Hyperkähler Floer theory as infinite dimensional Hamiltonian system

Sonja Hohloch
École Polytechnique Fédérale de Lausanne (EPFL)
sonja.hohloch@epfl.ch

December 11, 2014

Abstract

We reformulate the equation characterising the critical points of the hypersymplectic action functional as solutions of a Hamiltonian system on the iterated loop space. The intend is to gain more insight into dynamics of hyperkähler Floer theory.

1 Introduction

Floer theory was devised by Floer [Fl1], [Fl2], [Fl3] at the end of the 1980's with the intend to solve Arnold's Conjecture on the number of fixed points of a Hamiltonian diffeomorphism. Since then, Floer theory developed into a powerful tool in symplectic geometry, but its techniques are not limited to symplectic geometry. For instance, in order to study 3-dimensional manifolds, Ozsváth & Szabó associated a suitable even-dimensional manifold to a 3-dimensional manifold and imitated Floer theoretic methods. More generally, apart from the resulting Heegaard Floer theory, there exist a whole zoo of various Floer theories for 3-manifolds like Knot Floer homology, Seiberg-Witten Floer homology, Embedded Contact homology etc.

On hyperkähler manifolds, Floer theory and an analogue of Arnold's Conjecture was established by Hohloch & Noetzel & Salamon [HNS1], [HNS2] and reproved and generalized by Ginzburg & Hein [GH]. The important difference to the previous Floer settings is that hyperkähler Floer theory is not based on the Cauchy-Riemann resp. pseudo-holomorphic equation, but on a 'triholomorphic' equation called the Cauchy-Riemann-Fueter equation. In classical Floer theory, the critical points of the symplectic action functional are 1-periodic Hamiltonian solutions. In hyperkähler Floer homology, the critical points of the hypersymplectic action functional are certain 'triholomorphic 3-manifolds'.

The aim of this short note is to reformulate the 'triholomorphic 3-manifolds' appearing in hyperkähler Floer homology as 1-periodic solutions of a suitable Hamiltonian system on the (iterated) loop space in case the 'triholomorphic 3-manifold' is a torus.

In a subsequent work, we intend to use this approach to come up with a nice geometric interpretation for the index of the 'triholomorphic 3-manifolds' since their index in Hohloch & Noetzel & Salamon [HNS1] is given abstractly by the spectral flow. In classical Floer theory, the Conley-Zehnder index provides a nice geometric interpretation of the index and we hope to find an analogue using the Hamiltonian setting for hyperkähler Floer homology.

Acknowledgements

The author would like to thank Gregor Noetzel for many fruitful discussions during and after the 'hyperkähler Floer homology project' [HNS1], [HNS2]. At some point, we also discussed the possibility of reformulating hyperkähler Floer theory as an infinite dimensional Hamiltonian system. Moreover, the author is indebted to Tudor Ratiu for helpful suggestions.

2 Hyperkähler Floer theory

In this section, we recall the setting for hyperkähler Floer theory in case the 3-dimensional M is a torus.

A manifold is *symplectic* if it carries a nondegenerate closed 2-form. Such a 2-form is called a *symplectic form*. Finite dimensional symplectic manifolds are even dimensional.

Definition 1. *A manifold X is* ***hyperkähler*** *if there are three complex structures I_1, I_2 and I_3 and a metric $\langle \cdot, \cdot \rangle$ such that $I_1 I_2 = -I_2 I_1 = I_3$ and $\langle \cdot, \cdot \rangle = \langle I_i \cdot, I_i \cdot \rangle$ and $\omega_i := \langle I_i \cdot, \cdot \rangle$ are symplectic forms for $1 \leq i \leq 3$.*

In other words, $\langle \cdot, \cdot \rangle$ is Kähler w.r.t. I_1, I_2 and I_3. Finite dimensional hyperkähler manifolds are 4n-dimensional and the quaternions $\mathbb{H}$ with complex structures i, j and k are an example. The only compact 4-dimensional hyperkähler manifolds are the 4-torus and K3-surfaces. In Berger's classification, hyperkähler manifolds appear as those with holonomy group $Sp(n)$. They are Ricci-flat and thus Calabi-Yau such that in particular the first Chern class vanishes. Flat compact hyperkähler manifolds are 4n-tori modulo a finite group action.

Now consider the 3-torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with standard coordinates $t = (t_1, t_2, t_3)$, standard vector fields $\partial_1 := \frac{\partial}{\partial t_1}$, $\partial_2 := \frac{\partial}{\partial t_2}$ $\partial_3 := \frac{\partial}{\partial t_3}$ and volume form $\sigma := dt_1 \wedge dt_2 \wedge dt_3$. Let $(a_{ij}) \in GL(3, \mathbb{R})$ be a constant matrix and set $v_1 := \sum_{k=1}^{3} a_{1k} \partial_k$, $v_2 := \sum_{k=1}^{3} a_{2k} \partial_k$ and $v_3 := \sum_{k=1}^{3} a_{3k} \partial_k$. Note that the Lie derivative $\mathcal{L}_{v_i} \sigma$ vanishes for $1 \leq i \leq 3$, i.e. the (constant) vector fields v_1, v_2, v_3 are volume preserving. Let $(X, I_1, I_2, I_3, \langle \cdot, \cdot \rangle)$ be a hyperkähler manifold and set $\mathcal{F} := \{ f \in C^\infty(\mathbb{T}^3, X) \mid f \text{ contractible} \}$. As explained more detailed later in Section 3, its universal cover can be considered as

$$\widetilde{\mathcal{F}} = \left\{ (f, [F^1], [F^2], [F^3]) \;\middle|\; \begin{array}{l} f \in \mathcal{F}, \\ F^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1 \times \mathbb{S}^1, X),\ \ F^1|_{\mathbb{T}^3} = f, \\ F^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D} \times \mathbb{S}^1, X),\ \ F^2|_{\mathbb{T}^3} = f, \\ F^3 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}, X),\ \ F^3|_{\mathbb{T}^3} = f \end{array} \right\}$$

where $\mathbb{D}$ is the closed unit disk in $\mathbb{R}^2$. The equivalence class $[F^\lambda]$ is the homotopy class of F^λ relative to the boundary, i.e. $F^1(e^{2\pi i t_1}, t_2, t_3) = f(t_1, t_2, t_3)$ etc., and we abbreviate $F^1_{t_2 t_3} := F^1(\cdot, t_2, t_3)$ etc. For $1 \leq j, k \leq 3$, set

$$\mathscr{A}_{jk} : \widetilde{\mathcal{F}} \to \mathbb{R}, \qquad \mathscr{A}_{jk}(f) := -\int_0^1 \int_0^1 \int_{\mathbb{D}} (F^k_{t_\mu t_\nu})^* \omega_j dt_\mu dt_\nu.$$

Geometrically, $\mathscr{A}_{jk}$ is the symplectic action w.r.t. ω_j of the loop $t_k \mapsto f(t)$ averaged over the other two variables t_μ, t_ν. We define the **hypersymplectic action functional** $\mathscr{A}$ via

$$\mathscr{A} := \sum_{j,k=1}^{3} a_{jk} \mathscr{A}_{jk} : \widetilde{\mathcal{F}} \to \mathbb{R}.$$

Let $H : X \times \mathbb{T}^3 \to \mathbb{R}$ be a Hamiltonian function and define the **perturbed hypersymplectic action functional** $\mathscr{A}_H$ via

$$\mathscr{A}_H(f) := \mathscr{A}(f) - \int_{\mathbb{T}^3} H(f(t), t) dt.$$

Note that $\mathbb{T}^3 \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The definition of the hypersymplectic action functional on a cover of the 'torus loop space' $\mathcal{F}$ is motivated by the construction of the classical symplectic action functional on a cover of the standard loop space (see for example McDuff & Salamon [MS], p. 154).

Remark 2. *1) The action functional descends to $\mathcal{F}$ if and only if it is independent of the equivalence classes $(f, [F^1], [F^2], [F^3])$ of f which in turn is equivalent to X being **symplectically aspherical** w.r.t. ω_1, ω_2 and ω_3, i.e. $0 = \int_{\mathbb{S}^2} \mathfrak{s}^* \omega_i$ for all smooth $\mathfrak{s} : \mathbb{S}^2 \to X$ and $1 \leq i \leq 3$.*

2) For flat X, the action functional decends to $\mathcal{F}$, i.e. $\mathscr{A}$, $\mathscr{A}_H : \mathcal{F} \to \mathbb{R}$.

Proof. Item one is clear. It remains to prove item two: If X is flat, its universal cover is $\tau : \mathbb{R}^{4n} \to X$ and any map $\mathfrak{s} : \mathbb{S}^2 \to X$ lifts to a map $\tilde{\mathfrak{s}} : \mathbb{S}^2 \to \mathbb{R}^{4n}$ with $\tau \circ \tilde{\mathfrak{s}} = \mathfrak{s}$. Moreover, all symplectic forms on $\mathbb{R}^{4n}$ are exact, i.e. there is a 1-form α_i such that $\tau^* \omega_i = d\alpha_i$ and we get

$$0 = \int_{\partial \mathbb{S}^2} \tilde{\mathfrak{s}}^* \alpha_i = \int_{\mathbb{S}^2} \tilde{\mathfrak{s}}^* d\alpha_i = \int_{\mathbb{S}^2} \tilde{\mathfrak{s}}^* \tau^* \omega_i = \int_{\mathbb{S}^2} \mathfrak{s}^* \omega_i.$$

□

A vector field ξ along f can be seen as a section in the pullback bundle $\xi \in \Gamma^\infty(f^*TX)$. If we want to compute the critical points of, say, $\mathscr{A}_{12}$ we have to consider a variation of f and F^2 in direction ξ, i.e. we consider $s \mapsto f^s(t_1, t_1, t_3)$ with $\frac{d}{ds}|_{s=0} f^s = \xi$ and $s \mapsto (F^2)^s(t_1, z, t_3)$ with $\frac{d}{ds}|_{s=0}(F^2)^s|_{\mathbb{T}^3} = \xi$ and we calculate, using Cartan's formula and Stokes' theorem,

$$d\mathscr{A}_{12}(f, [F^1], [F^2], [F^3]).\xi = -\int_0^1\int_0^1\int_0^1 \omega_1|_{f(t)}(\xi(t), \partial_2 f(t)) dt_1 dt_2 dt_3 = \int_{\mathbb{T}^3} \langle \xi, I_1 \partial_2 f \rangle dt.$$

For $f \in \mathcal{F}$ and the vector fields v_1, v_2, v_3 on $\mathbb{T}^3$, we set $\partial_{v_i} f := df(v_i)$ for $1 \le i \le 3$. Thus for $\mathscr{A} = \sum_{j,k=1}^3 a_{jk}\mathscr{A}_{jk}$, the critical points $\mathrm{Crit}(\mathscr{A})$ are maps $f \in \widetilde{\mathcal{F}}$ with

$$\not{\partial} f := I_1 \partial_{v_1} f + I_2 \partial_{v_2} f + I_3 \partial_{v_3} f = 0 \tag{3}$$

and $f \in \mathrm{Crit}(\mathscr{A}_H)$ satisfies

$$\not{\partial}_H f := \not{\partial} f - \operatorname{grad} H(f) = 0 \tag{4}$$

where the gradient $\operatorname{grad} H(f)|_t := \operatorname{grad} H(f(t), t)$ is taken w.r.t. the X-valued variable of H and the metric $\langle \cdot, \cdot \rangle$. It is some kind of Dirac type equation. A solution is called *nondegenerate* if the linearized operator for this equation is bijective. By elliptic regularity, every $W^{1,p}$ solution of the above equations is in fact smooth (cf. Hohloch & Noetzel & Salamon [HNS1], Theorem 3.1).

Since the notion of a compact *Cartan hypercontact manifold* will appear only in the next two (quoted) theorems, we omit the definition here and refer the interested reader to Hohloch & Noetzel & Salamon [HNS1], [HNS2] for details. Note that the hypersymplectic action functional looks different in case of a Cartan hypercontact manifold, but its critical points are described by the same equation as above.

Theorem 5 (Hyperkähler Arnold Conjecture, [HNS1, HNS2])**.** *Let M be either a compact Cartan hypercontact 3-manifold (with Reeb vector fields v_i) or the 3-torus (with a constant frame v_i). Let X be a compact flat hyperkähler manifold. Then the space of solutions of $\not{\partial}_H f = 0$ is compact. Moreover, if the contractible solutions are all nondegenerate, then their number is bounded below by the sum of the $\mathbb{Z}_2$-Betti numbers of X. In particular, $\not{\partial}_H f = 0$ has a contractible solution for every H.*

Theorem 5 relies on the construction of Floer theory and its computation:

Theorem 6 ([HNS1, HNS2])**.** *Let M and X be as in Theorem 5 and fix a class $\tau \in \pi_0(C^\infty(M, X))$. Then, for a generic perturbation $H : X \times M \to \mathbb{R}$, there is a natural Floer homology group $HF_*(M, X, \tau; H)$ associated to a chain complex generated by the solutions of $\not{\partial}_H f = 0$. The Floer homology groups associated to different choices of H are naturally isomorphic. Moreover, for the component τ_0 of the constant maps there is a natural isomorphism $HF_*(M, X, \tau_0; H) \cong H_*(X; \mathbb{Z}_2)$.*

Theorem 5 inspired Ginzburg & Hein [GH] to reprove the hyperkähler Arnold conjecture on compact flat hyperkähler manifolds using Conley & Zehnder's method of finite dimensional approximation. They established also the degenerate version. Salamon [Sa] reformulated and relaxed certain conditions in the construction of Hohloch & Noetzel & Salamon [HNS1, HNS2].

3 The infinite dimensional Hamiltonian system on the iterated loop space

In this section, we will reformulate the solutions of (3) and (4) as 1-periodic solutions of a Hamiltonian system on the iterated loop space.

Let $(X, I_1, I_2, I_3, \langle\cdot,\cdot\rangle)$ be a 4n-dimensional hyperkähler manifold with symplectic forms $\omega_i := \langle I_i\cdot,\cdot\rangle$ for $1 \leq i \leq 3$. The space of smooth contractible loops of X is given by $\mathscr{L}(X) := \{c \in C^\infty(\mathbb{S}^1, X) \mid c \text{ contractible}\}$. Its universal cover can be seen as

$$\widetilde{\mathscr{L}}(X) = \big\{(c,[C]) \,\big|\, c \in C^\infty(\mathbb{S}^1, X),\ C \in C^\infty(\mathbb{D}, X),\ C|_{\mathbb{S}^1} = c\big\}$$

for the following reason: there exists the homotopy group relation $\pi_k(\mathscr{L}(X)) \simeq \pi_{k+1}(X)$ which reduces for $k=1$ to $\pi_1(\mathscr{L}(X)) \simeq \pi_2(X)$. Moreover, for the universal cover holds $\mathscr{L}(X) \simeq \widetilde{\mathscr{L}(X)}/\pi_1(\mathscr{L}(X))$ which simplifies to $\mathscr{L}(X) \simeq \widetilde{\mathscr{L}(X)}/\pi_2(X)$, i.e. $\pi_2(X)$ acts on $\widetilde{\mathscr{L}(X)}$ via building the connected sum of a 2-sphere with a disk which yields again a disk (see also McDuff & Salamon [MS], p. 154, for a 'smaller' covering space of the loop space used in classical Floer theory).

The twice iterated loop space can be seen as $\mathscr{L}(\mathscr{L}(X)) \simeq \mathscr{L}^2(X) = C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, X)$ and its universal cover is

$$\widetilde{\mathscr{L}^2}(X) = \left\{ (\gamma, [G^1], [G^2]) \;\middle|\; \begin{array}{l} \gamma \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, X), \\ G^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1, X),\ G^1|_{\mathbb{S}^1\times\mathbb{S}^1} = \gamma, \\ G^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D}, X),\ G^2|_{\mathbb{S}^1\times\mathbb{S}^1} = \gamma \end{array} \right\}.$$

$\mathscr{L}^2(X)$ carries three symplectic forms Ω_i with $1 \leq i \leq 3$ given by

$$\Omega_i|_\gamma(v,w) := \int_0^1\int_0^1 \omega_i|_{\gamma(s,r)}\big(v(s,r), w(s,r)\big)dsdr$$

where v and w are vector fields along γ, i.e. $v,\, w \in T_\gamma\widetilde{\mathscr{L}^2}(X) \simeq \Gamma^\infty(\gamma^*TX)$. Moreover, there are three complex structure $\mathcal{I}_i$ defined via

$$(\mathcal{I}_i|_\gamma v)(s,r) := I_i|_{\gamma(s,r)}v(s,r)$$

for $1 \leq i \leq 3$. We have a metric

$$\ll v, w \gg|_\gamma := \Omega_i|_\gamma(\mathcal{I}_i v, w)$$

for $1 \leq i \leq 3$. Replacing γ by $(\gamma, [G^1], [G^2])$ defines these symplectic forms, complex structures and the metric also on $\widetilde{\mathscr{L}^2}(X)$.

Theorem 7. *Equations* (3) *and* (4) *can be written as solutions of a Hamiltonian system on $\widetilde{\mathscr{L}^2}(X)$. If X is flat then the construction descends to $\mathscr{L}^2(X)$.*

Proof. Let $H : X \times \mathbb{T}^3 \to \mathbb{R}$ be a smooth Hamiltonian function and abbrevite $G^1_{t_i} := G^1(\cdot, t_i)$ and $G^2_{t_i} := G^2(t_i, \cdot)$. We define the (autonomous) Hamiltonian functions

$$\mathscr{H}_{jkl} : \widetilde{\mathscr{L}^2}(X) \times \mathbb{S}^1 \to \mathbb{R}, \quad \mathcal{H}_{jkl}((\gamma, [G^1], [G^2]), t_l) := -a_{jk} \int_0^1 \int_{\mathbb{D}} (G^k_{t_\mu})^* \omega_j dt_\mu$$

where $\mu \neq k$. Let $\xi \in \Gamma^\infty(\gamma^* TX)$ be a vector field along γ and $s \mapsto \gamma^s$ a variation of γ with $\frac{d}{ds}|_{s=0} \gamma^s = \xi$ and $s \mapsto (G^k)^s$ a variation with $\frac{d}{ds}|_{s=0} (G^k)^s|_{\mathbb{S}^1 \times \mathbb{S}^1} = \xi$. We compute

$$\begin{aligned} d\mathscr{H}_{jkl}(\gamma, [G^1], [G^2]).\xi &= -a_{jk} \frac{d}{ds}|_{s=0} \int_0^1 \int_{\mathbb{D}} ((G^k_{t_\mu})^s)^* \omega_j dt_\mu = -a_{jk} \int_0^1 \int_0^1 \omega_j(\xi, \partial_k \gamma) dt_k dt_\mu \\ &= \int_0^1 \int_0^1 \langle \xi, a_{jk} I_j \partial_k \gamma \rangle dt_k dt_\mu = -\Omega_l(\xi, I_l(a_{jk} I_j \partial_k \gamma)). \end{aligned}$$

Thus the Hamiltonian vector field of $\mathscr{H}_{jkl}$ w.r.t. Ω_l is $I_l(a_{jk} I_j \partial_k \gamma)$ and the Hamiltonian equation becomes

$$\partial_l f = I_l(a_{jk} I_j \partial_k f) \quad \Longleftrightarrow \quad a_{jk} I_j \partial_k f + I_l \partial_l f = 0$$

for $f : \mathbb{R} \to \widetilde{\mathscr{L}^2}(X)$, $t_l \mapsto f_{t_l}(\cdot, \cdot)$. In particular for the linear independent, constant vectors $w_1 = (a_{11}, a_{12}, 0)$, $w_2 := (a_{21}, a_{22}, 0)$, $w_3 := (0, 0, 1)$ and the Hamiltonian $\mathscr{H}_{113} + \mathscr{H}_{123} + \mathscr{H}_{213} + \mathscr{H}_{223}$ we obtain

$$I_1 \partial_{w_1} f + I_2 \partial_{w_2} f + I_3 \partial_{w_3} f = 0. \tag{8}$$

Since $\det(w_1, w_2, w_3) \neq 0 \neq \det(v_1, v_2, v_3)$ and the w_i and v_i preserve the volume form $dt_1 \wedge dt_2 \wedge dt_3$, we can make a change of variables transforming (8) into

$$\partial\!\!\!/ f = I_1 \partial_{v_1} f + I_2 \partial_{v_2} f + I_3 \partial_{v_3} f = 0.$$

If we consider instead of $\mathscr{H}_{jkl}$ the perturbed, nonautonomous Hamiltonian

$$\mathscr{H}_{jkl} - \int_0^1 \int_0^1 H(\gamma(t_1, t_2), (t_1, t_2, t_3)) dt_1 dt_2$$

the Hamiltonian equation becomes

$$a_{jk} I_j \partial_k f + I_l \partial_l f - \operatorname{grad} H(f) = 0.$$

If we consider in particular the nonautonomous Hamiltonian

$$\mathscr{H}_{113} + \mathscr{H}_{123} + \mathscr{H}_{213} + \mathscr{H}_{223} - \int_0^1 \int_0^1 H(\gamma(t_1, t_2), (t_1, t_2, t_3)) dt_1 dt_2$$

we obtain

$$I_1 \partial_{w_1} f + I_2 \partial_{w_2} f + I_3 \partial_{w_3} f - \operatorname{grad} H(f) = 0$$

which can be transformed into (4). Contractible 1-periodic solutions of this equation are elements of

$$\widetilde{\mathscr{L}}^3(X) = \left\{ [f, F^1, F^2, F^3] \;\middle|\; \begin{array}{l} f \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, X), \\ F^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1 \times \mathbb{S}^1, X),\ F^1|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f, \\ F^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D} \times \mathbb{S}^1, X),\ F^2|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f, \\ F^3 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}, X),\ F^3|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f \end{array} \right\}$$

which we can identify with the space $\widetilde{\mathcal{F}}$ on which our original action functional $\mathscr{A}$ lives. $\square$

References

[Fl1] Floer, A.: *A relative Morse index for the symplectic action.* Comm. Pure Appl. Math., Vol. **41** (1988), 393 – 407.

[Fl2] Floer, A.: *The unregularized gradient flow of the symplectic action.* Comm. Pure Appl. Math., Vol. **41** (1988), 775 – 813.

[Fl3] Floer, A.: *Morse theory for Lagrangian intersections.* J. Diff. Geom. **28** (1988), 513 – 547.

[GH] Ginzburg, V.; Hein, D.: *Hyperkähler Arnold conjecture and its generalizations.* Internat. J. Math. 23 (2012), no. 8, 1250077, 15 pp.

[HNS1] Hohloch, S.; Noetzel, G.; Salamon, D.: *Hypercontact structures and Floer homology.* Geometry & Topology 13 (2009), 2543 – 2617.

[HNS2] Hohloch, S.; Noetzel, G.; Salamon, D.: *Floer homoogy groups in hyperkähler geometry.* CRM proceedings and lecture notes, Volume **49** (2009), 251 – 261.

[MS] McDuff, D.; Salamon, D.: *J-holomorphic curves and quantum cohomology.* University Lecture Series, 6. American Mathematical Society, Providence, RI, 1994. viii+207 pp. ISBN: 0-8218-0332-8.

[Sa] Salamon, D.: *The three dimensional Fueter equation and divergence free frames*, Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg **82** (2013), 1 – 28.