of the associated *Hamiltonian equation* $z' = X^f(z)$ is called *Hamiltonian flow of* f and denoted by φ^f. It is a smooth map $\varphi^f : \mathbb{R} \times M \to M$ where we usually write $\varphi^f(t, p) =: \varphi^f_t(p)$ with $t \in \mathbb{R}$ and $p \in M$. Hence, for all $t \in \mathbb{R}$, we get a diffeomorphism $\varphi^f_t : M \to M$ that is in fact symplectic and often referred to as *time-t map*.

Given two smooth functions $f, g : M \to \mathbb{R}$, their *Poisson bracket induced by* ω is defined as $\{f, g\} := \omega(X^f, X^g) = -df(X^g) = dg(X^f)$. The smooth functions $f, g : M \to \mathbb{R}$ are said to *Poisson commute* if $\{f, g\} = 0$.

In our sign convention, the Lie bracket and the Poisson bracket are related via $[X^f, X^g] = X^{-\{f,g\}}$. Poisson commutativity of f and g implies $[X^f, X^g] = 0$ and thus commutativity of their Hamiltonian flows, i.e., $\varphi^f_s \circ \varphi^g_t = \varphi^g_t \circ \varphi^f_s$ for all $s, t \in \mathbb{R}$.

For more details, proofs, and further reading, we refer the interested reader e.g. to McDuff & Salamon [24].

2.2. **Integrable Hamiltonian systems.** Recall that $\dim M = 2n$. A smooth function $h := (h_1, \dots, h_n) : M \to \mathbb{R}^n$ is said to be a *(momentum map of a)* $2n$*-dimensional completely integrable Hamiltonian system* if $X^{h_1}, \dots, X^{h_n}$ are almost everywhere linearly independent and if $\{h_i, h_j\} = 0$ for all $1 \leq i, j \leq n$.

We briefly write (M, ω, h) for a completely integrable system on (M, ω) with momentum map h. We speak of a *compact* completely integrable system if we want to emphasize that the underlying symplectic manifold (M, ω) is compact.

We call $p \in M$ a *regular* point of a completely integrable systems (M, ω, h) if $\dim(\mathrm{Span}\{X^{h_1}(p), \dots, X^{h_n}(p)\}) = n$ and *singular* otherwise. We denote the set of regular points by M^{reg} and the set of singular points by M^{sing}. The property *almost everywhere* in the definition of an integrable system is understood w.r.t. to the measure μ_ω induced by the n-fold wedge product ω^n of ω (which in turn is continuous w.r.t. the Lebesgue measure and vice versa). Thus $\mu_\omega(M) = \mu_\omega(M^{reg})$ and $\mu_\omega(M^{sing}) = 0$.

Let $t := (t_1, \dots, t_n) \in \mathbb{R}^n$ and let α be a permutation of the set $\{1, \dots, n\}$. Commutativity of the flows of the component functions $h_1, \dots, h_n$ implies

$$\varphi^{h_1}_{t_1} \circ \dots \circ \varphi^{h_n}_{t_n} = \varphi^{h_{\alpha(1)}}_{t_{\alpha(1)}} \circ \dots \circ \varphi^{h_{\alpha(n)}}_{t_{\alpha(n)}}.$$

Thus a completely integrable system induces a welldefined (Hamiltonian) action of the abelian group $(\mathbb{R}^n, +)$ on M via

$$\mathbb{R}^n \times M \to M, \quad (t, p) \mapsto t.p := \varphi^h_t(p) := \varphi^{h_1}_{t_1} \circ \dots \circ \varphi^{h_n}_{t_n}.$$

For details, proofs, and further reading, we refer e.g. to Bolsinov & Fomenko [10].

2.3. **Toric systems.** Let $\mathbb{T}^1 = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}^n = (\mathbb{S}^1)^n$ and recall that a group action is said to be *effective* or *faithful* if the identity element is the only one acting trivially.

A $2n$-dimensional completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$ is *toric* if the action induced by the flow $\varphi^h : \mathbb{R} \times M \to M$ is an effective (Hamiltonian) $\mathbb{T}^n$-action, i.e., $\varphi^{h_k}_{2\pi} = \mathrm{Id}_M$ for all $k \in \{1, \dots, n\}$ and 2π is the 'minimal common period' of the n Hamiltonian circle actions induced by $h_1, \dots, h_n : M \to \mathbb{R}$.

For more details, proofs, and further reading, we refer e.g. to Cannas da Silva [11].

3. Optimal transport

Let M be a compact manifold and μ_- and μ_+ two (positive) measures with same total mass $\mu_-(M) = \mu_+(M) < \infty$. Let $c : M \times M \to \mathbb{R}^{\geq 0}$ be a 'sufficiently regular' function, usually referred to as *cost function*.

Introductory literature on optimal transport are e.g. the lecture notes by Ambrosio & Gigli [6] and Thorpe [35].

3.1. **The Monge transport problem.** Consider the *space of transport maps*

$$\mathcal{F}(\mu_-, \mu_+) := \{f : M \to M \mid f \text{ measurable}, f(\mu_-) = \mu_+\}$$

where $f(\mu_-)$ denotes the image or push forward measure of μ_- under f. Formulated in modern language, the French mathematician Monge [26] asked in 1781 if there is $f \in \mathcal{F}(\mu_-, \mu_+)$ minimizing

$$\int_M c(x, f(x))\, d\mu_-$$

over $\mathcal{F}(\mu_-, \mu_+)$. This is a nonlinear optimization problem and usually referred to as *Monge transport problem*.

Note that, if μ_- contains point measures, there does not necessarily exist a transport *map* since the requirement $f(\mu_-) = \mu_+$ may force the mass of *one* point to be distributed over *several distinct* points.

In 1979, Sudakov [34] proposed a proof of Monge's problem in $\mathbb{R}^n$ with the Euclidean distance as cost function. Unfortunately the proof turned out to have a gap (cf. Ambrosio [4, p. 137], [5, Chapter 6]) that can only be mended under stronger assumptions.

For more details and references, we refer the reader e.g. to the monographs by Rachev & Rüschendorf [32] and Villani [36].

3.2. **The Kantorovich transport problem.** Let $p_-, p_+ : M \times M \to M$ be the projections on the first and second factor respectively and consider the *space of transport measures*

$$\mathcal{M}(\mu_-, \mu_+) := \{\mu \text{ measure on } M \times M \mid p_-(\mu) = \mu_-,\ \ p_+(\mu) = \mu_+\}.$$

Then the search for a measure $\mu \in \mathcal{M}(\mu_-, \mu_+)$ minimizing

$$\int_{M \times M} c(x, y)\, d\mu$$

over $\mathcal{M}(\mu_-, \mu_+)$ is referred to as solving the *Kantorovich transport problem*. It is named after the Russian mathematician Kantorovich [19, 20] and it enjoys much more 'analysis friendly' properties than Monge's problem, for details see e.g. Rachev & Rüschendorf [32] and Villani [36].

Both transport problems are related as follows: If $f \in \mathcal{F}(\mu_-, \mu_+)$ is a solution of Monge's problem, then we have $(\mathrm{Id}_M \times f)(\mu_-) \in \mathcal{M}(\mu_-, \mu_+)$ and find

$$\begin{aligned}\inf_{f\in\mathcal{F}(\mu_-,\mu_+)} \int_M c(x, f(x))\, d\mu_- &= \inf_{f\in\mathcal{F}(\mu_-,\mu_+)} \int_{M\times M} c(x,y)\, d(\mathrm{Id}_M \times f)(\mu_-)\\ &\geq \inf_{\mu\in\mathcal{M}(\mu_-,\mu_+)} \int_{M\times M} c(x,y)\, d\mu.\end{aligned}$$

i.e., a solution of Kantorovich's problem gives a lower bound for Monge's problem. Under certain convexity and growth assumptions on the cost function, Gangbo & McCann [7] stated an explicit formula for the optimal transport map and showed that optimal transport measures 'lie in the graph' of the optimal transport map.

4. Characterization of toric systems by transport costs

To ensure that the Hamiltonian flow is defined for all times and that all appearing integrals are finite, we assume the $2n$-dimensional symplectic manifold (M, ω) to be compact throughout this section. M need not be connected.

We now consider the following modified transport problem: Given a cost function and a certain type of transport maps, how does the 'minimal' transport map *within this type of transport maps* look like?

4.1. **The cost functional.** Let (M, ω, h) be a completely integrable system. Let $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and recall that the time-t map $\varphi_t^h : M \to M$ is symplectic. The n-fold wedge product ω^n of ω is a volume form on M that is invariant under the time-t map, i.e., $\left(\varphi_t^h\right)^* \omega^n = \omega^n$. When we consider ω^n as measure on M we write μ_ω. The image measure under the time-t map satisfies $\varphi_t^h(\mu_\omega) = \mu_\omega$.

Let $c : M \times M \to \mathbb{R}^{\geq 0}$ be a continuous cost function and let $U \subseteq M$ be open. Define the parameter depending integral

$$C_t^h(U, c) := \int_U c(x, \varphi_t^h(x))\, d\mu_\omega.$$

The map

$$c \circ (\mathrm{Id}_M \times \varphi_t^h) \;:\; \mathbb{R} \to \mathbb{R}^{\geq 0}, \qquad t \mapsto c(x, \varphi_t^h(x))$$

is continuous for all $x \in M$ and so is $\mathbb{R} \to \mathbb{R}^{\geq 0}$, $t \mapsto C_t^h(U, c)$ for all open $U \subseteq M$.

4.2. **Characterization of toric systems.** We begin with

Definition 4.1. *Let M be a manifold. A function $c : M \times M \to \mathbb{R}^{\geq 0}$ is* metric-like *if*

1) *$c(x, y) = 0$ for $x, y \in M$ if and only if $x = y$.*
2) *$c(x, y) = c(y, x)$ for all $x, y \in M$.*

For instance, the Euclidean distance is a continuous metric-like cost function. Its square is a smooth metric-like cost function.

The functional $C_t^h(U, c)$ can be used to measure 'how periodic' a given completely integrable system is:

Definition 4.2. *Let (M, ω, h) be a $2n$-dimensional compact completely integrable system and $c : M \times M \to \mathbb{R}^{\geq 0}$ a continuous metric-like cost function and $T \in \mathbb{R}^n$. We call $C_T^h(M, c) \in [0, \infty[$ the T-*periodicity costs *of (M, ω, h) w.r.t. the cost function c.*

Now we will see what these notions mean for toric systems.

Proposition 4.3. *Let (M,ω,h) be a $2n$-dimensional compact completely integrable system and $c : M \times M \to \mathbb{R}^{\geq 0}$ a continuous metric-like cost function. Then (M,ω,h) is toric if and only if $C^h_{(2\pi,\dots,2\pi)}(M,c) = 0$ and $C^h_s(M,c) > 0$ for all $s \in \mathbb{R}^n \setminus \{(2\pi k,\dots,2\pi k) \mid k \in \mathbb{Z}\}$.*

Proof. (M,ω,h) being a toric system means that the flow φ^h is $(2\pi,\dots,2\pi)$-periodic and the action is effective, i.e., $(2\pi,\dots,2\pi)$ is minimal in the sense that $\varphi^h_s \neq \mathrm{Id}_M$ for all $s \in [0,2\pi]^n \setminus \{(0,\dots,0),(2\pi,\dots,2\pi)\}$.

'$\Rightarrow$': Since $\varphi^h_{(2\pi,\dots,2\pi)}(x) = x$ for all $x \in M$ we obtain $c(x,\varphi^h_{(2\pi,\dots,2\pi)}(x)) = 0$ for all $x \in M$ and thus $C^h_{(2\pi,\dots,2\pi)}(M,c) = 0$. Now let $s \in [0,2\pi]^n \setminus \{(0,\dots,0),(2\pi,\dots,2\pi)\}$. Since $\varphi^h_s \neq \mathrm{Id}_M$ there exists $y \in M$ with $\varphi^h_s(y) \neq y$. Continuity of φ^h_s implies the existence of an open neighbourhood $U \subseteq M$ of y with $\varphi^h_s(z) \neq z$ for all $z \in U$. Since c is metric-like, $c(z,\varphi^h_s(z)) > 0$ for all $z \in U$ and thus $0 < C^h_s(U,c) \leq C^h_s(M,c)$. The $(2\pi,\dots,2\pi)$-periodicity of φ^h implies that this is true for all $s \in \mathbb{R}^n \setminus \{(2\pi k,\dots,2\pi k) \mid k \in \mathbb{Z}\}$.

'$\Leftarrow$': Assume that $\varphi^h_{(2\pi,\dots,2\pi)} \neq \mathrm{Id}_M$. Then there exists $x \in M$ with $\varphi^h_{(2\pi,\dots,2\pi)}(x) \neq x$. Metric-likeness implies $c(x,\varphi^h_{(2\pi,\dots,2\pi)}(x)) > 0$. Hence, since c and φ^h are continuous, there exists an open neighbourhood U of x with $\varphi^h_{(2\pi,\dots,2\pi)}(y) \neq y$ for all $y \in U$ and thus $c(y,\varphi^h_{(2\pi,\dots,2\pi)}(y)) > 0$ for all $y \in U$. Therefore $0 < C^h_{(2\pi,\dots,2\pi)}(U,c) \leq C^h_{(2\pi,\dots,2\pi)}(M,c)$ $\lightning$.

Since $\varphi^h_{(2\pi,\dots,2\pi)} = \mathrm{Id}_M$, it suffices to show $\varphi^h_s \neq \mathrm{Id}_M$ for all $s \in [0,2\pi]^n \setminus \{(0,\dots,0),(2\pi,\dots,2\pi)\}$ to prove the claim for all $s \in \mathbb{R}^n \setminus \{(2\pi k,\dots,2\pi k) \mid k \in \mathbb{Z}\}$. Since $C^h_s(M,c) > 0$ there exists $V \subseteq M$ with $\mu_\omega(V) > 0$ and $c(z,\varphi^h_s(z)) > 0$ for all $z \in V$. Because of $\mu_\omega(M^{reg}) = \mu_\omega(M)$ we have $V \cap M^{reg} \neq \emptyset$ and $\mu_\omega(V \cap M^{reg}) = \mu_\omega(V) > 0$ and in particular $c(z,\varphi^h_s(z)) > 0$ for all $z \in V \cap M^{reg}$, i.e., $\varphi^h_s(z) \neq z$ for all $z \in V \cap M^{reg}$, i.e., $\varphi^h_s \neq \mathrm{Id}_M$. □

If a group does not act effectively then the set of elements acting trivially is a nontrivial normal subgroup. Dividing the original group by this normal subgroup yields a group inducing the same action but acting effectively. When considering torus actions and defining the acting torus in terms of periods then effectiveness translates into working with the torus with 'smallest possible' (= minimal) period for the given torus action.

Thus, if we do not work with the normalization $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, Proposition 4.3 amounts to

Corollary 4.4. *Proposition 4.3 holds true for toric systems induced by an effectively acting n-torus with period $T \in \mathbb{R}^n$ when replacing $(2\pi,\dots,2\pi)$ by $T = (T_1,\dots,T_n)$ and $(2\pi k,\dots,2\pi k)$ by $(T_1 k,\dots,T_n k)$.*

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