From

Floer theory in symplectic and hyperkähler geometry

via

Integrable Hamiltonian systems

and

Morse theory and *n*-category theory

to

Optimal mass transport and integer partitions

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DOCUMENT DE SYNTHÈSE EN VUE DE L'OBTENTION DU DIPLÔME
D'HABILITATION À DIRIGER DES RECHERCHES

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Meinen Eltern

(To my parents)

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Contents

Relevant Publications and Preprints

All publications and preprints, apart from *Homoclinic points and Floer homology*, J. Symplectic Geom. 11 (2013), no. 4, 645 – 701, are disjoint from my PhD thesis and Diploma thesis.

Submitted Preprints.

1) *Optimal transport and integer partitions*, 22 p., submitted.
2) *Floer homology and homoclinic dynamics*, 24 p., submitted.

Publications.

1) *Hyperkähler Floer theory as infinite dimensional Hamiltonian system*, 7 p., to appear in Proceedings of the AMS.
2) *From semitoric systems to Hamiltonian S^1-spaces*, (with S. Sabatini, D. Sepe), *to appear in* Discrete Contin. Dyn. Syst. Series A 35 (2015), no. 1, 247 – 281.
3) *Higher Morse moduli spaces and n-categories*, Homology, Homotopy and Applications 16 (2014) no. 2, 1 – 32.
4) *On the image of the almost strict Morse n-category under almost strict n-functors*, Theory and Applications of Categories 29 (2014), 21 – 47.
5) *Homoclinic points and Floer homology*, J. Symplectic Geom. 11 (2013), no. 4, 645 – 701.
6) *Transport, flux and growth of homoclinic Floer homology*, Discrete Contin. Dyn. Syst. Series A 32 (2012), no. 10, 3587 – 3620.
7) *Hypercontact structures and Floer homology*, (with G. Noetzel, D. Salamon), Geometry & Topology 13 (2009), 2543 – 2617.
8) *Floer homology groups in hyperkähler geometry*, (with G. Noetzel, D. Salamon), In *New Perspectives and Challenges in Symplectic Field Theory*, 251 – 261, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., Providence, RI, 2009.

PhD thesis and Diploma thesis.

1) *Floer homology for homoclinic tangles*, Ph.D. thesis, 153 p., University of Leipzig, 2008.
2) *Optimale Massebewegung im Monge-Kantorovich-Transportproblem (Optimal mass transportation in the Monge-Kantorovich transport problem)*, diploma thesis, 60 p., University of Freiburg, 2003.

CHAPTER 1

Floer theory in symplectic dynamics and hyperkähler geometry

Symplectic geometry is one of the most active topics in modern geometry. One rapidly developing part of symplectic geometry is Floer theory. This chapter will focus on applications of Floer theory to symplectic dynamics and hyperkähler geometry. Since symplectic geometry provides the framework for Hamiltonian dynamics, it is naturally linked to celestial mechanics and other topics in physics.

1. Introduction

A symplectic manifold (M, ω) is a smooth manifold M with a closed nondegenerate 2-form ω. For example $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$, cotangent bundles with the exterior derivative of the Liouville 1-form and surfaces equipped with their volume forms are symplectic. Note that symplectic manifolds are even dimensional, but not every even dimensional manifold is symplectic. Diffeomorphisms which leave the symplectic form invariant are called symplectomorphisms. They form the group

$$\mathrm{Symp}(M, \omega) := \{f \in \mathrm{Diff}(M) \mid f^*\omega = \omega\}.$$

The group of Hamiltonian diffeomorphisms

$$\mathrm{Ham}(M, \omega) \subseteq \mathrm{Symp}(M, \omega)$$

is a very important subgroup and is defined as follows. Take a smooth function $F : M \times \mathbb{S}^1 \to \mathbb{R}$ and abbreviate $F_t := F(\cdot, t)$ and define its (nonautonomous) Hamiltonian vector field X_t^F via

$$\omega(X_t^F, \cdot) = -dF_t(\cdot).$$

The associated (nonautonomous) ODE

$$\dot{z}(t) = X_t^F(z(t))$$

is called the (nonautonomous) Hamiltonian equation and its (nonautonomous) flow is called Hamiltonian flow and usually denoted by $\varphi_t = \varphi_t^F$. If we can write a symplectomorphism as time-1 map of a Hamiltonian flow we call it a *Hamiltonian diffeomorphism*. A Hamiltonian diffeomorphism is *nondegenerate* if its graph intersects the diagonal in $M \times M$ transversely.

The development of Floer theory was particularly stimulated by V.I. Arnold.

CONJECTURE 1.1 (**Arnold conjecture**, 1960s). *Let* (M, ω) *be a closed symplectic manifold and* $\varphi \in \mathrm{Ham}(M, \omega)$ *nondegenerate. Then*

$$|\mathrm{Fix}(\varphi)| \geq \sum_{i=0}^{\dim M} \mathrm{rk}\, H_i(M; \mathbb{Z})$$

Arnold linked in his conjecture the number of fixed points to the topology of the underlying manifold, more precisely to the sum over the Betti numbers. There are at least three formulations of the conjecture:

1) Counting fixed points of a Hamiltonian diffeomorphism φ.
2) Counting 1-periodic orbits of a Hamiltonian flow whose time-1 map coincides with φ.
3) Counting intersection points of graph φ with the diagonal in $M \times M$.

The reformulation as an intersection problem motivated Floer [**Fl1, Fl2, Fl3**] to devise some kind of infinite dimensional Morse theory for the 'L^2-gradient flow' of the symplectic action functional — nowadays known as **Floer theory**. Note that Floer theory was subsequently also defined for the other formulations.

Now let us briefly sketch the idea behind Floer theory. It took until 1983 for the first partial proof of Arnold's Conjecture when Conley & Zehnder [**CoZ**] showed it for the $2n$-torus. The breakthrough came a few years later with Floer's works [**Fl1, Fl2, Fl3**] where he identified the fixed points of a Hamiltonian diffeomorphism with intersection points of the graph of the Hamiltonian diffeomorphism with the diagonal in the symplectic manifold $(M \times M, \omega \oplus (-\omega))$, i.e. he turned the fixed point problem into an intersection problem. A distinguished class of submanifolds in symplectic geometry are Lagrangian submanifolds, i.e. submanifolds on which the symplectic form vanishes and whose dimension is half the dimension of M. As it turns out, the above mentioned diagonal and graph are *Lagrangian submanifolds* which is essential for analysis purposes. But even more important, Floer recognized the intersection points of the diagonal and the graph as critical points of the symplectic action functional which, in turn, he considered as some kind of Morse function on the path space. The associated 'infinite dimensional Morse theory' is nowadays called Floer theory and the associated homology theory is referred to as *Floer homology*.

Originally, Floer theory was developed to study Hamiltonian dynamical systems, more precisely, it was destined for counting the number of 1-periodic Hamiltonian solutions. But its ideas and techniques are not limited to periodic orbits or the symplectic framework. The aim of this chapter is to present two new types of Floer homology and their applications and ramifications:

(a) Four variants of Floer homologies for the study of *homoclinic* Hamiltonian orbits on symplectic manifolds.
(b) Two different settings for Floer homology on hyperkähler manifolds.

It is an almost hopeless task (and not the aim of this work) to give an overview of the archievements and applications of Floer theory in geometry, dynamics, algebra and many other areas of mathematics. We refer the interested reader to the excellent introduction by Audin & Damian [**AuD**] and the more advanced book by McDuff & Salamon [**McS**]. Just to mention a few speciemen in the now rapidly growing zoo of variants and applications, there are by now Rabinowitz-Floer homology, Heegaard Floer theory, Knot Floer homology, Seiberg-Witten Floer homology, (Embedded) Contact homology, Symplectic Field Theory etc. etc.

2. Homoclinic Floer theory

Let us first fix some notions in homoclinic dynamics, before we proceed to the definition of homolinic Floer homology.

2.1. Notions in homoclinic dynamics.

Consider a smooth manifold N and a diffeomorphism $f \in \mathrm{Diff}(N)$. A point $x \in N$ is an *m-periodic* if there exists $m \in \mathbb{N}$ such that $f^m(x) = x$. 1-periodic points are usually called *fixed points*. A fixed point x is *hyperbolic* if the eigenvalues of the linearization $Df(x)$ of f in x have modulus different from 1. The *stable manifold* of a hyperbolic fixed point x is defined by

$$W^s(f,x) := \{p \in N \mid \lim_{n\to\infty} f^n(p) = x\}$$

and the *unstable manifold* is given by

$$W^u(f,x) := \{p \in N \mid \lim_{n\to-\infty} f^n(p) = x\}.$$

The connected components of $W^s(f,x)\backslash\{x\}$ resp. $W^u(f,x)\backslash\{x\}$ are the *branches* of $W^s(f,x)$ resp. $W^u(f,x)$. A point $p \in N$ is called *homoclinic (w.r.t. x)* if $p \in W^s(f,x) \cap W^u(f,x)$, cf. Figure 1.1. We denote the set of homoclinic points of $x \in \mathrm{Fix}(f)$ by

$$\mathcal{H}(f,x) := W^s(f,x) \cap W^u(f,x).$$

The set $\{f^n(p) \mid n \in \mathbb{Z}\}$ is called the *orbit* of $p \in N$. The orbit of a periodic resp. homoclinic point is called a *periodic* resp. *homoclinic* orbit.

2.2. The motivation for homoclinic Floer theory. The study of a dynamical system usually starts with its fixed points and periodic points. The next more complicated type of points are homoclinic ones. Poincaré [**Poi1**], [**Poi2**] noticed the existence of homoclinic points around 1890 when working on the n-body problem. About 40 years later, Birkhoff [**Bi**] proved the existence of high-periodic points near homoclinic ones. The dynamics of homoclinic points were described by Smale [**Sm1, Sm2**] the 1960s by means of his 'horseshoe'. Later on, perturbation theory, calculus of variations and numerics became popular tools in the investigation of homoclinic points, but many questions remain unsolved.

Now, how are homoclinic points linked to Floer homology, or, more precisely, why does it make sense to think about Floer homology generated by homoclinic points?
The common link is their formulation as intersection problems: By definition, homoclinic points are the intersection set of stable and unstable manifold. And, as we saw above, Floer homology arises from the *Lagrangian* intersection problem 'diagonal $\cap$ graph'. Since being Lagrangian is crucial for the construction of Floer homology we have to check if the stable and unstable manifolds can be Lagrangians. It turns out that, for symplectomorphisms, the stable and unstable manifolds are always Lagrangians. This are good news. But there are also bad ones. The intersection of stable and unstable manifolds usually form a 'homoclinic tangle' (cf. Figure 1.1). More precisely, the stable and unstable manifold are usually only injectively immersed and give rise to an abundance of intersection points whereas in Floer's setting (and its generalizations), the Lagrangians are usually compact or at least 'sufficiently nice'. There are some techniques in classical Floer theory how to deal with 'too many' intersections points, but they fail for stable and unstable manifolds.

Up to our knowledge, not many people studied homoclinic points with symplectic techniques before. There is the work by Hofer & Wysocki [**HofW**] who use pseudo-holomorphic curves and Fredholm theory. Cieliebak & Séré [**CiS**] combine variational technics and pseudo-holomorphic curves. and finally Lisi [**Li**] uses Lagrangian embedding techniques to generalize Coti Zelati & Ekeland & Séré [**CZES**].

2.3. Four homoclinic Floer homologies. In our works [**Hoh1, Hoh2, Hoh3**], we developped four different types of *homoclinic* Floer homologies, i.e. Floer homologies generated by homoclinic points. Although their properties are very different, their construction is very similar such that we will line it out only in the case of so-called *primary homoclinic Floer homology*

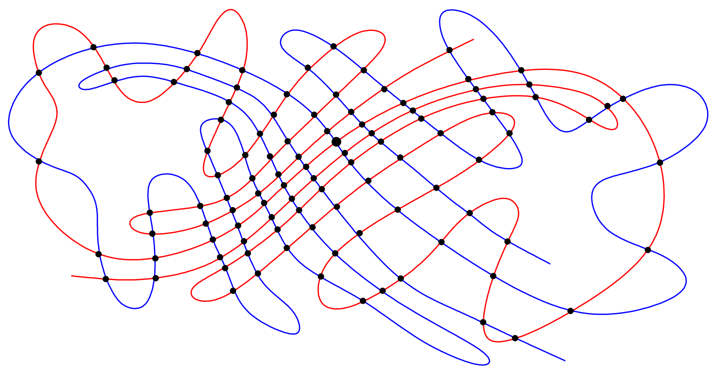


FIGURE 1.1. The intersection behaviour of transversely intersecting stable (red) and unstable (blue) manifold of a hyperbolic fixed point.

As already mentioned above, the (un)stable manifolds are 'highly noncompact'. This turns the analysis (Fredholm theory, Gromov compactness etc.) which is necessary to set up classical Floer theory into a hopeless task.
Fortunately several authors (cf. de Silva [**dS**], Felshtyn [**Fe**], Gautschi & Robbin & Salamon [**GaRS**]) noticed that one does not need analysis if one works on 2-dimensional manifolds. On 2-dimensional manifold, one can use instead combinatorics. In our situation, this gets rid of the analysis problems, but the difficulties related to the abundance of intersection points remain untouched.

Assume from now on that (M, ω) is $\mathbb{R}^2$ or a closed surface of genus $g \geq 1$ equipped with symplectic forms. Let φ be a symplectomorphisms with hyperbolic fixed point x and transversely intersecting (un)stable manifolds $W^s := W^s(\varphi, x)$ and $W^u := W^u(\varphi, x)$. Note that automatically $\dim W^s = \dim W^u = 1$. Let $p, q \in \mathcal{H} = W^s \cap W^u$ be homoclinic points and denote by $[p, q]_i$ the (one dimensional!) unoriented segment between p and q in W^i for $i \in \{s, u\}$. Iterating φ yields a $\mathbb{Z}$-action $\mathcal{H} \times \mathbb{Z} \to \mathcal{H}$, $(p, n) \mapsto \varphi^n(p)$. For transversely intersecting $W^s \cap W^u$, the sets $\mathcal{H}$ and $\mathcal{H}/\mathbb{Z}$ are both infinite. Let $c_p : [0, 1] \to W^u \cup W^s$ be a curve with $c_p(0) = x = c_p(1)$ which runs through $[x, p]_u$ to p and through $[p, x]_s$ back to x. $[p] := [c_p] \in \pi_1(M, x)$ denotes the *homotopy class* of p and

$$\mathcal{H}_{[x]} := \{p \in \mathcal{H} \mid [p] = [x]\}$$

is the set of *contractible* homoclinic points. It is in fact invariant under the action of φ. If $p, q \in \mathcal{H}$ with $[p] = [q]$, then there is a (relative) *Maslov index* $\mu(p, q) \in \mathbb{Z}$. In dimension two, one can think of it as follows. If the segments $[p, q]_s$ and $[p, q]_u$ intersect each other perpendicular at p and q, we can identify $\mu(p, q)$ with twice the winding number of the unit tangent vector of a loop starting in p, running through $[p,q]_u$ to q and through $[p,q]_s$ back to p (where we have to flip the tangent vector $+90°$ at q and $-90°$ at p). This yields a *grading*

$$\mu : \mathcal{H}_{[x]} \to \mathbb{Z}, \qquad \mu(p) := \mu(p,x)$$

and, for $p, q \in \mathcal{H}_{[x]}$, we have

$$\mu(p,q) = \mu(p,x) + \mu(x,q) = \mu(p,x) - \mu(q,x) = \mu(p) - \mu(q).$$

As we already indicated above, the set of homoclinic points is too big to generate a well-defined Floer homology. Thus we will focus on the following subsets.

DEFINITION 1.2. *$p \in \mathcal{H} \setminus \{x\}$ is* ***semi-primary*** *if $]x,p[_s \cap\,]x,p[_u = \emptyset$. A contractible $p \in \mathcal{H}_{[x]} \setminus \{x\}$ is called* ***primary*** *if $]p,x[_s \cap\,]p,x[_u \cap \mathcal{H}_{[x]} = \emptyset$ and the set of primary points is denoted by $\mathcal{H}_{pr}$.*

(Semi-)primary points have very nice properties.

LEMMA 1.3 (Hohloch [**Hoh1, Hoh2**]).

(i) *For a primary point p holds $\mu(p) = \mu(p,x) \in \{\pm 1, \pm 2, \pm 3\}$, i.e. the Maslov index is* ***bounded***.

(ii) *If W^s and W^u intersect transversely then $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/\mathbb{Z}$ is* ***finite***.

The equivalence class of $p \in \mathcal{H}_{pr}$ in $\tilde{\mathcal{H}}_{pr} = \mathcal{H}_{pr}/\mathbb{Z}$ is $\langle p \rangle$. Via

$$[\langle p \rangle] := [p] \quad \text{and} \quad \mu(\langle p \rangle, \langle q \rangle) := \mu(p,q) \quad \text{and} \quad \mu(\langle p \rangle) := \mu(p,x)$$

we obtain well-defined homotopy classes and Maslov indices on the quotient space.

After these preparations, we can line out the construction of *primary Floer homology*. We start with the boundary operator. Fix a convex 2-gon D in $\mathbb{R}^2$ with convex vertices at $(-1,0)$ and $(1,0)$, we call its lower edge B_u and its upper edge B_s. Given $p, q \in \mathcal{H}$ with $\mu(p) - \mu(q) = 1$, the moduli space $\mathcal{M}(p,q)$ is the space of smooth, immersed 2-gons $v : D \to M$ which are orientation preserving and satisfy $v(B_u) \subset W^u$, $v(B_s) \subset W^s$, $v(-1,0) = p$ and $v(1,0) = q$. If $G(D)$ is the group of orientation preserving diffeomorphisms of D which preserve the vertices, we set

$$\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/G(D).$$

Consider W_+^i and W_-^i, the branches of the (un)stable manifolds where $i \in \{s,u\}$. We put on them their 'jump direction' as orientation and we denote it by $o(W_+^i)$ resp. $o(W_-^i)$. There is also a way to impose an orientation on segments: Given $p, q \in \mathcal{H}_{pr}$ with $\mu(p,q) = 1$ and $v \in \mathcal{M}(p,q) \neq \emptyset$ we give

$\nu(B_i) = [p,q]_i$ the orientation induced by the parametrization direction from p to q and call it o_{pq}. This allows us to to define the following *signs*

$$m(p,q) := \begin{cases} 1 & \text{if } \mu(p,q) = 1,\ \mathcal{M}(p,q) \neq \emptyset,\ o(W_{pq}) = o_{pq}, \\ -1 & \text{if } \mu(p,q) = 1,\ \mathcal{M}(p,q) \neq \emptyset,\ o(W_{pq}) \neq o_{pq}, \\ 0 & \text{otherwise} \end{cases}$$

which is well-defined according to Hohloch [**Hoh1, Hoh2**]. These signs pass to the quotient via

$$m(\langle p\rangle, \langle q\rangle) := \sum_{n\in\mathbb{Z}} m(p, \varphi^n(q)) \quad \text{for} \quad \langle p\rangle, \langle q\rangle \in \tilde{\mathcal{H}}_{pr}.$$

Now we are ready to define the *primary Floer chain groups* with the associated *boundary operator*. We set

$$C_k := C_k(\varphi, x; \mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle\in\tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle)=k}} \mathbb{Z}\langle p\rangle, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle\in\tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle, \langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity. Note that the rank of all groups is finite (since there are only finitely many generators) and moreover $C_k = 0$ for $k \notin \{\pm1, \pm2, \pm3\}$ (cf. Lemma 1.3).

Theorem 1.4 (Hohloch [**Hoh1, Hoh2**]).

(i) $\partial \circ \partial = 0$*, i.e.* (C_*, ∂_*) *is a chain complex and*

$$H_k := H_k(\varphi, x; \mathbb{Z}) := \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}}$$

is called ***primary homoclinic Floer homology*** *of* φ *in* x*.*

(ii) We have $H_k = 0$ *for* $k \neq \pm1, \pm2, \pm3$.

The so-called *breaking and gluing procedure* (sketched for Morse trajectories in Figure 3.2) is the heart of the proof which in turn relies on the classification of $\widehat{\mathcal{M}}(p,q)$ and of immersions of relative Maslov index 2.

Remark 1.5. *Primary Floer homology is completely determined by a finite number of primary homoclinic points located in (possibly large) compact segments of the (un)stable manifolds centered around the fixed point.*

This means in particular that, given a decent plot of a tangle, we can always compute primary homoclinic Floer homology explicitly. Moreover, H_* is invariant under conjugation, thus making all constructions natural. But there is also another form of invariance.

Theorem 1.6 (Hohloch [**Hoh1, Hoh2**]). *H_* is invariant under so-called csi-isotopies (which are a quite large class of perturbations of the underlying symplectomorphism).*

Let us briefly summerize the other variants of homoclinic Floer theory before explaining some of their features in more detail.

THEOREM 1.7. *Apart from primary homoclinic Floer homology, there exist another three variants of homoclinic Floer homology.*

1) ***Semi-primary homoclinic Floer homology*** *(cf. Hohloch* [**Hoh1, Hoh2**]*) is defined analogously to primary homoclinic Floer homology, but uses semi-primary points as generators. It has weaker invariance properties, but can easier detect topology of the underlying manifold.*
2) ***Chaotic homoclinic Floer homology*** *(cf. Hohloch* [**Hoh1, Hoh2**]*) takes not only primary homoclinic points, but also higher periodic points into account. It gives rise to a zeta function.*
3) ***Cylinder homoclinic Floer homology*** *(cf. Hohloch* [**Hoh3**]*) is generated by noncontractible points on the cylinder. It is related to known quantities in dynamical systems like the absolute flux and Mather's difference in action.*

Apart from measuring algebraic properties of homoclinic points, homoclinic Floer homology has a physical meaning in terms of 'transport' as defined in MacKay & Meiss & Percival [**MacMP**]. A central notion in their paper is the *absolute flux* of a symplectomorphism φ w.r.t. a simply closed curve c. In the plane, it is defined as the volume of the set of points which are swept out of the interior of the curve:

$$\mathcal{F}lux_{\varphi}(c) = \mathrm{vol}_{\omega}(\varphi(\mathrm{Int}(c)) \cap \mathrm{Ext}(c)).$$

By associating a certain curve to a homoclinic point, MacKay & Meiss & Percival [**MacMP**] define the flux through a homoclinic point and recognize it (under certain assumptions) as Mather's [**Ma**] difference in action $\triangle W$.

THEOREM 1.8 (Hohloch [**Hoh3**]). *The relative symplectic action of two adjacent generators of primary and cylinder homoclinic Floer homology coincides (under certain natural assumptions) with their flux and Mather's difference in action* $\triangle W$.

The flux of a (semi-)primary point p transforms

$$\mathcal{F}lux_{\varphi^n}(p) = n\mathcal{F}lux_{\varphi}(p).$$

Therefore it has the same growth behaviour as the symplectic action and (mean) Maslov index in classical Floer theory (which are both invariant in our setting). Growth behaviour of Hamiltonian diffeomorphisms is an important ingredient in Polterovich's [**Pol**] Hamiltonian version of the Zimmer program. We proved for homoclinic Floer homology

THEOREM 1.9 (Hohloch [**Hoh3**]). *For the filtered primary and cylinder homoclinic Floer groups holds under certain natural assumptions*

$$\operatorname{rk}(H^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x)) = n \operatorname{rk}(H^{]b-\varepsilon,b+\varepsilon]}(\varphi, x)).$$

Call the data $(W^s, W^u, W^s \cap W^u)$ a *homoclinic tangle*. Inspired by Collins [**Co**] work on so-called *trellises* (which can be thought of as 'finite' parts of a homoclinic tangle) we find

THEOREM 1.10 (Hohloch [**Hoh4**]). *(a) Given a homoclinic tangle, it contains a trellis which already computes the homoclinic Floer homology of the homoclinic tangle.*
(b) The number of generators of primary homoclinic Floer homology in $\mathbb{R}^2$ *is greater or equal to twice the number of pairs of intersecting branches of the underlying tangle.*

Let us summarize the main properties of the homoclinic Floer homologies in a table (cf. Hohloch [**Hoh4**]).

	Primary FH	**Semi-pr. FH**	**Cylinder FH**	**Chaotic FH**
Generator	primary, contractible	semi-primary, contractible	'primary', noncontr.	primary, contractible
Invariance	strong	weak	strong	no
Growth φ vs. φ^n	Filtered FH: linear growth	examples of linear growth	Filtered FH: linear growth	'subsequence' $k \mapsto \hat{H}_*(x, (\varphi^n)^k)$
Transport	turnstiles, flux = action	turnstiles not involved	turnstiles, flux = action = $\triangle\mathscr{W}$	turnstiles not involved
Number theory				ζ-function

3. Homoclinic Floer homology: Future projects

3.1. Chaotic Floer homology and zeta functions. As already mentioned in Theorem 1.7, there is a version of homoclinic Floer homology which notices the higher periodic points near the homoclinic ones. It is called *chaotic homoclinic Floer homology* and it is denoted by $\hat{H}_*(x, \varphi^n)$ for $n \in \mathbb{N}$. To the sequence $n \mapsto \hat{H}_*(x, \varphi^n)$, we assign a symplectic zeta function via

$$\zeta_{x,\varphi}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\chi(\hat{H}_*(x, \varphi^n))}{n} z^n\right)$$

where $\chi(\hat{H}_*(x, \varphi^n))$ denotes the Euler characteristic of $\hat{H}_*(x, \varphi^n)$. We want to investigate the properties of this zeta function and plan to link them to the growth behaviour of symplectomorphisms and we intend to study its relation to Collins' [**Co**] zeta function.

3.2. Classification of homoclinic Floer homology. This project continues in the spirit of Theorem 1.10: In Hohloch [**Hoh4**], we compare Collins [**Co**] results for trellises to our homoclinic Floer theory. Among others, Collins is using so-called 'pruning isotopies' in oder to change and reduce the form of his trellises. Theorem 1.10 allows us to use parts of Collins techniques which in turn should allow us to prove

Conjecture 1.11. *(a) In a minimal trellis, there are exactly two primary points in each pair of intersecting branches.*
(b) Using (a), we can classify primary homoclinic Floer homologies on $\mathbb{R}^2$ *(and maybe also on other surfaces).*

4. Hyperkähler Floer theory

4.1. Introduction and definitions. Since Floer's seminal idea at the end of the 1980's, Floer theory has been applied to many different questions. Due to its success, there are now many variants of this theory, but only a few fundamentally different types. Hyperkähler Floer homology was devised in Hohloch & Noetzel & Salamon [**HNS1, HNS2**] and differs essentially from the previous types. It was reproved and generalized by Ginzburg & Hein [**GiH1, GiH2**] and Salamon [**Sa**]. Hohloch [**Hoh9**] rewrites it as an infinite dimensional Hamiltonian systems. As the name already suggests, hyperkähler Floer theory relates Floer theory to hyperkähler geometry.

Hyperkähler manifolds are manifolds with three complex structures satisfying certain compatibility conditions:

Definition 1.12. *A manifold X is **hyperkähler** if there are three complex structures I_1, I_2 and I_3 and a metric $\langle\cdot,\cdot\rangle$ such that $I_1I_2 = -I_2I_1 = I_3$ and $\langle\cdot,\cdot\rangle = \langle I_i\cdot, I_i\cdot\rangle$ and $\omega_i := \langle I_i\cdot,\cdot\rangle$ are symplectic forms for* $1 \leq i \leq 3$.

These manifolds are much more rigid than Kähler manifolds which only have one complex structure; in fact, $\langle\cdot,\cdot\rangle$ is Kähler w.r.t. I_1, I_2 and I_3. Whereas symplectic manifolds are always $2n$-dimensional, hyperkähler manifolds are always $4n$-dimensional. The standard example are the quaternions $\mathbb{H}$ with complex structures i, j and k. In dimension four, there are not many compact hyperkähler manifolds, only the 4-torus and K3-surfaces. Hyperkähler manifolds show up in Berger's holonomy group based classification as those having holonomy group $\mathrm{Sp}(n)$. They are Ricci-flat and thus Calabi-Yau such that in particular the first Chern class vanishes. Flat compact hyperkähler manifolds are $4n$-tori modulo a finite group action.

In physics, hyperkähler manifolds appear naturally in the supersymmetric σ-model which studies so-called triholomorphic maps between two hyperkähler manifolds.

4.2. Definition of hyperkähler Floer homology. The main difference to the classical Floer theories is that hyperkähler Floer theory is based on the analysis of a 'triholomorphic' equation (now also sometimes called *Fueter equation*) instead of the Cauchy-Riemann equation. Whereas, in classical Floer theory, the critical points of the symplectic actions functional are 1-periodic Hamiltonian solutions, hyperkähler Floer theory studies the critical points of the so-called hypersymplectic action functional which are certain 'triholomorphic 3-manifolds'. There are two main settings for hyperkähler Floer homology.

4.2.1. **Setting on the 3-torus**. Let X be a hyperkähler manifold with complex structures I_1, I_2, I_3 and symplectic forms $\omega_1, \omega_2, \omega_3$. Consider the 3-torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with standard coordinates $t = (t_1, t_2, t_3)$, standard vector fields $\partial_1 := \frac{\partial}{\partial t_1}$, $\partial_2 := \frac{\partial}{\partial t_2}$ $\partial_3 := \frac{\partial}{\partial t_3}$ and volume form $\sigma := dt_1 \wedge dt_2 \wedge dt_3$. Pick a (constant) matrix $(a_{ij}) \in \mathrm{GL}(3, \mathbb{R})$ and set

$$v_1 := \sum_{k=1}^{3} a_{1k}\partial_k, \quad v_2 := \sum_{k=1}^{3} a_{2k}\partial_k, \quad v_3 := \sum_{k=1}^{3} a_{3k}\partial_k.$$

The Lie derivative $\mathcal{L}_{v_i}\sigma$ vanishes for v_1, v_2 and v_3 meaning that the three (constant) vector fields v_1, v_2, v_3 are σ-volume preserving.

Now consider the space of maps $\mathcal{F} := \{f \in C^\infty(\mathbb{T}^3, X) \mid f \text{ contractible}\}$. Its universal cover can be identified with

$$\widetilde{\mathcal{F}} = \left\{ (f, [F^1], [F^2], [F^3]) \;\middle|\; \begin{array}{l} f \in \mathcal{F}, \\ F^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1 \times \mathbb{S}^1, X),\ F^1|_{\mathbb{T}^3} = f, \\ F^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D} \times \mathbb{S}^1, X),\ F^2|_{\mathbb{T}^3} = f, \\ F^3 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}, X),\ F^3|_{\mathbb{T}^3} = f \end{array} \right\}$$

where $\mathbb{D}$ is the closed unit disk in $\mathbb{R}^2$ and $[F^\lambda]$ is the homotopy class of F^λ relative to the boundary where $F^1(e^{2\pi i t_1}, t_2, t_3) = f(t_1, t_2, t_3)$ etc. Let us simplify the notation via $F^1_{t_2 t_3} := F^1(\cdot, t_2, t_3)$ etc. and define for $1 \le j, k \le 3$

$$\mathscr{A}_{jk} : \widetilde{\mathcal{F}} \to \mathbb{R}, \qquad \mathscr{A}_{jk}(f) := -\int_0^1 \int_0^1 \int_{\mathbb{D}} (F^k_{t_\mu t_\nu})^* \omega_j dt_\mu dt_\nu.$$

This is in fact the symplectic action w.r.t. ω_j of the loop $t_k \mapsto f(t)$ averaged over the other two variables t_μ, t_ν. Recalling the previously chosen matrix

$a = (a_{jk})$, we define the **hypersymplectic action functional**

$$\mathscr{A} := \mathscr{A}_a := \sum_{j,k=1}^{3} a_{jk}\mathscr{A}_{jk} : \widetilde{\mathcal{F}} \to \mathbb{R}.$$

Given a '$\mathbb{T}^3$-nonautonomous' Hamiltonian function $H : X \times \mathbb{T}^3 \to \mathbb{R}$, we obtain the **perturbed hypersymplectic action functional** $\mathscr{A}_H$ via

$$\mathscr{A}_H(f) := \mathscr{A}(f) - \int_{\mathbb{T}^3} H(f(t), t)dt.$$

Since $\mathbb{T}^3 \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ this construction on the 'torus loop space' is very similar to the construction of the classical symplectic action functional on a cover of the standard loop space (see for example McDuff & Salamon [**McS**], p. 154).

In order to obtain an equation for the critical points of $\mathscr{A}$, we calculate for instance

$$\begin{aligned} d\mathscr{A}_{12}(f, [F^1], [F^2], [F^3]).\xi &= -\int_0^1\int_0^1\int_0^1 \omega_1|_{f(t)}(\xi(t), \partial_2 f(t))dt_1dt_2dt_3 \\ &= \int_{\mathbb{T}^3} \langle \xi, I_1\partial_2 f\rangle dt \end{aligned}$$

where ξ is a vector field along f. If we abbreviate $\partial_{v_i} f := df(v_i)$ for $1 \le i \le 3$ we obtain for $\mathscr{A} = \sum_{j,k=1}^{3} a_{jk}\mathscr{A}_{jk}$ that the critical points $\mathrm{Crit}(\mathscr{A})$ are maps $f \in \widetilde{\mathcal{F}}$ with

$$\partial\!\!\!/ f := I_1\partial_{v_1} f + I_2\partial_{v_2} f + I_3\partial_{v_3} f = 0$$

and $f \in \mathrm{Crit}(\mathscr{A}_H)$ satisfies

$$\partial\!\!\!/_H f := \partial\!\!\!/ f - \operatorname{grad} H(f) = 0 \tag{1.13}$$

where the gradient $\operatorname{grad} H(f)|_t := \operatorname{grad} H(f(t), t)$ is taken w.r.t. the X-valued variable of H and the metric $\langle \cdot, \cdot \rangle$.

4.2.2. **Setting with hypercontact structures**. Let X be a hyperkähler manifold with complex structures I_1, I_2, I_3 and symplectic forms $\omega_1, \omega_2, \omega_3$. A 1-form α on a $2n+1$-dimensional manifold N is *contact* if $\alpha \wedge (d\alpha)^n$ is a volume form on N where $(d\alpha)^n$ is the n-fold wedge product of $d\alpha$. The vector field $v = v_\alpha$ is its *Reeb vector field* if $d\alpha(v, \cdot) = 0$ and $\alpha(v) = 1$.

Definition 1.14. *A triple of contact forms $(\alpha_1, \alpha_2, \alpha_3)$ on a three-dimensional manifold N is a **hypercontact structure** (or 'taut contact sphere' [**GeG1, GeG2**]), if $\alpha_i \wedge d\alpha_i = \alpha_j \wedge d\alpha_j$ for all i, j and $\alpha_i \wedge d\alpha_j =$*

$-\alpha_j \wedge d\alpha_i$ for $i \neq j$. It is called ***Cartan*** *if the associated Reeb vector fields v_i form a dual frame, i.e. $\alpha_i(v_j) = \delta_{ij}$.*

Cartan manifolds are (quotients of) $\mathrm{SU}(2) \simeq S^3$ as proved by Geiges & Gonzalo [**GeG1, GeG2**]. The (unperturbed) hypersymplectic action functional on such manifolds is given by

$$\mathcal{A} : C^\infty(N, X) \to \mathbb{R}, \quad \mathcal{A}(f) := -\int_N \alpha_1 \wedge f^*\omega_1 + \alpha_2 \wedge f^*\omega_2 + \alpha_3 \wedge f^*\omega_3.$$

Critical points $f \in C^\infty(N, X)$ of $\mathcal{A}$ are characterized by

$$\not{\partial} f := I_1 df(v_1) + I_2 df(v_2) + I_3 df(v_3) = 0$$

and $\not{\partial}$ is an elliptic operator of Dirac type. The equation of the negative gradient flow coincides in certain cases with the Cauchy-Riemann-Fueter equation of the supersymmetric σ-model. The negative gradient flow lines $u : \mathbb{R} \times M \to X$ of $\mathcal{A}$ are defined by $\partial_s u = -d\mathcal{A}(u)$ and turn out to satisfy

$$\partial_s u + \not{\partial} u = 0.$$

More details on the involved analysis and new bubbling-off phenomena are given in the Section *Hyperkähler Floer theory: Future projects*. Given a Hamiltonian function $H : X \times N \to \mathbb{R}$, the perturbed equation is

$$\not{\partial}_H(f) := \not{\partial} f - \operatorname{grad} H(f) = 0$$

where the gradient is computed w.r.t. the first argument.

4.3. Main results. Note that we obtain the same equation in both setting. A solution of $\not{\partial} f = 0$ or $\not{\partial}_H f = 0$ is called *nondegenerate* if the linearized operator for the equation is bijective.
By elliptic regularity, every $W^{1,p}$ solution of the above equations is in fact smooth (cf. Hohloch & Noetzel & Salamon [**HNS2**], Theorem 3.1).

Theorem 1.15 (**Hyperkähler Arnold Conjecture**, Hohloch & Noetzel & Salamon [**HNS1, HNS2**]). *Let N be either a compact Cartan hypercontact 3-manifold (with Reeb vector fields v_i) or the 3-torus (with a constant frame v_i). Let X be a compact flat hyperkähler manifold. Then the space of solutions of $\not{\partial}_H f = 0$ is compact. Moreover, if the contractible solutions are all nondegenerate, then their number is bounded below by the sum of the $\mathbb{Z}_2$-Betti numbers of X ('Hyperkähler Arnold conjecture'). In particular, $\not{\partial}_H f = 0$ has a contractible solution for every H.*

The proof of Theorem 1.15 is based on the construction of Floer theory and its computation which we will sketch briefly in the following. Assume the setting and notation of Theorem 1.15. For a generic H, define the chain complex

$$\mathrm{CF}_k(N,X;H;\mathbb{Z}/2\mathbb{Z}) := \bigoplus_{\substack{f\in\mathrm{Crit}(\mathscr{A}_H)\\ \mu_H(f)=k}} \mathbb{Z}/2\mathbb{Z}\, f$$

whose groups are finitely generated ([**HNS2**], Theorem 3.6) and where the index

$$\mu_H : \mathrm{Crit}(\mathscr{A}_H) \to \mathbb{Z}$$

is induced by the spectral flow ([**HNS2**], equation (50)). Given f^-, $f^+ \in \mathrm{Crit}(\mathscr{A}_H)$, the moduli space

$$\mathscr{M}(f^-, f^+; H)$$

consists of those $u : \mathbb{R}\times N \to X$ with $(s,x)\in\mathbb{R}\times N$ satisfying

$$\begin{cases} \partial_s u + \not\partial_H u := \partial_s u + I_1\partial_{v_1}u + I_2\partial_{v_2}u + I_3\partial_{v_3} - \mathrm{grad}\, H(u) = 0 \\ \lim\limits_{s\to\pm\infty} u(s,x) = f^{\pm}(x), \\ \lim\limits_{s\to\pm\infty} \mathscr{A}_H(u(s,\cdot)) = \mathscr{A}_H(f^{\pm}), \\ \sup\limits_{\mathbb{R}\times N} \|du\| < \infty. \end{cases}$$

For generic H, these moduli spaces are smooth manifolds of dimension $\mu_H(f^-) - \mu_H(f^+)$ ([**HNS2**], Theorem 4.3). Moreover, if $u \in \mathscr{M}(f^-, f^+; H)$, then, for all $\tau\in\mathbb{R}$, the function $(s,x)\mapsto u(s+\tau, x)$ lies also in $\mathscr{M}(f^-, f^+; H)$, i.e. there is an $\mathbb{R}$-action on the moduli space. If $\mu_H(f^-) - \mu_H(f^+) = 1$, then the cardinality of the quotient $\mathscr{M}(f^-, f^+; H)/\mathbb{R}$ is finite as Theorem 3.15 in [**HNS2**] assures. Thus, for $\mu_H(f^-) - \mu_H(f^+) = 1$, it makes sense to define the (mod 2) signs

$$n(f^-, f^+) := \#\mathscr{M}(f^-, f^+; H)/\mathbb{R} \ \text{ modulo } 2.$$

Now we can define the boundary operator

$$\partial^H : \mathrm{CF}_k(N,X;H;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{CF}_{k-1}(N,X;H;\mathbb{Z}/2\mathbb{Z}),$$
$$\partial^H f^- := \sum_{\substack{f\in\mathrm{Crit}(\mathscr{A}_H)\\ \mu_H(f^+)=k-1}} n(f^-, f^+) f^+$$

The following theorem states among others that $\partial^H\circ\partial^H = 0$. The resulting homology is called *hyperkähler Floer homology*.

Theorem 1.16 (Hohloch & Noetzel & Salamon [**HNS1, HNS2**]). *Let N and X be as in Theorem 1.15 and fix a class $\tau\in\pi_0(C^\infty(N,X))$. Then, for a generic perturbation $H : X\times N\to\mathbb{R}$, there is a natural Floer homology group $HF_*(N,X,\tau;H)$ associated to a chain complex generated by the solutions of $\not\partial_H f = 0$. The Floer homology groups associated to different choices* *of H are naturally isomorphic. Moreover, for the component τ_0 of the constant maps there is a natural isomorphism $HF_*(N, X, \tau_0; H) \cong H_*(X; \mathbb{Z}/2\mathbb{Z})$.*

As already mentioned at the beginning of this section, Salamon [**Sa**] reformulated and relaxed certain conditions in the construction. Theorem 1.15 inspired Ginzburg & Hein [**GiH1, GiH2**] to reprove the hyperkähler Arnold conjecture using Conley & Zehnder's method of finite dimensional approximation and to establish also the degenerate version.

In the torus setting, there exists a dynamical interpretation of the 'triholomorphic 3-tori': the solutions of $\not\partial f = 0$ and $\not\partial_H f = 0$ can be written as 1-periodic solutions of an infinite dimensional Hamiltonian system.

Theorem 1.17 (Hohloch [**Hoh7**]). *Let N be a 3-torus with constant frame (v_1, v_2, v_3). Then the solutions of $\not\partial f = 0$ and $\not\partial_H f = 0$ can be written as solutions of a suitable Hamiltonian system on the universal cover of the iterated loop space of X. For flat X, the construction descends to the iterated loop space.*

Theorem 1.17 exchanges a PDE on a finite dimensional manifold with an ODE in infinite dimensions. This looks like some trade-off, but studying infinite dimensional Hamiltonian systems is not a hopeless task as Kuksin [**Ku1, Ku2**] showed. The motivation for the reformulation was to come up with a nicer description of the index used in hyperkähler Floer homology. As already mentioned above, the index is abstractly defined by the spectral flow of a certain operator. But in classical Floer theory, the Conley-Zehnder index provides a nice geometric interpretation of the index and we hope to find an analogue using the Hamiltonian setting for hyperkähler Floer homology as sketched in the next section.

5. Hyperkähler Floer theory: Future projects

5.1. A geometric index for Hyperkähler Floer theory. In classical Floer theory, the Maslov index has a nice geometric interpretation, namely the Conley-Zehnder index. In Hohloch & Noetzel & Salamon [**HNS1, HNS2**], the Maslov index $\mu_H(f)$ of a solution f of $\not\partial_H f = 0$ is formally defined via the spectral flow of a certain Fredholm operator. Using the 3-torus setting, there may be a chance to give it a geometric meaning.

In finite dimensions, the Maslov index is an explicit isomorphism from the fundamental group of the Lagrangian Grassmannian to $\mathbb{Z}$. But in infinite dimensions, the corresponding space is contractible. Nevertheless, by restricting to the Fredholm Lagrangians, the fundamental group is again isomorphic to $\mathbb{Z}$. For such settings, the notion of a Maslov index with properties similar to those in finite dimensions had been established (Booß-Bavnbek & Lesch & Zhu [**BLZ**], Furutani & Otsuki [**FuO**] and others).

'Theorem' 1.18 (Hohloch [**Hoh9**]). *Assume the setting of Theorem 1.17. Then the Maslov index can be interpreted as a 'Fredholm Maslov index' of the associated Cauchy data space.*

5.2. Hyperkähler Floer theory: Bubbling-off in the nonflat case. The analysis of hyperkähler Floer homology relies on the energy identity

$$\mathcal{E}(f) := \frac{1}{2}\int_M |df|^2 \,\mathrm{vol}_M = \mathcal{A}(f) + \frac{1}{2}\int_M |\not{\partial} f|^2 \,\mathrm{vol}_M$$

which yields, in particular, $\mathcal{E}(f) = \mathcal{A}(f)$ for solutions $\not{\partial} f = 0$. In order to define Floer homology, we perturb the action functional with a Hamiltonian function H to $\mathcal{A}_H$ and consider the pertubed equation $\not{\partial}_H f = 0$. The energy density e_u of gradient flow lines $\partial_s u + \not{\partial}_H u = 0$ satisfies an a priori estimate

$$\mathcal{L}e_u + r_u \geq -A - B(e_u)^{\frac{3}{2}} \tag{1.19}$$

where $\mathcal{L}$ is a Laplace-Beltrami type operator, r_u a sum over certain sectional curvatures along u and $A, B > 0$ constants. In the classical Floer setting, the corresponding a priori estimate is $\Delta e \geq -A - Be^2$. In order to follow the classical constructions, we assume nonpositive sectional curvature for X and obtain $Le_u \geq -A - B(e_u)^{\frac{3}{2}}$. Since hyperkähler manifolds have vanishing Ricci curvature, nonpositive sectional curvature implies flatness for X.

The generalization to the nonflat case is not simply done by a better exploit of (1.19) and finer estimates — one also has to deal with a new bubbling-off phenomenon. Whereas in the classical Floer setting bubbling-off of pseudo-holomorphic spheres only takes place at isolated points, bubbling-off might happen here along codimension-2 subsets. This is analogous to Donaldson-Thomas theory where Walpuski [**Wa**] recently analysed the bubbling behaviour of so-called 'Fueter sections' in G_2-manifolds. We intend to use these new insights to study the bubbling phenomena in hyperkähler Floer theory.

Conjecture 1.20 (with Thomas Walpuski). *The bubbling happens along 'generalized Hopf circles' which may intersect each other.*

CHAPTER 2

Compact semi-toric integrable systems and Hamiltonian $\mathbb{S}^1$-actions

Integrable Hamiltonian systems are a very nice class of Hamiltonian systems since their phase space is foliated by invariant submanifolds. Examples are harmonic oscillators, Lagrange, Euler and Kovalevskaya tops, and any Hamiltonian system on a two dimensional manifold. In this chapter, we will focus on integrable Hamiltonian systems on compact 4-dimensional manifolds without boundary and their relation to Hamiltonian $\mathbb{S}^1$-actions. More precisely, we are linking Pelayo & Vũ Ngọc's [**PV1, PV2**] recent classification of semi-toric systems to Karshon's [**Ka**] classification of effective Hamiltonian $\mathbb{S}^1$-actions.

1. Introduction and definitions

Before we delve into the definition and properties of integrable systems, let us keep the following important fact in mind: When working with Floer theory, usually *nonautonomous* Hamiltonian systems are studied since autonomous systems cause problems for the involved analysis. In contrast, integrable systems are always *autonomous* systems.

1.1. Definitions and conventions. Let (M, ω) be a compact connected symplectic manifold of dimension $2n$ without boundary. Given a smooth function $F : M \to \mathbb{R}$, the (autonomous) Hamiltonian vector field X^F is defined via $\omega(X^F, \cdot) = -dF$. The (autonomous) Hamiltonian equation is given by

$$\dot{z} = X^F(z)$$

and its flow is denoted by φ_t^F. The Poisson bracket induced by ω is given by

$$\{\cdot,\cdot\} : C^\infty(M;\mathbb{R}) \times C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R}),$$
$$\{F_1, F_2\} := \omega(X^{F_1}, X^{F_2}).$$

A *Hamiltonian* $\mathbb{R}^k$*-action* on (M, ω) is a smooth map $\Phi := (F_1, \dots, F_k) : M \to \mathbb{R}^k$ such that $\{F_i, F_j\} = 0$ for all $1 \le i, j \le k$. The condition $\{F_i, F_j\} = 0$ is equivalent to $\varphi^{F_i} \circ \varphi^{F_j} = \varphi^{F_j} \circ \varphi^{F_i}$. The action is given by

$$\mathbb{R}^k \times M \to M, \quad (t = (t_1, \dots, t_k), x) \mapsto \varphi_{t_1}^{F_1} \circ \cdots \circ \varphi_{t_k}^{F_k}(x).$$

We will call the triple (M,ω,Φ) a *Hamiltonian* $\mathbb{R}^k$*-space* with *moment map* Φ. A **completely integrable system** is a Hamiltonian $\mathbb{R}^n$-space (M,ω,Φ) where $X^{F_1},\ldots,X^{F_n}$ are almost everywhere linearly independent. A *Hamiltonian* $\mathbb{T}^k$*-space* is a Hamiltonian $\mathbb{R}^k$-space (M,ω,Φ) where the flows $\varphi^{F_1}_{t_1}$, $\ldots$, $\varphi^{F_k}_{t_k}$ are periodic, the vector fields $X^{F_1},\ldots,X^{F_n}$ are almost everywhere linearly independent, and the induced torus action is effective. A **Hamiltonian** $\mathbb{S}^1$**-space** is a Hamiltonian $\mathbb{T}^1$-space.

Examples of completely integrable systems are harmonic oscillators, Lagrange, Euler and Kovalevskaya tops, n-toric $2n$-manifolds and any Hamiltonian system on a two dimensional manifold. The latter is based on the simple observation that Hamiltonian solutions $z:\mathbb{R}\to M$ stay in level sets of the associated Hamiltonian function H since

$$(H\circ z)'(t) = DH|_{z(t)}.\dot z(t) = -\omega(X^H(z(t)),\dot z(t)) = -\omega(X^H(z(t)),X^H(z(t))) = 0.$$

In the whole chapter, (M,ω) is a connected, *closed* symplectic manifold. Unless otherwise stated, group actions on manifolds are effective, i.e. there are no non-trivial elements of the group which act trivially on the whole space. We use the identification $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$.

1.2. Classifications of integrable systems. Integrable systems have been studied under various aspects. To give a complete overview is beyond the scope of this work such that we will only mention those aspects which are of interest to our work.

1.2.1. **Topological classifications**. There is the topological classification of completely integrable systems due to Fomenko [**Fo**] and his school who considered constant-energy surfaces and how they can be built.

1.2.2. **Local classification near singular points by normal forms**. There is the local classification of singularities by means of normal forms established by Eliasson [**El1, El1**] and Miranda & Zung [**MiZ**]. We sketch it briefly.

Let (M,ω,Φ) be an completely integrable system. A point $p\in M$ is *singular* or *critical* if $\operatorname{rk} D\Phi$ is not maximal at p. A *singular* or *critical value* for Φ is a value of Φ whose pre-image contains a singular point. By abuse of notation, we often do not distinguish between singular points and values. In the following, singular points are assumed to be nondegenerate in the sense of Williamson [**Wi**] which means a generalization of the Morse-Bott condition in the symplectic category (cf. also Zung [**Zu**]).
The local normal form due to Eliasson and Miranda & Zung states that there are nice new coordinates $(x,\xi) = (x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$ and new integrals $q_1,\ldots,q_n$ near a nondegenerate critical point of the following kind.

1) Elliptic component: $q_j(x,\xi) = \frac{1}{2}(x_j^2+\xi_j^2)$.

2) Hyperbolic component: $q_j(x, \xi) = x_j\xi_j$.
3) Focus-focus component (coming in pairs):
$$\begin{cases} q_{j-1}(x,\xi) = x_{j-1}\xi_j - x_j\xi_{j-1}, \\ \quad q_j(x,\xi) = x_{j-1}\xi_{j-1} + x_j\xi_j. \end{cases}$$
4) Nonsingular component: $q_j(x, \xi) = \xi_j$.

In this work, we are not dealing with hyperbolic singularities. Thus, in dimension four, only the following combinations remain which we summarize under the name of *almost toric* singular points.

1) *Elliptic-elliptic points:* $\Phi_{\mathrm{ee}} = (q_1, q_2)$ with $q_1 = \frac{1}{2}(x_1^2+\xi_1^2)$, $q_2 = \frac{1}{2}(x_2^2+\xi_2^2)$.
2) *Focus-focus points:* $\Phi_{\mathrm{ff}} = (q_1, q_2)$ with $q_1 = x_1\xi_2 - x_2\xi_1$, $q_2 = x_1\xi_1 + x_2\xi_2$.
3) *Elliptic-regular orbits:* $\Phi_{\mathrm{er}} = (q_1, q_2)$ with $q_1 = \frac{1}{2}(x_1^2 + \xi_1^2)$, $q_2 = \xi_2$.

The proofs of Theorem 2.8 and Theorem 2.9 will rely a lot on these normal forms.

1.2.3. **Toric manifolds**. Toric manifolds are a very special class of completely integrable systems as we will see in a moment.

A convex polygon in $\mathbb{R}^n$ is *simple* if there are exactly n edges meeting at each vertex. A simple polygon is *rational* if all edges have rational slope. A vertex of a simple, rational polygon is *smooth* if its primitive tangent vectors generate the lattice $\mathbb{Z}^n$. A *Delzant polygon* is a convex, simple, rational polygon whose vertices are all smooth.

These polygons play an essential roll in the classification of toric manifolds.

Theorem 2.1 (Delzant [**De**]). *Let (M, ω) be a $2n$-dimensional connected, closed symplectic manifold and let $\Phi : M \to \mathbb{R}^n$ be the moment map of an effective torus action. Then $\Phi(M) =: \Delta$ is a simple, rational, smooth polygon, called Delzant polygon. Any such polygon determines a symplectic toric manifold (up to symplectomorphisms preserving the momentum map).*

The proofs of Theorem 2.8 and Theorem 2.9 will use as much features of toric systems as possible.

1.2.4. **Semi-toric systems**. Pelayo & Vũ Ngọc [**PV1, PV2**] classified so-called semi-toric systems on (not necessarily compact) 4-dimensional manifolds. For an overview, we refer to Pelayo & Vũ Ngọc [**PV3**]. Since we will relate their classification to Karshon's classification of Hamiltonian $\mathbb{S}^1$-spaces we will always assume the manifolds to be compact. For reasons, which become later apparent, let us take a more category theoretic aproach.

Definition 2.2. *Let (M, ω) be a compact connected symplectic 4-dimensional manifold. The category of compact semi-toric systems $\mathcal{ST}$ is given by:*

(1) The objects are completely integrable Hamiltonian systems $(M, \omega, \Phi = (J, H))$ whose singular points are almost toric with

exactly one focus-focus point in a focus-focus fibre and such that (M, ω, J) is a Hamiltonian $\mathbb{S}^1$-space. Such systems are called ***compact semi-toric systems****.*

(2) *The morphisms are pairs (Ψ, ψ), where $\Psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism and $\psi : \Phi_1(M_1) \subset \mathbb{R}^2 \to \Phi_2(M_2) \subset \mathbb{R}^2$ is a locally defined diffeomorphism of the form $\psi(x, y) = (\psi^{(1)}, \psi^{(2)})(x, y) = (x, \psi^{(2)}(x, y))$ making the following diagram commute*

$$\begin{tikzcd} (M_1, \omega_1) \arrow[r, "\Psi"] \arrow[d, "\Phi_1"'] & (M_2, \omega_2) \arrow[d, "\Phi_2"] \\ \mathbb{R}^2 \arrow[r, "\psi"'] & \mathbb{R}^2. \end{tikzcd}$$

We will use the short notation $(\Psi, \psi) : (M_1, \omega_1, \Phi_1) \to (M_2, \omega_2, \Phi_2)$ and call them ***isomorphisms of compact semi-toric systems****.*

Having exactly one focus-focus point in a focus-focus fibre is a generic property and also assumed in Pelayo & Vũ Ngọc [**PV2**]. For technical convenience (it simplifies the definition of signs and cuts in a semi-toric polygon), we moreover assume that any map $f : B = \Phi(M) \to \mathbb{R}^2$ whose image yields a polygon is orientation preserving.

Semi-toric systems induce an $\mathbb{S}^1 \times \mathbb{R}$-action which places them 'between' general integrable systems which have an $\mathbb{R} \times \mathbb{R}$-action and toric systems with an $\mathbb{S}^1 \times \mathbb{S}^1$-action. Semi-toric systems differ from toric systems by the existence of focus-focus singularities. But they do not carry an arbitrary number of them:

Proposition 2.3 (Vũ Ngọc [**Vu2**], Cor. 5.10). *A semi-toric system has only a finite number of focus-focus critical values.*

Pelayo & Vũ Ngọc [**PV1**] use the following five invariants to classify semi-toric systems. Let $(M, \omega, \Phi = (J, H))$ be a (not necessarily compact) semi-toric system. Its list of *semi-toric invariants* consists of the following items.

(1) The number m_f of focus-focus points $\mathfrak{c}_1, \ldots, \mathfrak{c}_{m_f}$.

(2) m_f Taylor series $S_i := S_i(\mathfrak{c}_i) \in \mathbb{R}[[X, Y]]$ for $1 \leq i \leq m_f$ associated to the focus-focus points.
The Taylor series S_i of the focus-focus point $\mathfrak{c}_i$ is a *local* invariant of $\mathfrak{c}_i$ and does not interact with any of the other Taylor series or invariants in the list. Roughly, Vũ Ngọc constructs a 'generating function' $\mathcal{S}_i$ for the Lagrangian fibration near the focus-focus point $\mathfrak{c}_i$ which extends smoothly over the focus-focus singularity. Its Taylor series is S_i and its constant term vanishes due to construction. For details see Vũ Ngọc [**Vu1**].

(3) The (equivalence class of a) semi-toric polygon $[\mathcal{P}^{\Phi}_{\varepsilon}] := [(\mathcal{P}^{\Phi}, (l_i)_{i=1}^{m_f}, (\varepsilon_i)_{i=1}^{m_f})]$ where the l_i are vertical cuts into the polygon through the focus-focus points and the ε_i are associated signs.
$\mathcal{P}^{\Phi}_{\varepsilon}$ is obtained by a 'straightening procedure' of the 'curved polygon' $\Phi(M)$.

(4) The heights $0 < h_i < length(l_i \cap \mathcal{P}^{\Phi})$ for $1 \le i \le m_f$ of the focus-focus points in the semi-toric polygon.

(5) The twisting index is (an equivalence class of) $(\mathcal{P}^{\Phi}, (l_i)_{i=1}^{m_f}, (\varepsilon_i)_{i=1}^{m_f}, (k_i)_{i=1}^{m_f})$ where the k_i are integers which describe the twistedness locally around $\mathfrak{c}_i$ w.r.t. the globally chosen toric momentum map. Thus it is a global invariant, but k_i is independent of k_j for $1 \le i \ne j \le m_f$. For details consult Pelayo & Vũ Ngọc [**PV2, PV1**].

Theorem 2.4 (Pelayo & Vũ Ngọc [**PV1, PV2**]).

1) Two semi-toric integrable systems are isomorphic if and only if they have the same semi-toric invariants (cf. [**PV1**]*, Theorem 6.2).*

2) Given a list of 'semi-toric ingredients', there exists a 4*-dimensional semi-toric system having the semi-toric ingredients as semi-toric invariants (cf.* [**PV2**]*).*

An important observation is the fact that the underlying manifold is compact if and only if the semi-toric polygon is compact.

1.3. Classification of 4**-dimensional Hamiltonian** $\mathbb{S}^1$**-spaces.** For later purposes, let us consider Hamiltonian $\mathbb{S}^1$-spaces from a more category theoretical point of view.

Definition 2.5. *Let (M, ω) be a compact symplectic* 4*-dimensional manifold. The category $\mathcal{H}am_{\mathbb{S}^1}$ is defined by:*

(1) Objects*: Hamiltonian $\mathbb{S}^1$-spaces (M, ω, J).*

(2) Morphisms*: symplectomorphisms $\Psi\colon (M_1, \omega_1) \to (M_2, \omega_2)$ making the following diagram*

$$\begin{tikzcd} (M_1,\omega_1) \arrow[rr, "\Psi"] \arrow[dr, "J_1"'] & & (M_2,\omega_2) \arrow[dl, "J_2"] \\ & \mathbb{R} & \end{tikzcd}$$

*commute. We use the notation $\Psi : (M_1, \omega_1, J_1) \to (M_2, \omega_2, J_2)$ and and call them **isomorphisms of Hamiltonian $\mathbb{S}^1$-spaces**.*

Denote by $M^{\mathbb{S}^1}$ the fixed point set of the $\mathbb{S}^1$-action. Karshon [**Ka**] assigns to each (isomorphism class of a) Hamiltonian $\mathbb{S}^1$-space (M, ω, J) a **labeled,**

directed graph $\Gamma = (V, E)$ consisting of the set of vertices V and the set of edges E which are obtained as follows.

Vertex set V: For every connected component in $M^{\mathbb{S}^1}$ draw a vertex. Those associated to fixed surfaces are drawn as 'fat vertices'.

Labeling of V: Each vertex is labeled by the value of J on the corresponding component of the fixed point set.
A fat vertex is additionally labeled by the genus of the corresponding surface Σ and its normalized symplectic area $\frac{1}{2\pi}\int_\Sigma \omega$.

Edge set E: Every $\mathbb{Z}_k$-sphere, $k \geq 2$, (cf. Karshon [**Ka**]) gives rise to a directed edge going from its south to its north pole.

Labeling of E: Label each edge with the isotropy weight of the corresponding $\mathbb{Z}_k$-sphere if $k \geq 2$.

These labeled, directed graphs classify Hamiltonian $\mathbb{S}^1$-spaces:

THEOREM 2.6 (Karshon [**Ka**], Theorem 4.1). *Two 4-dimensional Hamiltonian $\mathbb{S}^1$-spaces are isomorphic if and only if their directed labeled graphs are equal.*

The following statement turned out to be essential for our work.

THEOREM 2.7 (Karshon [**Ka**], Prop. 5.16 and 5.21). *Given a 4-dimensional Hamiltonian $\mathbb{S}^1$-space (M, ω, J), the following are equivalent:*

(1) The $\mathbb{S}^1$-action extends to an effective Hamiltonian 2-torus action with moment map given by $(J, H) : M \to \mathbb{R}^2$, i.e. the triple $(M, \omega, (J, H))$ is a symplectic toric manifold.

(2) Each fixed surface has genus zero and each non-extremal level set of J contains at most two non-free orbits.

(3) Each fixed surface has genus zero and there is a compatible metric for which there are no more than two non-trivial chains of gradient spheres.

Therefore we call a Hamiltonian $\mathbb{S}^1$-space (M, ω, J) **extendable** if it satisfies any of the conditions of Theorem 2.7. Otherwise it is **nonextendable**.

2. From compact semi-toric systems to Hamiltonian $\mathbb{S}^1$-actions

Let (M, ω) be a connected, closed, symplectic 4-manifold. In this section, we will show how Pelayo & Vũ Ngọc's invariants of a compact semi-toric system $(M, \omega, \Phi = (J, H))$ interact with Karshon's labeled, directed graph of the underlying Hamiltonian $\mathbb{S}^1$-space (M, ω, J).

First notice that we have a functor

$$\mathcal{F} : \mathcal{ST} \to \mathcal{H}am_{\mathbb{S}^1}, \qquad \begin{aligned} &(M, \omega, \Phi = (J, H)) \mapsto (M, \omega, J), \\ &(\Psi, \psi) \mapsto \Psi, \end{aligned}$$

i.e. we obtain the underlying space (M, ω, J) from $(M, \omega, \Phi = (J, H))$ by 'forgetting' H and this is compatible with the isomorphisms of semi-toric systems and Hamiltonian $\mathbb{S}^1$-spaces.

Let us call a semi-toric polygon decorated with focus-focus critical values ('points'), cuts and signs a *labeled convex polygon*. Pelayo & Vũ Ngọc [**PV1**] ask for the minimal set of invariants of $(M, \omega, \Phi = (J, H))$ in order to recover the labeled directed graph of (M, ω, J). The answer is

THEOREM 2.8 (Hohloch & Sabatini & Sepe [**HSS**]). *Any labeled convex polygon associated to a semi-toric system $(M, \omega, \Phi = (J, H))$ yields the labeled directed graph associated to the underlying Hamiltonian $\mathbb{S}^1$-space (M, ω, J).*

Thus the first and third invariant in Pelayo & Vũ Ngọc's classification encode all information about the underlying Hamiltonian $\mathbb{S}^1$-space.

The proof of Theorem 2.8 makes use of the toric features of a semi-toric system. Toric systems are completely classified by the image of their moment map, i.e. their Delzant polygon, see Theorem 2.1. How to pass from a Delzant polygon to the labeled directed graph is nicely described by Karshon. We mimic her ideas as much as possible, but needed new methods to deal with the focus-focus points. Here the local normal form by Eliasson and Miranda & Zung is important as well as the connectedness of the fibers of a semi-toric system. We summarize the situation in Figure 2.1.

We call a semi-toric system **adaptable** if its underlying Hamiltonian $\mathbb{S}^1$-action can be extended to an effective Hamiltonian $\mathbb{T}^2$-action. The study of adaptable versus nonadaptable semi-toric systems leads to the following observation.

THEOREM 2.9 (Hohloch & Sabatini & Sepe [**HSS**]).

(a) A semi-toric system (M, ω, Φ) is adaptable if and only if one of its associated labeled convex polygons is Delzant.

(b) Let $(M, \omega, \Phi = (J, H))$ be an adaptable system and denote by (M, ω, J) its underlying Hamiltonian $\mathbb{S}^1$-space. The family of labeled convex polygons associated to $(M, \omega, \Phi = (J, H))$ contains all Delzant polygons whose corresponding symplectic toric manifolds have (M, ω, J) as their associated Hamiltonian $\mathbb{S}^1$-space.

This gives a nice and useful criterion for distinguishing between adaptable and nonadaptable systems. Moreover, Example 4.12 in Hohloch & Sabatini

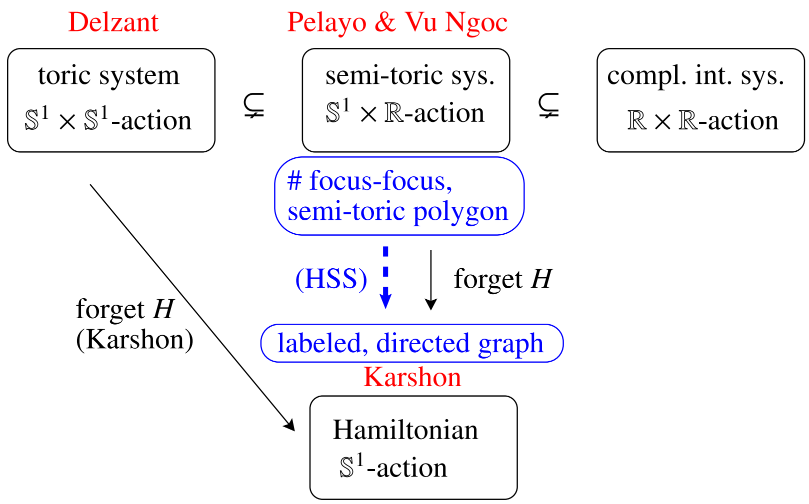


FIGURE 2.1. Overview.

& Sepe [**HSS**] shows the construction of a nonadaptable system which, to the best of our knowledge, is the first such example in the literature. We consider the moment polytope of a toric system on $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at two points (Figure 2.2 (a)), apply a nodal trade (Figure 2.2 (b)) and a suitable piecewise integral affine transformation (Figure 2.2 (c)) and blow up the vertex $(0, 0)$ to obtain a nonadaptable system in Figure 2.2 (d). For details we refer the reader to Hohloch & Sabatini & Sepe [**HSS**].

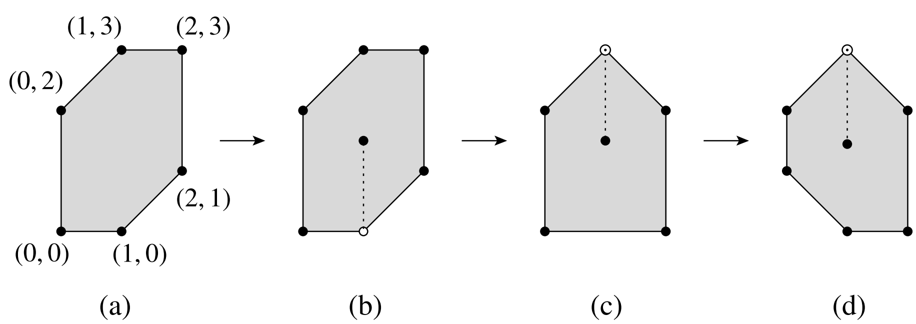


FIGURE 2.2. Example 4.12 in Hohloch & Sabatini & Sepe [**HSS**].

3. Future projects

In contrast to the earlier work Hohloch & Sabatini & Sepe [**HSS**], where the transition from compact semi-toric systems to Hamiltonian $\mathbb{S}^1$-spaces is studied, this new joint project Hohloch & Sabatini & Sepe & Symington

[**HSSS**] sheds light on the transition from Hamiltonian $\mathbb{S}^1$-spaces to compact semi-toric systems. This is motivated by Karshon [**Ka**] who describes under which conditions a Hamiltonian $\mathbb{S}^1$-spaces extends to a $\mathbb{T}^2$-action. We are investigating when it extends to a compact semi-toric system, i.e. an $\mathbb{S}^1 \times \mathbb{R}$-action with 'nice' singularities. The situation is sketched in Figure 2.3.

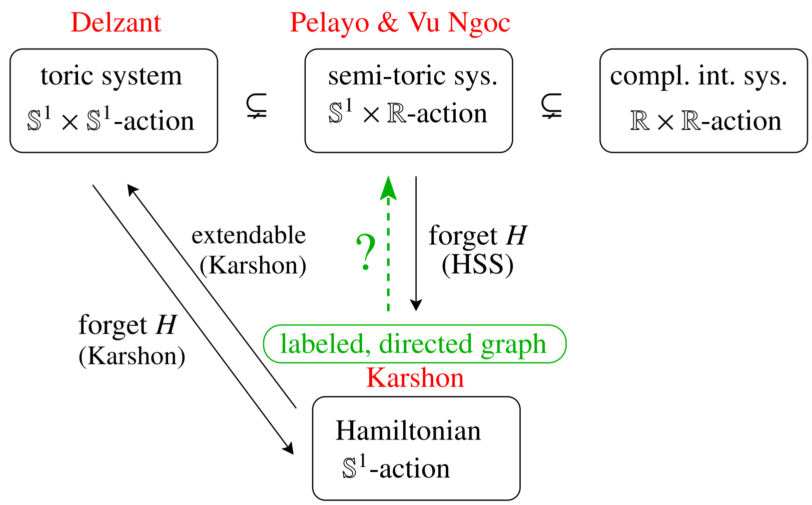


FIGURE 2.3. Future projects.

'THEOREM' 2.10 (Hohloch & Sabatini & Sepe & Symington [**HSSS**]). *Given a 'weakly extendable' Karshon graph, there exists a compact semi-toric polygon (and thus a semi-toric system) which yields the Karshon graph when forgetting the $\mathbb{R}$-action.*

The proof of this theorem involves the definition of 'semi-toric blow-ups' and 'semi-toric nodal trades and slides' which exist in the literature for $\mathbb{S}^1$- and $\mathbb{T}^2$-equivariant settings. Karshon [**Ka**] proved that all Hamiltonian $\mathbb{S}^1$-spaces can be obtained from three different types of spaces using $\mathbb{S}^1$-equivariant blow-ups. We intend to prove an analogous statement in the semi-toric world.

CONJECTURE 2.11 (Hohloch & Sabatini & Sepe & Symington [**HSSS**]). *Compact semi-toric systems can be obtained from a finite number of 'minimal' compact semi-toric systems using the blow-up and nodal trade and slide operations.*

CHAPTER 3

Higher Morse moduli spaces and n-categories

In this chapter, we show a connection between Morse theory and higher category theory. More precisely, we will construct a so-called *almost strict n-category* by repeatedly doing Morse theory on Morse moduli spaces. The resulting n-category $\mathcal{X}$ generalizes Cohen & Jones & Segal's [**CJS**] *Flow category* and looks very complicated. Thus we looked for n-category functors from $\mathcal{X}$ to 'easier' almost strict n-categories in order to imitate representation theory of groups where one studies homomorphisms ('representations') from a given group into a 'nicer' group. We found two other almost strict n-categories $\mathcal{V}$ and $\mathcal{W}$ and two almost strict n-category functors $\mathcal{F} : \mathcal{X} \to \mathcal{V}$ and $\mathcal{G} : \mathcal{X} \to \mathcal{W}$ which are based on the dimension of the Morse moduli spaces and the Morse index.

1. Almost strict n-categories

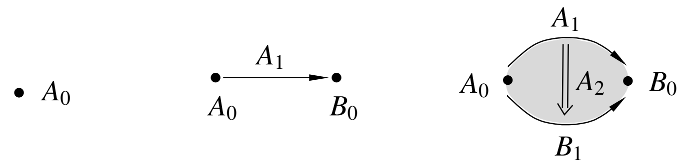
There is a whole zoo of notions and definitions of higher categories in the literature. The main distinction is between 'weak' versus 'strict' categories where 'weak' in contrast to strict means that certain properties only have to hold 'up to some deformation', for instance instead of 'associativity', one only has 'associativity up to homotopy'. Leinster's book [**Le**] gives a good overview and introduction to higher category theory and we stick to its notions and conventions. We will first repeat the definition of strict n-categories (which goes back to Ehresmann) before we introduce the new notion of 'almost strict' n-categories.

Definition 3.1. *Given $n \in \mathbb{N}$, an **n-globular set** Y is a collection of sets $\{Y(\ell) \mid 0 \le \ell \le n\}$ together with **source** and **target functions** $s, t : Y(\ell) \to Y(\ell-1)$ for $1 \le \ell \le n$ satisfying $s \circ s = s \circ t$ and $t \circ s = t \circ t$. Elements $A_\ell \in Y(\ell)$ are called **ℓ-cells**.*

Figure 3.1 suggests to think of ℓ-cells as ℓ-dimensional disks. For $0 \le p < \ell \le n$, the set

$$Y(\ell) \times_p Y(\ell) := \{(C, A) \in Y(\ell) \times Y(\ell) \mid s^{\ell-p}(C) = t^{\ell-p}(A)\}$$

describes those ℓ-cells A and C which can be composed along a p-cell, i.e. it encodes the matching conditions for the composition of cells.



(a) (b) (c)

FIGURE 3.1. **(a)** 0-cell $A_0 \in Y(0)$, **(b)** displays a 1-cell $A_1 \in Y(1)$ with $s(A_1) = A_0 \in Y(0)$ and $t(A_1) = B_0 \in Y(0)$, **(c)** sketches a 2-cell $A_2 \in Y(2)$ with $s(A_2) = A_1$, $t(A_2) = B_1 \in Y(1)$ and therefore $s(A_1) = s(B_1) = A_0$ and $t(A_1) = t(B_1) = B_0$.

DEFINITION 3.2. *Let $n \in \mathbb{N}$. A **strict n-category** $\mathcal{Y}$ is an n-globular set Y equipped with*

- *a function $\circ_p : Y(l) \times_p Y(l) \to Y(l)$ for all $0 \le p < l \le n$. We set $\circ_p(C_l, A_l) =: C_l \circ_p A_l$ and call it **composite** of A_l and C_l.*
- *a function $\mathbf{1} : Y(l) \to Y(l+1)$ for all $0 \le l < n$. We set $\mathbf{1}_{A_l} := \mathbf{1}(A_l)$ and call it the **identity** on A_l.*

These have to satisfy the following axioms:

(a) ***(Sources and targets of composites)*** *For $0 \le p < l \le n$ and $(C_l, A_l) \in Y(l) \times_p Y(l)$ we require*

$$\begin{aligned} \text{for } p = l-1: \qquad & s(C_l \circ_p A_l) = s(A_l), \\ & t(C_l \circ_p A_l) = t(C_l), \\ \text{for } p \le l-2: \qquad & s(C_l \circ_p A_l) = s(C_l) \circ_p s(A_l), \\ & t(C_l \circ_p A_l) = t(C_l) \circ_p t(A_l). \end{aligned}$$

(b) ***(Sources and targets of identities)*** *For $0 \le l < n$ and $A_l \in Y(l)$ we require*

$$s(\mathbf{1}_{A_l}) = A_l = t(\mathbf{1}_{A_l}).$$

(c) ***(Associativity)*** *For $0 \le p < l \le n$ and A_l, C_l, $E_l \in Y(l)$ with (E_l, C_l), $(C_l, A_l) \in Y(l) \times_p Y(l)$ we require*

$$(E_l \circ_p C_l) \circ_p A_l = E_l \circ_p (C_l \circ_p A_l).$$

(d) ***(Identities)*** *For $0 \le p < l \le n$ and $A_l \in Y(l)$ we require*

$$\mathbf{1}^{l-p}(t^{l-p}(A_l)) \circ_p A_l = A_l = A_l \circ_p \mathbf{1}^{l-p}(s^{l-p}(A_l)).$$

(e) ***(Binary interchange)*** *For $0 \le q < p < l \le n$ and A_l, C_l, E_l, $H_l \in Y(l)$ with*

$$(H_l, E_l), (C_l, A_l) \in Y(l) \times_p Y(l) \text{ and } (H_l, C_l), (E_l, A_l) \in Y(l) \times_q Y(l)$$

we require

$$(H_l \circ_p E_l) \circ_q (C_l \circ_p A_l) = (H_l \circ_q C_l) \circ_p (E_l \circ_q A_l).$$

(f) ***(Nullary interchange)*** *For $0 \le p < l < n$ and $(C_l, A_l) \in Y(l) \times_p Y(l)$ we require $\mathbf{1}_{C_l} \circ_p \mathbf{1}_{A_l} = \mathbf{1}_{C_l \circ_p A_l}$.*

If $\mathcal{Y}$ and $\mathcal{Z}$ are strict n-categories we define a ***strict n-functor*** *$\mathcal{F}$ as a map $\mathcal{F} : Y \to Z$ of the underlying n-globular sets commuting with composition and identities. This defines a category of strict n-categories.*

Strict categories are rare in real life. The following definition relaxes the requirements slightly.

Definition 3.3 (Hohloch [**Hoh5**]). *An* ***almost strict n-category*** *satisfies the requirements of a strict n-category up to canonical isomorphism. Let $\mathcal{A}$ and $\mathcal{B}$ be two almost strict n-categories with n-globular sets A and B. An* ***almost strict n-category functor*** *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, briefly an* ***n-functor****, is a map $\mathcal{F} : A \to B$ of the underlying n-globular sets commuting with composition and identities. This defines the category of almost strict n-categories.*

In geometry, being identical ‘up to canonical isomorphism’ is often identified with ‘being identical’ since otherwise for instance the cartesian product would have to be called ‘associative up to canonical isomorphism’ which nobody does. Our main interest lies in geometry, thus we voted for calling the above n-category ‘almost strict’ instead of ‘not very weak’.

2. Morse moduli spaces on smooth manifolds without boundary.

In the following, we recall briefly the approach to Morse theory via dynamical systems, i.e. using the negative gradient flow of a Morse function instead of handle attachements. For details, we refer the reader to the books by Schwarz [**Sch**] and Audin & Damian [**AuD**].

Let M be a compact m-dimensional manifold without boundary. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if its Hessian $D^2 f$ is nondegenerate at the critical points $\mathrm{Crit}(f) := \{x \in M \mid Df(x) = 0\}$. The *Morse index* $\mathrm{Ind}(x)$ of a critical point x is the number of negative eigenvalues of $D^2 f(x)$. For a Riemannian metric g on M, we denote by $\mathrm{grad}_g f$ the gradient of f w.r.t. the metric g. The *negative gradient flow* φ_t of the pair (f, g) is given by the (autonomous) ODE

$$\dot{\varphi}_t = -\,\mathrm{grad}_g f(\varphi_t).$$

By

$$W^s(f, x) := W^s(f, g, x) := \{p \in M \mid \lim_{t \to +\infty} \varphi_t(p) = x\}$$

we denote the *stable manifold* of a critical point $x \in \mathrm{Crit}(f)$ and by

$$W^u(f,x) := W^u(f,g,x) := \{p \in M \mid \lim_{t\to-\infty} \varphi_t(p) = x\}$$

its *unstable manifold*. A pair (f,g) is *Morse-Smale* if $W^s(f,g,x)$ and $W^u(f,g,y)$ intersect transversely for all critical points $x, y \in \mathrm{Crit}(f)$. The *Morse moduli space* between $x, y \in \mathrm{Crit}(f)$ is the space of smooth curves

$$\mathcal{M}(x,y) := \mathcal{M}(x,y,f,g) := \left\{ \gamma : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \dot\gamma(t) = -\operatorname{grad}_g f(\gamma(t)), \\ \lim_{t\to-\infty} \gamma(t) = x, \\ \lim_{t\to+\infty} \gamma(t) = y \end{array} \right\}.$$

The elements of $\mathcal{M}(x,y)$ are those negative gradient flow lines which join x to y. The space can also be seen as the intersection $W^u(x,f) \cap W^s(f,y)$. If (f,g) is Morse-Smale the moduli space $\mathcal{M}(x,y)$ is a smooth manifold of dimension $\mathrm{Ind}(x) - \mathrm{Ind}(y)$. If $\mathrm{Ind}(y) > \mathrm{Ind}(x)$ then $\mathcal{M}(x,y)$ is empty. Since we are dealing with an autonomous ODE the moduli space carries a natural $\mathbb{R}$-action. More precisely, given $\gamma \in \mathcal{M}(x,y)$ and $\sigma \in \mathbb{R}$, the curve γ_σ defined by $\gamma_\sigma(t) := \gamma(t+\sigma)$ is also a negative gradient flow line. Thus there is an action $\mathbb{R} \times \mathcal{M}(x,y) \to \mathcal{M}(x,y)$, $(\gamma,\sigma) \mapsto \gamma_\sigma$. The quotient space by the action is the *unparametrized* moduli space $\mathcal{M}(x,y)/\mathbb{R}$.

An *m-dimensional manifold with corners* is an m-dimensional manifold which is locally modeled on $\mathbb{R}^m_+ := (\mathbb{R}_{\geq 0})^m$. If $\psi = (\psi_1, \dots, \psi_m) : U \subseteq N \to \mathbb{R}^m_+$ is a chart of an m-dimensional manifold with corners N and $x \in U$, set

$$depth(x) := \#\{i \mid \psi_i(x) = 0,\ 1 \leq i \leq m\}.$$

A *face* of N is the *closure* of a connected component of $\{x \in N \mid depth(x) = 1\}$. If k is the number of faces, we fix an *order* of the faces and denote them by $\partial_1 N, \dots, \partial_k N$. The connected components of $\{x \in N \mid depth(x) = l\} =: D_{\dim N - l}$ are called the *$(\dim N - l)$-strata* of N.

Definition 3.4. *Let N be an m-dimensional manifold with corners having k faces $\partial_1 N, \dots, \partial_k N$. We call N a **$\langle k\rangle$-manifold** if*

(a) Each $x \in N$ lies in $depth(x)$ faces.
(b) $\partial_1 N \cup \dots \cup \partial_k N = \partial N$.
(c) For all $1 \leq i, j \leq k$ with $i \neq j$ the intersection $\partial_i N \cap \partial_j N$ is a face of both $\partial_i N$ and $\partial_j N$.

Note that in this convention the faces $\partial_i N \subset N$ are manifolds with corners, but the boundary ∂N is not. There are several conventions in the literature and we have chosen Joyce's [**Jo**] definition where the integer $\langle k\rangle$ has a priori nothing to do with the dimension m of the manifold N. If one admits $\partial_i N$ to be a union of faces (cf. Laures [**Laur**]) one can enforce $k = \dim N$.

In order to compactify unparametrized moduli spaces we briefly have to discuss the phenomenon of 'breaking' and 'gluing' of Morse trajectories. Consider a smooth compact manifold M with Morse-Smale pair (f, g) and $x, y, z \in \mathrm{Crit}(f)$ with $\mathrm{Ind}(x) > \mathrm{Ind}(y) > \mathrm{Ind}(z)$. Figure 3.2 (a) displays a sequence of trajectories $(\gamma_n)_{n\in\mathbb{N}}$ from x to z which 'break' in the limit into trajectories γ_{xy} from x to y and γ_{yz} from y to z. This phenomenon is called 'breaking'. Unparametrized moduli space are compactified by adding 'broken trajectories' as boundary points and this compactification of $\mathcal{M}(x,z)/\mathbb{R}$ via adding broken trajectories is denoted $\widehat{\mathcal{M}}(x,z) := \overline{\mathcal{M}(x,z)/\mathbb{R}}$. Since we wish to give the compactified moduli spaces the structure of a manifold with corners, we have to be a little bit careful in the choice of the metric. Thus we introduce the following notation. Let f be a Morse function. Then a metric g is *f-euclidean* if it is euclidean near the critical points of f. Moreover, given $x, y \in \mathrm{Crit}(f)$ with $x \neq y$, we write $x > y$ if $\mathcal{M}(x,y) \neq \emptyset$.

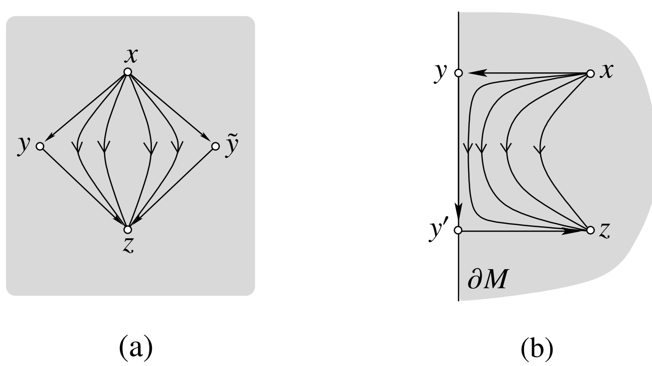


FIGURE 3.2. Breaking of trajectories: (a) in the interior, (b) on the boundary.

THEOREM 3.5 (Burghelea [**Bu**], Wehrheim [**Weh**], Qin [**Qi1**], [**Qi2**]). *Let M be compact and (f, g) be Morse-Smale and assume g to be f-euclidean. Let $x, z \in \mathrm{Crit}(f)$ with $x > z$. Then there exists $k \in \mathbb{N}_0$ such that $\widehat{\mathcal{M}}(x,z)$ is an $(\mathrm{Ind}(x)-\mathrm{Ind}(z)-1)$-dimensional $\langle k\rangle$-manifold with corners and its boundary is given by*

$$\partial\widehat{\mathcal{M}}(x,z) = \bigcup_{\substack{(\mathrm{Ind}(x)-\mathrm{Ind}(z)-1)\geq l\geq 0\\ x>y_1>\cdots>y_l>z}} \widehat{\mathcal{M}}(x,y_1)\times\widehat{\mathcal{M}}(y_1,y_2)\times\ldots\times\widehat{\mathcal{M}}(y_{l-1},y_l)\times\widehat{\mathcal{M}}(y_l,z)$$

where $y_1,\ldots,y_l \in \mathrm{Crit}(f)$. There is a canonical smooth structure on $\widehat{\mathcal{M}}(x,z)$.

Given broken trajectories $(\gamma_{xy}, \gamma_{yz})$ in $\widehat{\mathcal{M}}(x,y)\times\widehat{\mathcal{M}}(y,z)$ we can 'glue' them to a Morse trajectory from x to z. Later on, we will have to glue multiply broken trajectory $(\gamma_1, \dots, \gamma_{l+1}) \in \widehat{\mathcal{M}}(x, y_1) \times \dots \times \widehat{\mathcal{M}}(y_l, z)$ such that the question of associativity of the gluing procedure arises. Qin [**Qi3**] and Wehrheim [**Weh**] showed that gluing is indeed associative such that our constructions in later sections are well-defined.

3. Morse moduli spaces on $\langle k \rangle$-manifolds.

In the previous section, we recalled Morse theory on manifolds without boundary. This section now deals with Morse theory on manifolds with corners. If the manifold has a smooth boundary (i.e. no corners), Akaho [**Ak**] and Kronheimer & Mrowka [**KrM**] showed how the negative gradient approach works for Morse functions with gradient vector fields which are tangent to the boundary. More generally, Ludwig [**Lu**] defined Morse theory with tangential gradient vector field on stratified spaces, but her theory does not cover higher dimensional moduli spaces in the way we need it later.

On a smooth compact manifold M, consider a Morse-Smale pair (f_0, g_0) consisting of a Morse function f_0 with f_0-euclidean metric g_0. Given distinct critical points x_0, $z_0 \in \operatorname{Crit}(f_0)$, the space $\widehat{\mathcal{M}}(x_0, z_0, f_0)$ is, according to Theorem 3.5, a manifold (possibly) with corners (if it is not empty). Its boundary is of the form

$$\partial \widehat{\mathcal{M}}(x_0, z_0, f_0) = \bigsqcup_{\substack{(\operatorname{Ind}(x_0) - \operatorname{Ind}(z_0) - 1) \geq l \geq 0 \\ x_0 > y_0^1 > \dots > y_0^l > z_0}} \widehat{\mathcal{M}}(x_0, y_0^1, f_0) \times \dots \times \widehat{\mathcal{M}}(y_0^l, z_0, f_0)$$

with $y_0^1, \dots, y_0^l \in \operatorname{Crit}(f_0)$ which we can reformulate to

$$\partial \widehat{\mathcal{M}}(x_0, z_0, f_0) = \bigsqcup_{y_0 \in \operatorname{Crit}(f_0)} \widehat{\mathcal{M}}(x_0, y_0, f_0) \times \widehat{\mathcal{M}}(y_0, z_0, f_0).$$

Keep in mind that a moduli space may have several connected components. Choosing an ordering for the components of depth one, we give $\widehat{\mathcal{M}}(x_0, z_0, f_0)$ the structure of a $\langle k \rangle$-manifold. The space $\widehat{\mathcal{M}}(x_0, z_0, f_0)$ might share strata with other moduli spaces $\widehat{\mathcal{M}}(\tilde{x}_0, \tilde{z}_0, f_0)$ for $\tilde{x}_0, \tilde{z}_0 \in \operatorname{Crit}(f_0)$.

THEOREM 3.6 (Hohloch [**Hoh5**]). *Let f be a Morse function on a compact $\langle k \rangle$-manifold whose negative gradient flow is tangential to the boundary strata and flows from higher dimensional to lower dimensional strata, but not from lower to higher ones. Assume the metric to be euclidean near the critical points. Let $x, z \in \operatorname{Crit}(f)$ with $x > z$. Then there exists $k \in \mathbb{N}_0$ such that $\widehat{\mathcal{M}}(x, z)$ is an $(\operatorname{Ind}(x) - \operatorname{Ind}(z) - 1)$-dimensional $\langle k \rangle$-manifold with* *corners and its boundary is given by*

$$\partial\widehat{\mathcal{M}}(x,z) = \bigcup_{\substack{(\mathrm{Ind}(x)-\mathrm{Ind}(z)-1)\geq l\geq 0 \\ x>y_1>\dots>y_l>z}} \widehat{\mathcal{M}}(x,y_1)\times\widehat{\mathcal{M}}(y_1,y_2)\times\ldots\times\widehat{\mathcal{M}}(y_{l-1},y_l)\times\widehat{\mathcal{M}}(y_l,z)$$

where $y_1,\ldots,y_l \in \mathrm{Crit}(f)$. *There is a canonical smooth structure on* $\widehat{\mathcal{M}}(x,z)$.

The condition *'…flows from higher dimensional to lower dimensional strata, but not from lower to higher ones'* is not just technical, but essential. If we drop it, the breaking behaviour gets much wilder, see Figure 3.2 (b). In particular, one may have to glue simultanously *more than two* trajectories in order to obtain a connecting trajectory. This phenomenon is part of the forthcoming work Hohloch & Ludwig [**HohL**].

4. The almost strict Morse n-category.

This section summarizes the construction of the almost strict Morse n-category from Hohloch [**Hoh5**] to which we refer for details. Throughout this section, we require the Morse functions to satisfy:

1) The gradient vector field is tangential to the boundary strata.
2) The Morse function is compatible with the cartesian product structure of the boundary of a Morse moduli space.
3) The negative gradient flow passes from higher dimensional into lower dimensional strata, but never from lower to higher dimensional strata, i.e. flowing into the boundary and later back into the interior of the manifold like in Figure 3.2 (b) does not happen.

Consider a compact n-dimensional $\langle k\rangle$-manifold M with a Morse function f_0 and an f_0-euclidean metric g_0. Let us define those sets which turn out to be an n-globular set. First set

$$X(0) := \{x_0 \mid x_0 \in \mathrm{Crit}(f_0)\}.$$

For x_0, $y_0 \in \mathrm{Crit}(f_0)$ choose on the space $\widehat{\mathcal{M}}(x_0,y_0,f_0)$ a Morse function $f_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]}$ with $f_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]}$-euclidean metric $g_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]}$. This leads to the definition of

$$X(1) := \{(x_1,\widehat{\mathcal{M}}(x_0,y_0,f_0)) \mid x_0,y_0\in\mathrm{Crit}(f_0),\ x_1\in\mathrm{Crit}(f_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]})\}.$$

Let us briefly explain the notation. The first entry in the lower index of the Morse function $f_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]}$ or metric $g_{1\left[\begin{smallmatrix}x_0\\y_0\end{smallmatrix}\right]}$ is the number of the 'iteration level' on which the function or metric lives. Then comes, in brackets, the (history of) critical points which gave rise to the moduli space. In the brackets, the upper row lists the source points and the lower row the target points. For our construction, we need to keep in mind the 'history' of a moduli space. Iterating leads to

$$X(l) := \left\{ \left(x_l, \widehat{\mathcal{M}}(x_{l-1}, y_{l-1}, f_{l-1\left[\substack{x_{l-2},\dots,x_0\\ y_{l-2},\dots,y_0}\right]}) \right) \middle| \begin{array}{l} 0 \le j \le l-1, \\ x_j, y_j \in \mathrm{Crit}(f_{j\left[\substack{x_{j-1},\dots,x_0\\ y_{j-1},\dots,y_0}\right]}), \\ x_l \in \mathrm{Crit}(f_{l\left[\substack{x_{l-1},\dots,x_0\\ y_{l-1},\dots,y_0}\right]}) \end{array} \right\}$$

for $2 \le l \le n$. *Source* and *target functions* are given by

$$s : X(l) \to X(l-1) \qquad \text{and} \qquad t : X(l) \to X(l-1)$$

for $2 \le l \le n$ via

$$s\left(x_l, \widehat{\mathcal{M}}(x_{l-1}, y_{l-1}, f_{l-1\left[\substack{x_{l-2},\dots,x_0\\ y_{l-2},\dots,y_0}\right]}) \right) := \left(x_{l-1}, \widehat{\mathcal{M}}(x_{l-2}, y_{l-2}, f_{l-2\left[\substack{x_{l-3},\dots,x_0\\ y_{l-3},\dots,y_0}\right]}) \right),$$
$$t\left(x_l, \widehat{\mathcal{M}}(x_{l-1}, y_{l-1}, f_{l-1\left[\substack{x_{l-2},\dots,x_0\\ y_{l-2},\dots,y_0}\right]}) \right) := \left(y_{l-1}, \widehat{\mathcal{M}}(x_{l-2}, y_{l-2}, f_{l-2\left[\substack{x_{l-3},\dots,x_0\\ y_{l-3},\dots,y_0}\right]}) \right)$$

and we set for $s, t : X(1) \to X(0)$

$$s\left(a_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)\right) := x_0 \quad \text{and} \quad t\left(a_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)\right) := y_0.$$

LEMMA 3.7 (Hohloch [**Hoh5**]). *$X := \{X(l) \mid 0 \le l \le n\}$ is an n-globular set.*

Remember that the matching condition for the composition of l-cells along p-cells was given by

$$X(l) \times_p X(l) := \{(C_l, A_l) \in X(l) \times X(l) \mid s^{l-p}(C_l) = t^{l-p}(A_l)\}.$$

There is actually a nice way to write a tupel $(C_l, A_l) \in X(l) \times_p X(l)$ such that one can see exactly where a and C match:

$$A_l = \left(a_l, \widehat{\mathcal{M}}(a_{l-1}, b_{l-1}, f_{l-1\left[\substack{a_{l-2},\dots,a_{p+1},x_p,\alpha_{p-1},\dots,\alpha_0\\ b_{l-2},\dots,b_{p+1},y_p,\beta_{p-1},\dots,\beta_0}\right]}) \right),$$
$$C_l = \left(c_l, \widehat{\mathcal{M}}(c_{l-1}, d_{l-1}, f_{l-1\left[\substack{c_{l-2},\dots,c_{p+1},y_p,\alpha_{p-1},\dots,\alpha_0\\ d_{l-2},\dots,d_{p+1},z_p,\beta_{p-1},\dots,\beta_0}\right]}) \right).$$

In words, this means that both l-cells arise, up to level $(p-1)$, from the same critical points $\left[\substack{\alpha_{p-1},\dots,\alpha_0\\ \beta_{p-1},\dots,\beta_0}\right]$. At level p, they match via $\substack{\left[\substack{x_p\\ y_p}\right]\\ \left[\substack{y_p\\ z_p}\right]}$. There are no additional conditions on the critical points on the higher levels $\substack{\left[\substack{a_{l-2},\dots,a_{p+1}\\ b_{l-2},\dots,b_{p+1}}\right]\\ \left[\substack{c_{l-2},\dots,c_{p+1}\\ d_{l-2},\dots,d_{p+1}}\right]}$ apart from the ones required in the definition of $X(l)$. The whole expression $\left[\substack{a_{l-2},\dots,a_{p+1},x_p,\alpha_{p-1},\dots,\alpha_0\\ b_{l-2},\dots,b_{p+1},y_p,\beta_{p-1},\dots,\beta_0}\right]$ is called the *history* of A_l up to level $(l-2)$. If $j = 1$ in the two expressions above then there are no a's and b's resp. c's and d's in the index of the function.

The identity functions

$$\mathbf{1} : X(l) \to X(l+1)$$

are motivated now. Consider $x_0 \in X(0)$ and identify x_0 with the moduli space $\widehat{\mathcal{M}}(x_0, x_0, f_0)$. Then in turn identify $\widehat{\mathcal{M}}(x_0, x_0, f_0)$ with the only critical point $x_1 \in \mathrm{Crit}(f_{1\left[\substack{x_0 \\ x_0}\right]})$ on $\widehat{\mathcal{M}}(x_0, x_0, f_0)$. Thus we have $x_1 \simeq \widehat{\mathcal{M}}(x_0, x_0, f_0) \simeq x_0$ which motivates

$$\mathbf{1}_{x_0} := \mathbf{1}(x_0) := (x_0, \widehat{\mathcal{M}}(x_0, x_0, f_0)).$$

If $l > 0$, we set for $A_l = \left(a_l, \widehat{\mathcal{M}}(a_{l-1}, b_{l-1}, f_{l-1\left[\substack{a_{l-2},\dots,a_0 \\ b_{l-2},\dots,b_0}\right]})\right) \in X(l)$

$$\begin{aligned}
\mathbf{1}_{A_l} &:= \mathbf{1}\left(a_l, \widehat{\mathcal{M}}(a_{l-1}, b_{l-1}, f_{l-1\left[\substack{a_{l-2},\dots,a_0 \\ b_{l-2},\dots,b_0}\right]})\right) \\
&:= \left(a_l, \widehat{\mathcal{M}}(a_l, a_l, f_{l\left[\substack{a_{l-1},\dots,a_0 \\ b_{l-1},\dots,b_0}\right]})\right) \\
&:= \left(a_{l+1}, \widehat{\mathcal{M}}(a_l, a_l, f_{l\left[\substack{a_{l-1},\dots,a_0 \\ b_{l-1},\dots,b_0}\right]})\right)
\end{aligned}$$

where we again identified $a_{l+1} \simeq a_l$. For $0 \leq l \leq n-1$, this yields functions

$$\mathbf{1} : X(l) \to X(l+1)$$

which will be the identity functions in Theorem 3.8. We now introduce the composite $\circ_p$ for $l > p \geq 0$.

Case $l \in \mathbb{N}$ and $p = 0$: There are no α's and β's such that the 'history index' starts with x_0, y_0, z_0. We set

$$\begin{aligned}
&\left(c_l, \widehat{\mathcal{M}}(c_{l-1}, d_{l-1}, f_{l-1\left[\substack{c_{l-2},\dots,c_1,y_0 \\ d_{l-2},\dots,d_1,z_0}\right]})\right) \circ_0 \left(a_l, \widehat{\mathcal{M}}(a_{l-1}, b_{l-1}, f_{l-1\left[\substack{a_{l-2},\dots,a_1,x_0 \\ b_{l-2},\dots,b_1,y_0}\right]})\right) \\
&:= \left((a_l, c_l), \widehat{\mathcal{M}}((a_{l-1}, c_{l-1}), (b_{l-1}, d_{l-1}), f_{l-1\left[\substack{(a_{l-2},c_{l-2}),\dots,(a_1,c_1),x_0 \\ (b_{l-2},d_{l-2}),\dots,(b_1,d_1),z_0}\right]})\right).
\end{aligned}$$

Case $l \in \mathbb{N}$ and $l-2 \geq p \geq 1$: We set

$$\begin{aligned}
&\left(c_l, \widehat{\mathcal{M}}(c_{l-1}, d_{l-1}, f_{l-1\left[\substack{c_{l-2},\dots,c_{p+1},y_p,\alpha_{p-1},\dots,\alpha_0 \\ d_{l-2},\dots,d_{p+1},z_p,\beta_{p-1},\dots,\beta_0}\right]})\right) \\
&\quad \circ_p \left(a_l, \widehat{\mathcal{M}}(a_{l-1}, b_{l-1}, f_{l-1\left[\substack{a_{l-2},\dots,a_{p+1},x_p,\alpha_{p-1},\dots,\alpha_0 \\ b_{l-2},\dots,b_{p+1},y_p,\beta_{p-1},\dots,\beta_0}\right]})\right) \\
&:= \left((a_l, c_l), \widehat{\mathcal{M}}((a_{l-1}, c_{l-1}), (b_{l-1}, d_{l-1}), f_{l-1\left[\substack{(a_{l-2},c_{l-2}),\dots,(a_{p+1},c_{p+1}),x_p,\alpha_{p-1},\dots,\alpha_0 \\ (b_{l-2},d_{l-2}),\dots,(b_{p+1},d_{p+1}),z_p,\beta_{p-1},\dots,\beta_0}\right]})\right).
\end{aligned}$$

Case $l \in \mathbb{N}$ and $p = l - 1$: There are no a's, b's, c's and d's in the 'history index' which ends with $x_{l-1}, y_{l-1}, z_{l-1}$. We set

$$\left(c_l, \widehat{\mathcal{M}}(y_{l-1}, z_{l-1}, f_{l-1\left[\begin{smallmatrix}\alpha_{l-2},\dots,\alpha_0\\ \beta_{l-2},\dots,\beta_0\end{smallmatrix}\right]})\right) \circ_{l-1} \left(a_l, \widehat{\mathcal{M}}(x_{l-1}, y_{l-1}, f_{l-1\left[\begin{smallmatrix}\alpha_{l-2},\dots,\alpha_0\\ \beta_{l-2},\dots,\beta_0\end{smallmatrix}\right]})\right)$$
$$:= \left((a_l, c_l), \widehat{\mathcal{M}}(x_{l-1}, z_{l-1}, f_{l-1\left[\begin{smallmatrix}\alpha_{l-2},\dots,\alpha_0\\ \beta_{l-2},\dots,\beta_0\end{smallmatrix}\right]})\right).$$

The whole construction depends on the choice of a family of Morse functions $F := \{f_0, f_{1\left[\begin{smallmatrix}x_0\\ y_0\end{smallmatrix}\right]}, \dots\}$ and metrics $G := \{g_0, g_{1\left[\begin{smallmatrix}x_0\\ y_0\end{smallmatrix}\right]}, \dots\}$. We intend to address the question how the constructions with different Morse datas are related in a future work.

THEOREM 3.8 (Hohloch [**Hoh5**]). *The above defined n-globular set $X = \{X(l) \mid 0 \leq l \leq n\}$ together with the above defined identity functions* $\mathbf{1}$ *and composites $\circ_p$ is an almost strict n-category $\mathcal{X} := \mathcal{X}(F, G)$, called the* ***almost strict Morse n-category****.*

5. Functors to the almost strict n-categories $\mathcal{V}$ and $\mathcal{W}$

In this section, we intend to gain a better understanding of the almost strict Morse n-category. Assume the setting from the previous section, i.e. M is a smooth compact manifold and $\mathcal{X} = \mathcal{X}(F, G)$ depends on the Morse data $F = \left(f_{0,[\dots]}, \dots\right)$ and $G = \left(g_{0,[\dots]}, \dots\right)$.
We will now define two new almost strict n-categories $\mathcal{V}$ and $\mathcal{W}$ and n-functors $\mathcal{F} : \mathcal{X} \to \mathcal{V}$ and $\mathcal{F} : \mathcal{X} \to \mathcal{W}$. Both new categories are considerably easier to access than the Morse n-category. Thus we hope to gain knowledge of $\mathcal{X}$ by studying its image under $\mathcal{F}$ and $\mathcal{G}$. This is motivated by representation theory of groups where homomorphisms from a given group into a usually well known one are studied.

5.1. The almost strict n-categories $\mathcal{V}$ and $\mathcal{W}$. Now let us define the elements of the n-globular set $V = \{V(\ell) \mid 0 \leq \ell \leq n\}$. We begin with

$$V(0) := \{\mathbb{R}^{i_0} \mid i_0 \in \mathbb{N}_0\}$$

followed by

$$V(1) := \left\{(\mathbb{R}^{i_1}, \mathrm{Hom}(\mathbb{R}^{i_0}, \mathbb{R}^{j_0})) \;\middle|\; \begin{matrix} 0 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{matrix} \right\}$$

and

$$V(2) := \left\{(\mathbb{R}^{i_2}, \mathrm{Hom}(\mathbb{R}^{i_1}, \mathbb{R}^{j_1}), \mathrm{Hom}(\mathbb{R}^{i_0}, \mathbb{R}^{j_0})) \;\middle|\; \begin{matrix} 0 \leq i_2 < i_1 - j_1, \\ 0 \leq j_1 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{matrix} \right\}$$

and we set in general for $n \geq \ell \geq 1$

$$V(\ell) := \left\{ (\mathbb{R}^{i_\ell}, \mathrm{Hom}(\mathbb{R}^{i_{\ell-1}}, \mathbb{R}^{j_{\ell-1}}), \dots, \mathrm{Hom}(\mathbb{R}^{i_0}, \mathbb{R}^{j_0})) \;\middle|\; \begin{array}{l} 0 \leq i_\ell < i_{\ell-1} - j_{\ell-1} \\ 0 \leq j_{\ell-1} \leq i_{\ell-1} < i_{\ell-2} - j_{\ell-2} \\ \qquad \vdots \\ 0 \leq j_1 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{array} \right\}.$$

Before we define the source and target functions, let us introduce the elements of the second n-globular set $W = \{W(\ell) \mid 0 \leq \ell \leq n\}$. For $k \in \mathbb{N}_0$, abbreviate $\mathbb{N}_0^k := (\mathbb{N}_0)^k$ and define

$$W(0) := \mathbb{N}_0$$

and

$$W(1) := \left\{ \left(i_1, \begin{bmatrix} i_0 \\ j_0 \end{bmatrix} \right) \in \mathbb{N}_0 \times \mathbb{N}_0^2 \;\middle|\; \begin{array}{l} 0 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{array} \right\}$$

and

$$W(2) := \left\{ \left(i_2, \begin{bmatrix} i_1, i_0 \\ j_1, j_0 \end{bmatrix} \right) \in \mathbb{N}_0 \times (\mathbb{N}_0^2)^2 \;\middle|\; \begin{array}{l} 0 \leq i_2 < i_1 - j_1, \\ 0 \leq j_1 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{array} \right\}$$

and generally for $n \geq \ell \geq 1$

$$W(\ell) := \left\{ \left(i_\ell, \begin{bmatrix} i_{\ell-1}, \dots, i_0 \\ j_{\ell-1}, \dots, j_0 \end{bmatrix} \right) \in \mathbb{N}_0 \times (\mathbb{N}_0^2)^\ell \;\middle|\; \begin{array}{l} 0 \leq i_\ell < i_{\ell-1} - j_{\ell-1} \\ 0 \leq j_{\ell-1} \leq i_{\ell-1} < i_{\ell-2} - j_{\ell-2} \\ \qquad \vdots \\ 0 \leq j_1 \leq i_1 < i_0 - j_0, \\ 0 \leq j_0 \leq i_0 \end{array} \right\}.$$

The reason why we briefly interrupt the construction of $\mathcal{V}$ and $\mathcal{W}$ is the following simplifying observation.

Remark 3.9 (Hohloch [**Hoh6**]). *Although V and W are clearly different sets, we can abbreviate elements in $V(\ell)$ by means of elements in $W(\ell)$ via identifying*

$$(\mathbb{R}^{i_\ell}, \mathrm{Hom}(\mathbb{R}^{i_{\ell-1}}, \mathbb{R}^{j_{\ell-1}}), \dots, \mathrm{Hom}(\mathbb{R}^{i_0}, \mathbb{R}^{j_0})) \mathrel{\widehat{=}} \left(i_\ell, \begin{bmatrix} i_{\ell-1}, \dots, i_0 \\ j_{\ell-1}, \dots, j_0 \end{bmatrix} \right)$$

which simplifies the notation considerably. Since the dimensions of the vector spaces in V satisfy the same constraints as the integers in W we can even use this short notation in proofs, keeping in mind the different canonical isomorphims of V and W.

Using this simplifying notion, the source and target functions

$$s : V(\ell) \to V(\ell - 1) \quad \text{and} \quad t : V(\ell) \to V(\ell - 1)$$

and

$$s : W(\ell) \to W(\ell - 1) \quad \text{and} \quad t : W(\ell) \to W(\ell - 1)$$

are given by

$$s\left(i_\ell, \begin{bmatrix} i_{\ell-1}, \dots, i_0 \\ j_{\ell-1}, \dots, j_0 \end{bmatrix}\right) := \left(i_{\ell-1}, \begin{bmatrix} i_{\ell-2}, \dots, i_0 \\ j_{\ell-2}, \dots, j_0 \end{bmatrix}\right),$$
$$t\left(i_\ell, \begin{bmatrix} i_{\ell-1}, \dots, i_0 \\ j_{\ell-1}, \dots, j_0 \end{bmatrix}\right) := \left(j_{\ell-1}, \begin{bmatrix} i_{\ell-2}, \dots, i_0 \\ j_{\ell-2}, \dots, j_0 \end{bmatrix}\right)$$

for $\ell > 1$ and

$$s\left(i_1, \begin{bmatrix} i_0 \\ j_0 \end{bmatrix}\right) := i_0, \qquad t\left(i_1, \begin{bmatrix} i_0 \\ j_0 \end{bmatrix}\right) := j_0$$

for $\ell = 1$.

LEMMA 3.10 (Hohloch [**Hoh6**]). *With the above defined s and t as source and target functions, V and W are n-globular sets.*

The remaining construction of the n-categories $\mathcal{V}$ and $\mathcal{W}$ will be done only for $\mathcal{W}$ since it carries over to $\mathcal{V}$ immediately by Remark 3.9. We define the identity functions $\mathbf{1} : W(l) \to W(l+1)$ by setting on $W(0)$

$$\mathbf{1}(i_0) := \left(0, \begin{bmatrix} i_0 \\ i_0 \end{bmatrix}\right)$$

and on $W(l)$ with $l > 0$

$$\mathbf{1}\left(i_l, \begin{bmatrix} i_{l-1}, \dots, i_0 \\ j_{l-1}, \dots, j_0 \end{bmatrix}\right) := \left(0, \begin{bmatrix} i_l, i_{l-1} \dots, i_0 \\ i_l, j_{l-1}, \dots, j_0 \end{bmatrix}\right).$$

Now we get to the composite. A tuple $(R_l, Q_l) \in W(l) \times_p W(l)$ is of the form

$$Q_l = \left(i_l, \begin{bmatrix} i_{l-1}, \dots, i_{p+1}, u_p, \rho_{p-1}, \dots, \rho_0 \\ j_{l-1}, \dots, j_{p+1}, v_p, \sigma_{p-1}, \dots, \sigma_0 \end{bmatrix}\right),$$
$$R_l = \left(\mu_l, \begin{bmatrix} \mu_{l-1}, \dots, \mu_{p+1}, v_p, \rho_{p-1}, \dots, \rho_0 \\ \nu_{l-1}, \dots, \nu_{p+1}, w_p, \sigma_{p-1}, \dots, \sigma_0 \end{bmatrix}\right)$$

and we define

$$\begin{aligned} &R_l \circ_p Q_l \\ &= \left(\mu_l, \begin{bmatrix} \mu_{l-1}, \dots, \mu_{p+1}, v_p, \rho_{p-1}, \dots, \rho_0 \\ \nu_{l-1}, \dots, \nu_{p+1}, w_p, \sigma_{p-1}, \dots, \sigma_0 \end{bmatrix}\right) \circ_p \left(i_l, \begin{bmatrix} i_{l-1}, \dots, i_{p+1}, u_p, \rho_{p-1}, \dots, \rho_0 \\ j_{l-1}, \dots, j_{p+1}, v_p, \sigma_{p-1}, \dots, \sigma_0 \end{bmatrix}\right) \\ &:= \left((i_l + \mu_l), \begin{bmatrix} (i_{l-1} + \mu_{l-1}), \dots, (i_{p+1} + \mu_{p+1}), u_p, \rho_{p-1}, \dots, \rho_0 \\ (j_{l-1} + \nu_{l-1}), \dots, (j_{p+1} + \nu_{p+1}), w_p, \sigma_{p-1}, \dots, \sigma_0 \end{bmatrix}\right) \end{aligned}$$

THEOREM 3.11 (Hohloch [**Hoh6**]). *The n-globular sets $V = \{V(l) \mid 0 \leq l \leq n\}$ and $W = \{W(l) \mid 0 \leq l \leq n\}$ together with the above defined identity functions $\mathbf{1}$ and the composites $\circ_p$ yield two almost strict n-category $\mathcal{V}$ and $\mathcal{W}$.*

5.2. The functors $\mathcal{F} : \mathcal{X} \to \mathcal{V}$ and $\mathcal{G} : \mathcal{X} \to \mathcal{W}$. We assume the setting of the previous section when we now define the n-functors $\mathcal{F} : \mathcal{X} \to \mathcal{V}$ and $\mathcal{G} : \mathcal{X} \to \mathcal{W}$. Recall that $\mathrm{Ind}(x)$ is the Morse index of a critical point. Let us begin with

$$\mathcal{G} : X(0) = \mathrm{Crit}(f_0) \to W(0) = \mathbb{N}_0, \quad x_0 \mapsto \mathrm{Ind}(x_0)$$

and

$$\begin{aligned} \mathcal{G} : X(1) = \{(x_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)) \mid \dots\} &\to W(1) \\ (x_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)) \quad &\mapsto \quad \left(\mathrm{Ind}(x_1), \begin{bmatrix} \mathrm{Ind}(x_0) \\ \mathrm{Ind}(y_0) \end{bmatrix}\right) \end{aligned}$$

and generally for $1 \leq \ell \leq n$

$$\mathcal{G} : X(\ell) \to W(\ell)$$

$$\left(a_\ell, \widehat{\mathcal{M}}(a_{\ell-1}, b_{\ell-1}, f_{\ell-1\left[\substack{a_{\ell-2},\dots,a_0 \\ b_{\ell-2},\dots,b_0}\right]})\right) \mapsto \left(\mathrm{Ind}(a_\ell), \left[\substack{\mathrm{Ind}(a_{\ell-1}),\dots,\mathrm{Ind}(a_0) \\ \mathrm{Ind}(b_{\ell-1}),\dots,\mathrm{Ind}(b_0)}\right]\right).$$

The other n-functor is given by

$$\mathcal{F} : X(0) = \mathrm{Crit}(f_0) \to V(0) = \{\mathbb{R}^{i_0} \mid i_0 \in \mathbb{N}_0\}, \quad x_0 \mapsto \mathbb{R}^{\mathrm{Ind}(x_0)}$$

and

$$\begin{aligned} \mathcal{F} : X(1) = \{(x_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)) \mid \dots\} &\to V(1) = \{(\mathbb{R}^{i_1}, \mathrm{Hom}(\mathbb{R}^{i_0}, \mathbb{R}^{j_0})) \mid \dots\} \\ (x_1, \widehat{\mathcal{M}}(x_0, y_0, f_0)) \quad &\mapsto \quad (\mathbb{R}^{\mathrm{Ind}(x_1)}, \mathrm{Hom}(\mathbb{R}^{\mathrm{Ind}(x_0)}, \mathbb{R}^{\mathrm{Ind}(y_0)})) \end{aligned}$$

and generally for $1 \leq \ell \leq n$ using short notation

$$\mathcal{F} : X(\ell) \to V(\ell)$$

$$\left(a_\ell, \widehat{\mathcal{M}}(a_{\ell-1}, b_{\ell-1}, f_{\ell-1\left[\substack{a_{\ell-2},\dots,a_0 \\ b_{\ell-2},\dots,b_0}\right]})\right) \mapsto \left(\mathrm{Ind}(a_\ell), \left[\substack{\mathrm{Ind}(a_{\ell-1}),\dots,\mathrm{Ind}(a_0) \\ \mathrm{Ind}(b_{\ell-1}),\dots,\mathrm{Ind}(b_0)}\right]\right).$$

THEOREM 3.12. *$\mathcal{F}$ is an almost strict n-functor from $\mathcal{X}$ to $\mathcal{V}$ and $\mathcal{G}$ is an almost strict n-functor from $\mathcal{X}$ to $\mathcal{W}$.*

6. Examples

In Hohloch [**Hoh5**], we computed $\mathcal{X}$ for a choice of Morse data on the standard sphere and the deformed sphere. In Hohloch [**Hoh6**], we did it for the 2-torus (with Morse-Smale data) and we also calculated its image under $\mathcal{F}$ and $\mathcal{G}$. The calculation of $\mathcal{X}$ is in all cases lengthy, even for the pretty trivial case of the 2-sphere, such that we refer the interested reader to the articles Hohloch [**Hoh5, Hoh6**].

But we give an impression how $\mathcal{G}$ simplifies things by recalling the image of $\mathcal{G}$ for the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the flat metric and the Morse function $f_0(x, y) = \cos(2\pi x)+\cos(2\pi y)$ whose critical points are $\{(\frac{k}{2}, \frac{l}{2}) \mid k, l \in \mathbb{Z}\}$. We work on the fundamental domain $[0, 1]\times[0, 1]$ which has four critical points $w := w_0 = (0, 0) = (1, 0) = (0, 1) = (1, 1)$ and $x := x_0 = (\frac{1}{2}, 0) = (\frac{1}{2}, 1)$ and $y := y_0 = (0, \frac{1}{2}) = (1, \frac{1}{2})$ and $z := z_0 = (\frac{1}{2}, \frac{1}{2})$ as in Figure 3.3. We suppress the level indices in w_0, x_0, y_0, z_0 since it would complicate the notation.

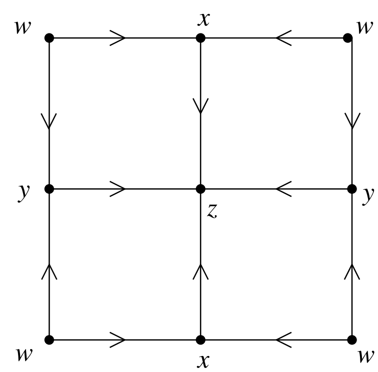


FIGURE 3.3. Morse trajectories on $\mathbb{T}^2$.

For $X(0) = \{w, x, y, z\}$, we obtain

$$\mathcal{G}(w) = 2, \quad \mathcal{G}(x) = 1, \quad \mathcal{G}(y) = 1, \quad \mathcal{G}(z) = 0.$$

which are the indices of the critical points. Next we get

$$\mathcal{G}(X(1)) = \left\{ \left(0 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right), \left(0 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right), \left(0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(0 \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right), \left(1 \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \right\}$$

and

$$\mathcal{G}(X(2)) = \left\{ = \left(0 \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\right), = \left(0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), = \left(0 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \right\}.$$

Thus, $\mathcal{G}$ displays the history of the indices of the critical points resp. the dimension of the involved moduli spaces.

7. Future projects

For the construction of the almost strict n-category in Theorem 3.8, we used Morse functions whose negative gradient flow always flows from higher to lower dimensional strata, but never back.

If we do allow flowing back into higher dimensional strata, we obtain breaking phenomena different from usual Morse theory: For example (cf. Figure 3.2), given critical points x and z in the interior of the manifold with Morse index $\mathrm{Ind}(x) = \mathrm{Ind}(z)+2$, the connecting Morse trajectories may break *twice* with intermediate critical points y_1, y_2 on the boundary whereas under the previous assumptions there would only be one intermediate critical point possible. Moreover, we cannot glue the trajectories from x to y_1 with the trajctories from y_1 to y_2 if we do not simultanously also glue with the trajectories from y_2 to z since *there is no trajectory from x to y_2*, cf. Figure 3.2. This phenomenon prevents us from obtaining an n-category — instead we get a so-called opetope (cf. for instance Kock & Joyal & Batanin & Mascari [**KoJBM**]) which allows the 'simultanous' composition of several 'arrows'.

'THEOREM' 3.13 (Hohloch & Ludwig [**HohL**]). *Under the above assumptions on the Morse function, the space of higher dimensional Morse moduli spaces carries the structure of opetopes.*

CHAPTER 4

Optimal transport and integer partitions

In this chapter, we outline a link between optimal transport and integer partitions. Both are classical topics which have already been of interest since the 18th century. Up to our knowledge, these topics have not been linked before.

1. Optimal transport

The motivation for optimal transport is very old: in 1781, Monge [**Mo**] raised the following question sketched in Figure 4.1: Consider two heaps of sand μ^- and μ^+ with same volume $\mathrm{vol}(\mu^-) = \mathrm{vol}(\mu^+)$. Is there a map $\varphi : \mu^- \to \mu^+$ mapping ('transporting') μ^- to μ^+ which minimizes the sum of the transport distances of the sand grains? And what happens if we consider the sand as 'continuous' matter instead of grains? What if we study not just a distance, but a more general 'cost function'?

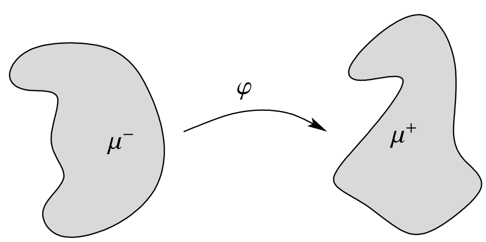


FIGURE 4.1. Transporting μ^- into μ^+.

In modern language, Monge proposes to study the following optimization problem. Consider a finite dimensional manifold M and its Borel σ-algebra $\mathfrak{S}(M)$ and denote by $\mathfrak{M}(M)$ the space of finite, positive Borel measures on M. For a measurable map $\psi : M \to M$ and $\mu \in \mathfrak{M}(M)$, the *image* or *push forward measure* $\psi(\mu)$ is given by $\psi(\mu(B)) := \mu(\psi^{-1}(B))$ for all measurable $B \subset M$.

PROBLEM 4.1 (Monge).

Given: *$\mu^-, \mu^+ \in \mathfrak{M}(M)$ with $\mu^-(M) = \mu^+(M)$ and a measurable '**cost function**' $c : M \times M \to \mathbb{R}^{\geq 0}$.*

Wanted: *A measurable '**optimal map**' $\varphi : M \to M$ which realizes the minimum of*

$$C(\mu^-, \mu^+) := \inf \left\{ \int_M c(x, \varphi(x)) d\mu^-(x) \,\middle|\, \varphi \text{ Borel, } \varphi(\mu^-) = \mu^+ \right\}.$$

For point measures $\mu^- = \sum_{i=1}^l \delta_{x_i}$ and $\mu^+ = \sum_{i=1}^l \delta_{y_i}$, the integral turns into the sum $\sum_{i=1}^l c(x_i, \varphi(x_i))$ which looks nice and easy to solve, but we have to be cautious: There may not exist an optimal map if μ^- is a point measure.

Example 4.2. *Let be x, y_1 and y_2 be distinct points on M. Then for $\mu^- = \delta_x$ and $\mu^+ = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$ there is no map φ with $\varphi(\mu^-) = \mu^+$ since φ would have to satisfy $\varphi(x) = y_1$* and *$\varphi(x) = y_2$.*

Monge's problem is hard to attack since it is for instance neither linear nor convex. In 1979, Sudakov [**Su**] published a proof on $\mathbb{R}^m$ with the euclidean distance as a cost function, but unfortunately his work contained a gap (cf. Ambrosio [**Am1**, p. 137], [**Am2**, chapt. 6]) which can only be fixed under stronger assumptions.

In the middle of the 20th century, another approach to Monge's problem was found by Kantorovich [**Ka1, Ka2**] which is much more accessible. Consider the projections $p_- : M \times M \to M$, $p_-(x, y) = x$ and $p_+ : M \times M \to M$, $p_+(x, y) = y$ and let μ^-, $\mu^+ \in \mathfrak{M}(M)$ with $\mu^-(M) = \mu^+(M)$. Now introduce the space

$$\mathfrak{M}(\mu^-, \mu^+) := \{\mu \in \mathfrak{M}(M \times M) \mid p_-(\mu) = \mu^- \text{ and } p_+(\mu) = \mu^+\}$$

on which we study the following problem.

Problem 4.3 (Kantorovich).

Given: *μ^-, $\mu^+ \in \mathfrak{M}(M)$ and a '**cost function**' $c : M \times M \to \mathbb{R}^{\geq 0}$.*

Wanted: *An '**optimal measure**' $\mu \in \mathfrak{M}(\mu_-, \mu_+)$ which realizes the minimum of*

$$K(\mu^-, \mu^+) := \inf \left\{ \int_{M \times M} c(x, y) d\mu(x, y) \,\middle|\, \mu \in \mathfrak{M}(\mu^-, \mu^+) \right\}.$$

Kantorovich's problem is much easier to handle since it is linear in μ and the space $\mathfrak{M}(\mu^-, \mu^+)$ is convex. Thus, under reasonable assumptions on μ^-, μ^+ and c, there exists always an optimal measure on M which follows from a standard compactness argument using the calculus of variations.

Convex problems can be dualized, i.e. Kantorovich's problem can be reformulated as

$$\min_{\mu\in\mathfrak{M}(m^-,\mu^+)} \int\limits_{M\times M} c(x,y)d\mu(x,y) = \sup\left\{\int\limits_M h_-(x)d\mu^-(x) + \int\limits_M h_+(y)d\mu^+(y)\right\}$$

where the supremum is taken over all $(h_-, h_+) \in L^1(\mu^-)\times L^1(\mu^+)$ with $h_-(x)+h_+(y) \leq c(x,y)$.

Are Monge's and Kantorovich's problems related? The answer is clearly yes. How are they linked? In case there is a measurable map $\varphi : M \to M$ with $\varphi(\mu^-) = \mu^+$, then set

$$\mathrm{Id}\times\varphi : M \to M \times M, \quad x \mapsto (x, \varphi(x)).$$

and note that $(\mathrm{Id}\times\varphi)(\mu^-) \in \mathfrak{M}(\mu^-, \mu^+)$ and that its support lies in the graph of φ. Now a calculation yields

$$\begin{aligned}\inf_{\varphi \text{ with } \varphi(\mu^-)=\mu^+} \int\limits_M c(x,\varphi(x))d\mu^-(x) &= \inf_{\varphi \text{ with } \varphi(\mu^-)=\mu^+} \int\limits_{M\times M} c(x,y)d(\mathrm{Id}\times\varphi)(\mu^-)(x,y)\\ &\geq \min_{\mu\in\mathfrak{M}(\mu^-,\mu^+)} \int\limits_{M\times M} c(x,y)d\mu(x,y),\end{aligned}$$

i.e. Kantorovich's problem is a lower bound for Monge's problem.

Gangbo & McCann [**GaM**] studied and solved Kantorivich's and Monge's problem for convex and concave cost functions. Under some natural assumptions, they find a (unique) optimal $\mu \in \mathfrak{M}(\mu^-, \mu^+)$ for Kantorovich's problem which is in fact of the form $(\mathrm{Id}\times\varphi)(\mu^-)$ such that they also obtain an optimal map for Monge's problem. This optimal map is even given by an explicit formula. In case of strictly concave cost functions, the cost function induces a metric such that a minimal measure does not 'move' the intersection set of the support of μ^- and μ^+.

By now there exist several overviews on mass transportation problems like the books by Villani [**Vi**] or Rachev & Rüschendorf [**RR**] where more details and additional references can be found.

2. Integer partitions

An 1**-dimensional (integer) partition** π of $n \in \mathbb{N}$ is an (ordered) tuple $\pi = (n_1, \ldots, n_{k(\pi)})$ with $n \geq n_1 \geq \cdots \geq n_{k(\pi)} \geq 1$ and $\sum_{i=1}^{k(\pi)} n_i = n$. We call $\mathscr{P}(n) := \mathscr{P}_1(n)$ the set of 1-dimensional partitions of n and $p(n) := p_1(n)$ its cardinality. 1-dimensional partitions are usually just called partitions.

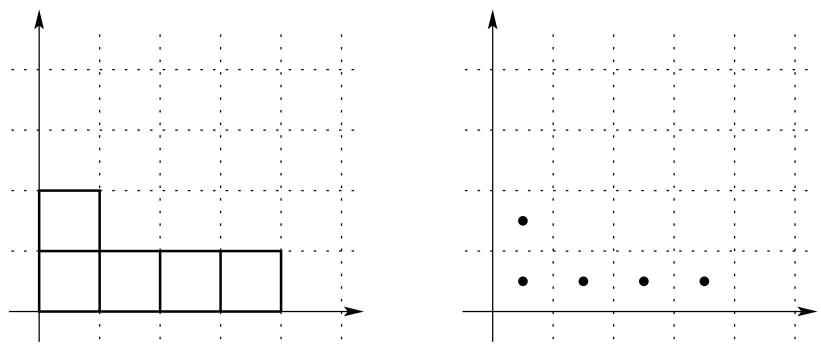
For example, the number 3 can be written as

$$3 = 2 + 1 = 1 + 1 + 1$$

and thus has 3 partitions, namely

$$\mathscr{P}(3) = \{(3), (2, 1), (1, 1, 1)\}.$$

There are also several ways to draw partitions. Usually Young tableaux and Ferrer graphs are used as sketched in Figure 4.2 where a given partition $\pi = (n_1, \ldots, n_{k(\pi)})$ is displayed by $k(\pi)$ columns of height n_i resp. by dots over the positive real axis. This suggests the name 'one dimensional' partitions.

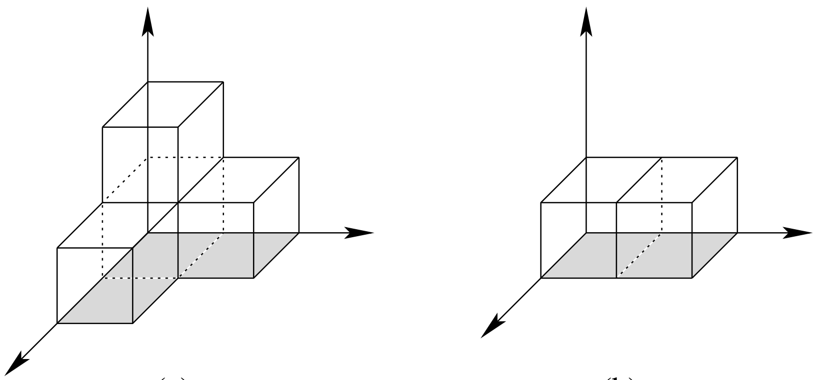


(a) (b)

FIGURE 4.2. (a) Young tableau and (b) Ferrer graph of $(2, 1, 1, 1) \in \mathscr{P}(5)$.

Consequently, **two dimensional partitions** of an integer n are characterized by Ferrer graphs or Young tableaux over the two dimensional plane $\mathbb{R}^2$. More precisely, let $n \in \mathbb{N}$. A **two dimensional** or **plane partition** of n is an array consisting of $n_{ij} \in \mathbb{N}$ where $1 \le i \le k$ and $1 \le j \le l$ for some integers $1 \le k, l \le n$ such that $n_{1j} \ge \cdots \ge n_{kj}$ for all j and $n_{i1} \ge \cdots \ge n_{il}$ for all i and $\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} = n$. $\mathscr{P}_2(n)$ denotes the **set of two dimensional partitions** and $p_2(n) := |\mathscr{P}_2(n)|$ denotes its cardinality.
Figure 4.3 (a) shows the Young tableau of $\left[\begin{smallmatrix} 1 & \\ 2 & 1 \end{smallmatrix}\right] \in \mathscr{P}_2(4)$ and Figure 4.3 (b) shows the Young tableau of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \in \mathscr{P}_2(2)$.

This formulation is easily generalized to m dimensions: the set of m-dimensional partitions is called $\mathscr{P}_m(n)$ with $n \in \mathbb{N}$ and defined as follows. We abbreviate multi-indices like $\begin{pmatrix} 1 \le i_1 \le k_1 \\ \vdots \\ 1 \le i_m \le k_m \end{pmatrix}$ by $1 \le i_1, \ldots, i_m \le k_1, \ldots, k_m$. Let $n \in \mathbb{N}$. An **m-dimensional partition** of n is an array consisting of $n_{i_1 \ldots i_m} \in \mathbb{N}$ where $1 \le i_1, \ldots, i_m \le k_1, \ldots, k_m$ for some integers $1 \le k_1, \ldots, k_m \le n$ such that for each index $i_j = 1, \ldots, k_j$ with $1 \le j \le m$ the integers



(a) (b)

FIGURE 4.3. (a) and (b) display Young tableaux of 2-dimensional partitions.

$n_{i_1 \dots i_m}$ are a monotone decreasing sequence with $n \geq \max_{i_j \in \{1,\dots,k_j\}} n_{i_1,\dots,i_m}$ and $\min_{i_j \in \{1,\dots,k_j\}} n_{i_1,\dots,i_m} \geq 1$ and $\sum_{i_1=1}^{k_1} \cdots \sum_{i_m=1}^{k_m} n_{i_1 \dots i_m} = n$. $\mathscr{P}_m(n)$ denotes the **set of m-dimensional partitions** and $p_m(n) := |\mathscr{P}_m(n)|$ denotes its cardinality.

One classical method for the study of integer partitions are generating functions. Euler used them to great success. For example, he proved that the elements of the sequence $(p(n))_{n \in \mathbb{N}}$ are the coefficients of the expansion

$$\prod_{i \geq 1} \frac{1}{1 - x^i} = \sum_{n \geq 0} p(n) x^n$$

which is called the *generating function* of $p(n)$. Moreover, let $\mathscr{P}(n \mid A) \subseteq \mathscr{P}(n)$ be the subset of partitions with property A and $p(n \mid A)$ its cardinality. Euler also showed that

$$p(n \mid \text{all } n_i \text{ odd}) = p(n \mid \text{all } n_i \text{ mutually distinct}). \tag{4.4}$$

For two-dimensional partitions (also called 'plane partitions'), the associated generating function is

$$\prod_{k \geq 1} \frac{1}{(1 - x^k)^k} = \sum_{n \geq 0} p_2(n) x^n$$

which was found by MacMahon.

For m-dimensional partitions with $m \geq 3$, no generating function is known which complicates their study considerably. Research on 1-dimensional partitions has been very popular during the last two centuries such that the literature is vast. Good overviews are the books Andrews [**An**] and Andrews & Eriksson [**AnE**]. 2-dimensional partitions have been less studied. Notably there is not much on higher dimensional partitions to be found. In the following section, we outline a new approach to integer partitions which may make higher dimensional partitions more accessible.

3. Optimal transport and integer partitions

The link between optimal transport and integer partitions is motivated by interpreting m-dimensional partitions as *measures* in $\mathbb{R}^{m+1}$: Let us demonstrate this for 1-dimensional partitions. The idea is astonishingly simple: just consider (a translation of) the Ferrer graph of a partition as a sum of point measures associated to the given partition. More precisely, we identify $\pi = (n_1, \dots, n_{k(\pi)}) \in \mathscr{P}(n)$ with the measure

$$\delta_\pi := \sum_{i=1}^{k} \sum_{\alpha=1}^{n_i} \delta_{(i,\alpha)}$$

where $\delta_{(i,\alpha)}$ denotes the point measure at $(i, \alpha) \in \mathbb{R}^2$ with mass one. If continuous measures are preferred we can as well work with the Lebesgue measure restricted to the boxes of the according Young tableau.

Now comes another important observation: Given two partitions $\pi^-, \pi^+ \in \mathscr{P}(n)$, we certainly have

$$\delta_{\pi^-}(\mathbb{R}^2) = n = \delta_{\pi^+}(\mathbb{R}^2) \tag{4.5}$$

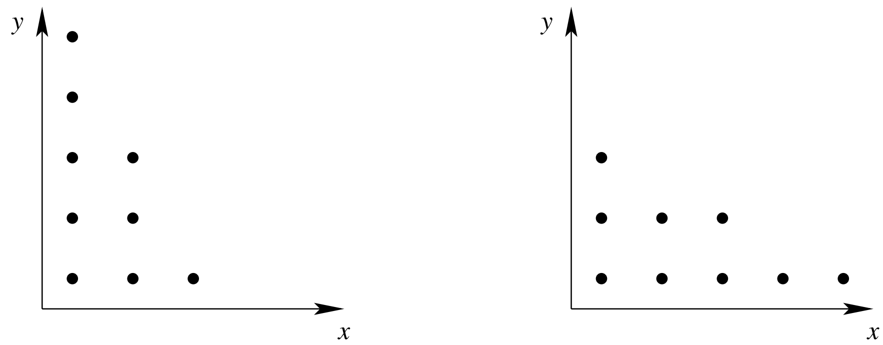
since their supports consist of exactly n distinct points. This means that $\mu^- := \delta_{\pi^-}$ and $\mu^+ := \delta_{\pi^+}$ fulfill the setting of Monge's problem. Thus we may look for a map φ 'transforming' π^- into π^+ in an optimal way w.r.t. a given cost function c, i.e. we look for φ with $\varphi(\delta_{\pi^-}) = \delta_{\pi^+}$ and $\int_{\mathrm{spt}(\delta_{\pi^-})} c((x, y), \varphi(x, y)) d\delta_{\pi^-}(x, y)$ minimal. Since the support of the involved measures is finite there is always a map realizing the minimum.

This works analogously for higher dimensional partitions. Just identify the higher dimensional Ferrer graph or young tableau with a measure. Equation (4.5) holds as well such that Monge's setting applies. For more details see Hohloch [**Hoh8**].

Up to our knowledge, we are the first to consider partitions from this point of view.

4. The main results

There are some types of partitions which invite the application of optimal transport in particular. For instance, denote by $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (y, x)$ the reflection on the diagonal $x = y$. Given $\pi \in \mathscr{P}_1(n)$ with Ferrer graph $\Gamma(\pi)$, **the symmetric partition** $sym(\pi) \in \mathscr{P}_1(n)$ of π is the partition with Ferrer graph $\Gamma(sym(\pi)) = T(\Gamma(\pi))$ as sketched in Figure 4.4. Partitions with $sym(\pi) = \pi$ are called **self-symmetric**.



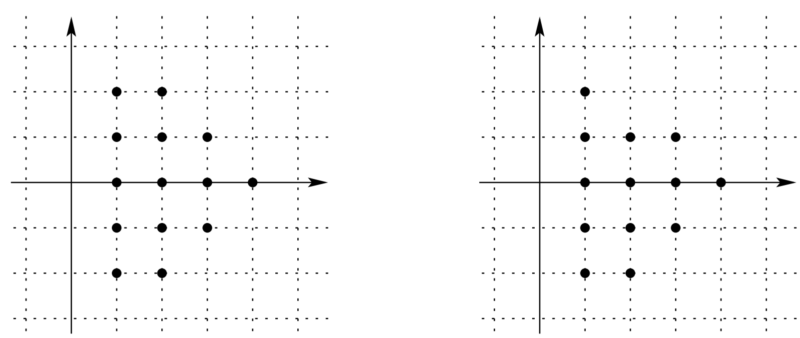
(a) (b)

FIGURE 4.4. (a) $\pi \in \mathscr{P}(9)$ and (b) its symmetric partition $sym(\pi)$.

We call a cost function c **metric-like** if c has the properties of a metric.

THEOREM 4.6 (Hohloch [**Hoh8**]). *1) Let $\pi \in \mathscr{P}_1(n)$ and let c be the euclidean distance. Then f given by $f = \mathrm{Id}$ on $\mathrm{spt}(\delta_\pi) \cap \mathrm{spt}(\delta_{sym(\pi)})$ and $f = T$ elsewhere attains the minimum for $C(\delta_\pi, \delta_{sym(\pi)})$.*
2) $\pi \in \mathscr{P}_1(n)$ is self-symmetric if and only if $C(\delta_\pi, \delta_{sym(\pi)}) = 0$ for metric-like cost functions.

We can also characterize nicely Euler's identity described in (4.4). For this we associate a new measure $\hat{\delta}_\pi$ to a partition π by centering the dot columns of its Ferrer graph on the x-axis as sketched in Figure 4.5.



(a) (b)

FIGURE 4.5. (a) The support of $\hat{\delta}_\pi$ for $\pi = (5, 5, 3, 1)$. (b) The support of $\hat{\delta}_\pi$ for $\pi = (5, 4, 3, 1)$.

Proposition 4.7 (Hohloch [**Hoh8**]). *Let* $S : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, -y)$ *be the reflection on the x-axis and let c be a metric-like cost function. Then*

$$\pi \in \mathscr{P}_1(n \mid \textit{all } n_i \textit{ odd}) \quad \Longleftrightarrow \quad S(\hat{\delta}_\pi) = \hat{\delta}_\pi \quad \Longleftrightarrow \quad C(\hat{\delta}_\pi, S(\hat{\delta}_\pi)) = 0.$$

In order to approach the other half of Euler's identity, we introduce another measure. Let $Perm(k)$ be the permutation group of the set $\{1, \dots, k\}$. For $\pi = (n_1, \dots, n_k) \in \mathscr{P}_1(n)$ and $\sigma \in Perm(k)$, we set

$$\delta_\pi^\sigma := \sum_{i=1}^{k} \sum_{\alpha=1}^{n_i} \delta_{(\sigma(i),\alpha)}.$$

For $\sigma = \mathrm{Id}$, we have $\delta_\pi^{\mathrm{Id}} = \delta_\pi$.

Proposition 4.8 (Hohloch [**Hoh8**]). *Let c be a metric-like cost function. Then*

(1) $\pi \in \mathscr{P}_1(n \mid \textit{not all } n_i \textit{ distinct})$.
$\Leftrightarrow$ *There is* $\sigma \in Perm(k(\pi)) \setminus \{\mathrm{Id}\}$ *with* $\delta_\pi = \delta_\pi^\sigma$.
$\Leftrightarrow$ *There is* $\sigma \in Perm(k(\pi))) \setminus \{\mathrm{Id}\}$ *with* $C(\delta_\pi, \delta_\pi^\sigma) = 0$.
(2) $\pi \in \mathscr{P}_1(n \mid \textit{all } n_i \textit{ distinct})$.
For all $\sigma \in Perm(k(\pi)) \setminus \{\mathrm{Id}\}$ *holds* $\delta_\pi \neq \delta_\pi^\sigma$.
For all $\sigma \in Perm(k(\pi)) \setminus \{\mathrm{Id}\}$ *holds* $C(\delta_\pi, \delta_\pi^\sigma) \neq 0$.

Andrews & Eriksson [**AnE**] prove Euler's identity by means of an explicit algorithm which turns a partition with distinct n_i into a partition with only odd n_i. One can find a cost function for which this algorithm is optimal. More precisely, one can reformulate the algorithm as a bijection φ between $\mathscr{P}_1(n \mid \textit{all } n_i \textit{ distinct}) =: \mathcal{D}$ and the slightly generalized space $\mathscr{P}_1^{perm}(n \mid \textit{all } n_i \textit{ odd}) =: \mathcal{O}$ for whose exact definition we refer to Hohloch [**Hoh8**].

Theorem 4.9 (Hohloch [**Hoh8**]). *Denote by* $\mathscr{F}(\mathcal{D}, \mathcal{O})$ *the space of maps from* $\mathcal{D}$ *to* $\mathcal{O}$. *Then there is a cost function* $\mathscr{C} : \mathcal{D} \times \mathcal{O} \to \mathbb{R}_+$ *such that* $\mathscr{C}(\pi, \varphi(\pi)) = \min\{C(\pi, \psi(\pi)) \mid \psi \in \mathscr{F}(\mathcal{D}, \mathcal{O})\}$.

5. Future projects

We believe that the importance of the approach to partitions via optimal transportation lies in its independence of the dimension in contrast to the generating function method which only works for two and three dimensional partitions.

Recall that a partition $\pi \in \mathscr{P}_\ell(n)$ is a measure δ_π on $\mathbb{R}^{\ell+1}$ given by $\delta_\pi = \sum_{i_1=1}^{k_1} \cdots \sum_{i_\ell}^{k+\ell} \sum_{\alpha=1}^{n_{i_1\dots i_\ell}} \delta_{(i_1,\dots,i_\ell,\alpha)}$ where the $n_{i_1\dots i_\ell}$ are monotone decreasing in each coordinate as explained in detail in Hohloch [**Hoh8**]. We can study optimal transport for two partitions $\pi^-, \pi^+ \in \mathscr{P}_\ell(n)$ by setting

$$\mu^- := \delta_{\pi^-} \quad \text{and} \quad \mu^+ := \delta_{\pi^+}$$

and look for a map φ with $\varphi(\delta_{\pi^-}) = \delta_{\pi^+}$ sending π^- to π^+ in an optimal way, i.e. minimizing $\int_{\mathrm{spt}(\delta_{\pi^-})} c(z, \varphi(z)) d\delta_{\pi^-}(z)$.

An immediate application should be the study of 'generalized' symmetric partitions: Let $\sigma \in Perm(k+1)$ and $T_\sigma : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be linear with matrix $T_\sigma = (e_{\sigma(1)}, \ldots, e_{\sigma(k+1)})$ w.r.t. the standard basis $e_1, \ldots, e_{k+1}$ of $\mathbb{R}^{k+1}$. The **σ-symmetric partition** of $\pi \in \mathscr{P}_k(n)$ is the partition $sym_\sigma(\pi)$ with Ferrer graph $\Gamma(sym_\sigma(\pi)) = T_\sigma(\Gamma(\pi))$. We call $\pi \in \mathscr{P}_k(n)$ with $\pi = sym_\sigma(\pi)$ **σ-selfsymmetric**.

Conjecture 4.10. *The map T_σ induces an optimal transport map for π and $sym_\sigma(\pi)$ with the euclidean distance as a cost function. Moreover, $\pi \in \mathscr{P}_k(n)$ is σ-selfsymmetric if and only if $C(\delta_\pi, \delta_{sym_\sigma(\pi)}) = 0$ where c is a metric-like cost function.*

But there should be many more results to discover about non σ-symmetric partitions.

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