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TRANSPORT, FLUX AND GROWTH OF HOMOCLINIC FLOER HOMOLOGY

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Abstract. We point out an interesting relation between transport in Hamiltonian dynamics and Floer homology. We generalize homoclinic Floer homology from $\mathbb{R}^2$ and closed surfaces to two-dimensional cylinders. The relative symplectic action of two homoclinic points is identified with the flux through a turnstile (as defined in MacKay & Meiss & Percival [19]) and Mather's [20] difference in action $\triangle W$. The Floer boundary operator is shown to annihilate turnstiles and we prove that the rank of certain filtered homology groups and the flux grow linearly with the number of iterations of the underlying symplectomorphism.

1. Introduction.

Before we approach the actual topic of this article, let us fix some notation.

Symplectic dynamics. A smooth manifold M is *symplectic* if there exists a non-degenerate, closed 2-form. Such a 2-form is called a *symplectic form* and we will usually denote it by ω or Ω. Note that symplectic manifolds are always even dimensional. On a symplectic manifold (M,ω), $f \in \mathrm{Diff}(M)$ is called *symplectic* or a *symplectomorphism* if it preserves ω, i.e. $f^*\omega = \omega$. The group of symplectomorphisms of (M,ω) is denoted by $\mathrm{Symp}(M,\omega)$ which we often shorten to $\mathrm{Symp}(M)$. A submanifold $L \subset M$ is *Lagrangian* if $\dim L = \frac{1}{2}\dim M$ and $\omega|_L = 0$. Given a smooth function $H : M \times S^1 \to \mathbb{R}$, we set $H_t := H(\cdot, t)$ and define its (non-autonomous) *Hamiltonian vector field* X_t^H via $\omega(X_t^H, \cdot) = -dH_t(\cdot)$. The (non-autonomous) Hamiltonian equation is given by $\dot z(t) = X_t^H(z(t))$ and the associated flow is called the Hamiltonian flow. A *Hamiltonian diffeomorphism* is a symplectomorphism which can be written as the time-1 map φ_1 of a Hamiltonian flow φ_t. We denote by $\mathrm{Ham}^c(M,\omega)$ the group of compactly supported Hamiltonian diffeomorphisms and shorten it to $\mathrm{Ham}^c(M)$ if convenient. A Hamiltonian diffeomorphism is *non-degenerate* if its graph intersects the diagonal in $M \times M$ transversely.

The fixed point set of a diffeomorphism f is denoted by $\mathrm{Fix}(f) := \{x \in M \mid f(x) = x\}$. A periodic point of f with period k is a fixed point of f^k. A fixed point $x \in \mathrm{Fix}(f)$ is *hyperbolic* if the modulus of all eigenvalues of $Df|_x$ is different from 1. Given a hyperbolic fixed point x, its *stable manifold* $W^s(f,x) :=$

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$\{p \in M \mid \lim_{n\to+\infty} f^n(p) = x\}$ and *unstable manifold* $W^u(f,x) := \{p \in M \mid \lim_{n\to-\infty} f^n(p) = x\}$ are injectively immersed submanifolds. The set of intersection points $\mathcal{H}(f,x) := W^s(f,x) \cap W^u(f,x)$ is called the set of *homoclinic points*. Note that if f is a symplectomorphism then the stable and unstable manifolds are Lagrangian submanifolds.

Homoclinic points and Floer theory. Homoclinic points were discovered by Poincaré [24], [25] around 1889 when he studied the n-body problem. But it took quite some time until first properties of homoclinic points and their impact on dynamical systems were sucessfully investigated: In 1935, Birkhoff [2] proved the existence of periodic points with high periods near homoclinic ones. And in 1963, Smale [35], [36] described the dynamics of homoclinic points by his horseshoe formalism. Since then, homoclinic points have been studied under different view points like approximation methods via integrable systems, genericity questions, calculus of variations etc. But many questions about homoclinic points are still open.

In the 1960s, V.I. Arnold conjectured that, on a closed symplectic manifold (M,ω), the number of fixed points of a non-degenerate Hamiltonian diffeomorphism is greater or equal to the sum over the Betti numbers of M.

For the $2n$-torus, the conjecture was proven in 1983 by Conley & Zehnder [4], but their methods do not generalize. In the late 1980's, Floer [8], [9], [10] devised a different approach. Roughly, Floer did the following: He identified the fixed points of the Hamiltonian diffeomorphism with the intersection points of its graph with the diagonal in the symplectic manifold $(M\times M, \omega\oplus(-\omega))$. The graph and the diagonal are Lagrangian submanifolds. Thus Floer turned the fixed point problem into a so-called 'Lagrangian intersection problem'. Then he identified the intersection points with the critical points of the symplectic action functional $\mathscr{A}$. He considered the action functional $\mathscr{A}$ as a 'Morse function' and constructed for it some kind of 'infinite dimensional' Morse theory (the involved Fredholm analysis needs the intersecting submanifolds to be Lagrangian). The resulting 'Morse homology' is nowadays called *Floer homology*. Its 'generators' are the intersection points. Then Floer showed that this new homology can be identified with the actual Morse homology of M, which enabled him to prove the Arnold conjecture (under certain conditions on M). A good, short introduction to Floer theory is Salamon [33].

Apart from leading to a proof of the Arnold conjecture, Floer's ideas and methods found many other applications in symplectic geometry and dynamical systems such that Floer theory is vividly studied nowadays.

The present paper links homoclinic points and Floer theory. The construction of 'homoclinic Floer homology' was motivated by the fact that the (un)stable manifolds of a hyperbolic fixed point of a symplectomorphism are Lagrangian submanifolds. Thus homoclinic points are associated to a 'Lagrangian intersection problem' and one may hope to adapt the techniques used in Floer theory to the intersection problem of stable and unstable manifolds. In the earlier work Hohloch [16], the existence of 'homoclinic Floer homology' was established on $\mathbb{R}^2$ and on closed surfaces. In this paper, we will generalize homoclinic Floer homology to (infinite) two-dimensional cylinders. But the main part of this paper is dedicated to dynamical interpretations and applications of 'homoclinic Floer theory'.

Homoclinic Floer theory. Omitting assumptions and details, the basic idea of homoclinic Floer homology is outlined in the following paragraph. The details and exact definitions are given in Section 2 and in the earlier work Hohloch [16].

For simplicity, consider a symplectomorphism f in the symplectic plane $(\mathbb{R}^2, \omega)$ with hyperbolic fixed point x. Then $W^s := W^s(f,x)$ and $W^u := W^u(f,x)$ are one-dimensional submanifolds. Assume that the associated set of homoclinic points $\mathcal{H} := W^s \cap W^u$ is not empty. Given $p,\ q \in \mathcal{H}$, we denote by $]p,q[_u \subset W^u$ resp. $]p,q[_s \subset W^s$ the unoriented, (un)stable segments between p and q. Moreover, $\mathbb{Z}$ acts on $\mathcal{H}$ via $\mathbb{Z} \times \mathcal{H} \to \mathcal{H}$, $(n,p) \mapsto f^n(p)$ and we denote the equivalence class of p by $\langle p \rangle$. Based on the 'winding number' of the 'loop' $]p,q[_u \cup\,]p,q[_s$, one can define a 'grading' of $\mathcal{H}$, i.e. a function $\mu : \mathcal{H} \to \mathbb{Z}$ with certain properties, which descends to the quotient $\mathcal{H}/\mathbb{Z}$.

To obtain a homology, we need a chain complex. Classical Floer theory suggests the set of intersection points $\mathcal{H}$ as generator set of the chain groups. However, $\mathcal{H}$ turned out to be 'too large' to yield a well-defined homology theory. But certain subsets of $\mathcal{H}$ can serve as generator sets for the chain groups. For simplicity, let us focus here on the subset of *primary points* $\mathcal{H}_{pr}$ which are the points in $p \in \mathcal{H} \backslash \{x\}$ which satisfy $]p,x[_s \cap\,]p,x[_u = \emptyset$. Denote by $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/\mathbb{Z}$ the set of equivalence classes of primary points. It turns out to be a finite set. The kth chain group is defined via

$$C_k(f,x) := \operatorname{Span}_{\mathbb{Z}}\{\langle p \rangle \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr},\ \mu(\langle p \rangle) = k\}.$$

The boundary operator $\partial : C_*(f,x) \to C_{*-1}(f,x)$ is defined in the following way: Given two primary points p and q with $\mu(q) = \mu(p) - 1$, consider the moduli space $\mathcal{M}(p,q)$ consisting of certain immersed 2-gons joining p to q whose boundaries lie in the stable resp. unstable manifold (the precise definition can be found after Figure 2). Denote by $m(p,q)$ the cardinality of $\mathcal{M}(p,q)$ up to orientation and vertex preserving diffeomorphisms and set $m(\langle p \rangle, \langle q \rangle) := \sum_{n \in \mathbb{Z}} m(p, f^n(q))$. Then we define

$$\partial \langle p \rangle := \sum_{\substack{\langle q \rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q \rangle) = \mu(\langle p \rangle) - 1}} m(\langle p \rangle, \langle q \rangle) \langle q \rangle$$

and prove $\partial \circ \partial = 0$. The associated homology of $(C_*(f,x), \partial)$ is given by $H_*(f,x) := \ker \partial_* / \operatorname{Im} \partial_{*+1}$ and is called *primary Floer homology.*

Primary Floer homology is defined on $\mathbb{R}^2$ and on closed, oriented surfaces with genus $g \geq 1$ as lined out in Hohloch [16] where also topological properties, like invariance under certain perturbations of the underlying symplectomorphism, are studied. But even on surfaces with genus, primary Floer homology always uses *contractible* homoclinic points as generators, i.e. homoclinic points p where the loop $]p,x[_s \cup\,]p,x[_u$ is contractible. But many physically important systems like the pendulum on the cylinder have noncontractible homoclinic points. For that reason, we will construct in this paper a version of homoclinic Floer homology on (infinite) 2-dimensional cylinders which is called *Cylinder Floer homology* and denoted by $\mathscr{H}_*(f,x)$. It admits noncontractible homoclinic points as generators and thus applies e.g. to the pendulum. The construction of $\mathscr{H}_*(f,x)$ is deduced from the construction of primary Floer homology, mainly by adjusting the proofs to the new generators. Cylinder Floer homology will be of interest in particular for the dynamical meaning of homoclinic Floer homology.

In general, by varying the generator set and the definition of the boundary operator, one can define several versions of homoclinic Floer homology — for an overview see Hohloch [17].

Dynamical properties of homoclinic Floer homology. In this paper, we focus on the dynamical properties of homoclinic Floer homology, i.e. its relation to other known dynamical invariants and the behaviour of $H_*(f,x)$ in comparison to $H_*(f^n,x)$ for $n \in \mathbb{N}$. First steps to investigate this behavour had already been taken in Hohloch [16].

MacKay & Meiss & Percival [19] study 'transport' in Hamiltonian systems which, in their terminology, is the motion of points under (many) iterations of the symplectomorphism. They investigate how and, in particular, *where* points are mapped from one region to another. A central notion in their paper is the *absolute flux* (briefly *flux*) of a symplectomorphism f w.r.t. a simply closed curve c. In $(\mathbb{R}^2,\omega)$, the flux is defined via $\mathcal{F}lux_f(c) = \mathrm{vol}_\omega(f(\mathrm{Int}(c)) \cap \mathrm{Ext}(c))$, i.e. the volume of the set of points which are swept out of the interior of the curve. For instance, an f-invariant curve γ satisfies $\mathcal{F}lux_f(\gamma) = 0$ and thus forms a 'complete barrier' for the transport of points under f, i.e. no points leave the interior of the curve. Partially invariant curves admit non-zero transport and are 'partial barriers' with a unique transport scheme as we will see in a moment. Note that the definition of the flux here is different from the flux homomorphism in symplectic geometry (cf. McDuff & Salamon [22], Polterovich [27]) which, roughly speaking, considers the difference between $f(\mathrm{Int}(c)) \cap \mathrm{Ext}(c)$ and $f(\mathrm{Ext}(c)) \cap \mathrm{Int}(c)$.

Partially invariant curves were of particular interest to MacKay & Meiss & Percival [19] as they noticed a unique behaviour concerning the flux. Let $f \in \mathrm{Symp}(\mathbb{R}^2,\omega)$ and let c be a simply closed curve in $(\mathbb{R}^2,\omega)$ which is 'partially invariant' under f: For simplicity, assume the picture in Figure 1 (a) where the range of the curve c and the range of $f \circ c$ coincide for a large part. Since f is volume preserving, the interiors of c and $f \circ c$ have the same volume. Thus we obtain a 'turnstile-like' shape for the ranges of c and $f \circ c$, i.e. (at least) one lobe 'sweeps points out' and (at least) one lobe 'sweeps points in'. The simplest scenario is sketched in Figure 1 (a) where the 'turnstile' consists of the 'pivot' q and the 'frame' p and $\tilde{p}$. The flux arises from the shaded region.

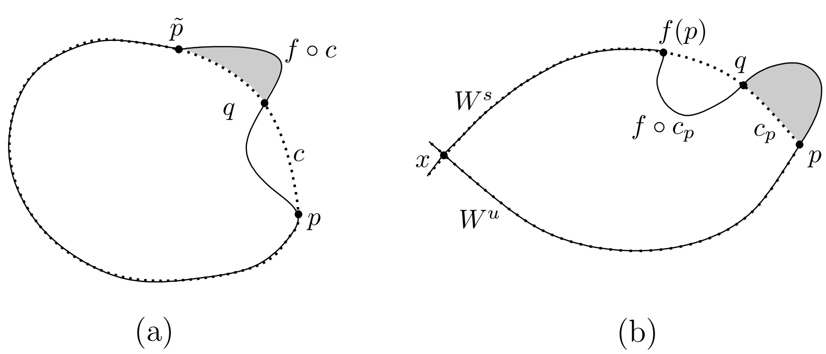


FIGURE 1. Flux and turnstile: (a) of a closed curve c in the plane, (b) of a homoclinic point p

MacKay & Meiss & Percival [19] also define the flux through periodic points, homoclinic points and cantori. For instance, given $f \in \mathrm{Symp}(\mathbb{R}^2, \omega)$ with hyperbolic fixed point x and homoclinic point $p \in W^s(f,x) \cap W^u(f,x)$, the flux $\mathcal{F}lux_f(p)$ of f through p is defined as the flux through a curve c_p which parametrizes the loop $]p,x[_s \cup]p,x[_u$. The associated curve c_p is partially invariant under f and the noninvariant part forms a 'turnstile' as displayed in Figure 1 (b). The shaded region yields $\mathcal{F}lux_f(p)$.

Whereas MacKay & Meiss & Percival [19] only consider turnstiles with one pivot, we are interested in more general turnstiles for homoclinic points. Depending on the intersection behaviour of the stable and unstable manifold between the 'frame' p and $f(p)$, we will distinguish later turnstiles in 'true', 'overtwisted' and 'generalized' turnstiles. The distinction in 'true', 'overtwisted' and 'generalized' turnstiles also corresponds to the different Reidemeister moves in the invariance proof of primary Floer homology in Hohloch [16]. If the stable and unstable manifold only intersect once between a primary point and its iterate we call the symplectomorphism *x-simple* (see Definition 5.5).

Denote the infinite symplectic cylinder by $(\mathcal{Z}, \Omega)$. By being related to the flux, turnstiles are linked to the dynamics of a symplectomorphism. But they also have an 'algebraic' meaning, more precisely, turnstiles show up in the boundary operator ∂ of primary Floer homology and cylinder Floer homology. In case of x-simple symplectomorphisms, turnstiles are annihilated by the boundary operator:

Proposition 1.1. *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple. Let $q \in W^s(\varphi,x) \cap W^u(\varphi,x)$ be primary and $\{p\} =]q,\varphi(q)[_s \cap]q,\varphi(q)[_u$ and w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $\langle p\rangle$ and $\langle q\rangle$ give rise to two distinct (families of) turnstiles, more precisely p is the pivot of a turnstile with frame q and $\varphi(q)$ and q is the pivot of a turnstile with frame $\varphi^{-1}(p)$ and p. The first turnstile enters the boundary operator via*

$$\partial\langle p\rangle = \langle q\rangle - \langle q\rangle + \sum_{\substack{\langle q\rangle \neq \langle \tilde q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle \tilde q\rangle) = \mu(\langle p\rangle)-1}} m(\langle p\rangle, \langle \tilde q\rangle)\langle \tilde q\rangle = \sum_{\substack{\langle q\rangle \neq \langle \tilde q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle \tilde q\rangle) = \mu(\langle p\rangle)-1}} m(\langle p\rangle, \langle \tilde q\rangle)\langle \tilde q\rangle.$$

If $\mathcal{M}(p,\tilde q) = \emptyset$ for all $\langle q\rangle \neq \langle \tilde q\rangle \in \tilde{\mathcal{H}}_{pr}$ the turnstile with pivot p lies in the kernel of the boundary operator, i.e. the pivot is a cycle.

As mentioned above, classical Floer theory can be seen as some kind of Morse theory with the symplectic action functional $\mathscr{A}$ as Morse function. Whereas the definition of the symplectic action functional $\mathcal{A}$ associated to homoclinic points on the cylinder involves some notation, it is easy to define it for homoclinic points in $(\mathbb{R}^2, \omega)$. Given a homoclinic point p of some $f \in \mathrm{Symp}(\mathbb{R}^2)$ we define its *symplectic action* $\mathcal{A}(p)$ as the (signed) symplectic area enclosed by the loop $[p,x]_u \cup [p,x]_s$. For two homoclinic points p and q, we define the *relative action* $\mathcal{A}(p,q) = \mathcal{A}(p) - \mathcal{A}(q)$. The action is compatible with the $\mathbb{Z}$-action on the set of homoclinic points, thus we can set $\mathcal{A}(\langle p\rangle) = \mathcal{A}(p)$.

MacKay & Meiss & Percival [19] established a relation between the flux and another action functional W under the following conditions: A *monotone twist map* on a cylinder or annulus with coordinates $(s,t) \in \mathbb{R} \times S^1$ is a volume preserving map f with $f(s,t) = (\tilde s, \tilde t)$ satisfying $\frac{\partial \tilde t}{\partial s} > 0$ for all s and t. Mather [20] studied the (non)existence of invariant circles for monotone twist maps using the calculus of variations with a certain action functional W. He denoted the difference in action between an action maximizing orbit and its associated minimax orbit by $\triangle W$. MacKay & Meiss & Percival [19] proved the flux to coincide with Mather's difference in action $\triangle W$. In our homoclinic situation, we can additionally identify the flux with the relative action:

Theorem 1.2. *Let $f \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple and, in addition, a monotone twist map. Consider a true turnstile with frame p, $f(p) \in \mathcal{H}_{pr}$ and pivot $q \in \mathcal{H}_{pr}$ and assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $v \in \mathcal{M}(p,q) \neq \emptyset$ and*

$$\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle p\rangle, \langle q\rangle) = \int_v \omega = \mathcal{F}lux_f(\langle p\rangle) = \triangle W_{p,q}.$$

For x-simple $f \in \mathrm{Symp}(\mathbb{R}^2)$ we have under analogous assumptions

$$\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle p\rangle, \langle q\rangle) = \int_v \omega = \mathcal{F}lux_f(\langle p\rangle).$$

An important feature in symplectic geometry and classical Floer theory is the action spectrum which consists of the action values of the critical points of the action functional i.e. the action values of the generators of the Floer complex. The boundary operator in classical Floer theory counts the cardinality of moduli spaces of so-called 'pseudo-holomorphic curves' between critical points. These 'pseudo-holomorphic curves' can be seen as (negative) gradient flow lines of the symplectic action functional. Thus the action value decreases along the boundary operator, i.e. if p is a critical point and ∂ the boundary operator, then the action of a critical point showing up in the expression ∂p is smaller than the action of p. Thus one can consider a *filtered* Floer complex, which is generated by critical points with action less than a certain value, without interfering with the definition of the boundary operator. This idea eventually leads to *filtered Floer homology* where the action of the generators lies in an interval. For applications of the action spectrum and filtered Floer homology see e.g. Schwarz [34].

This line of thoughts also goes through in our homoclinic situation, i.e. there is a *(homoclinic) action spectrum* $\mathrm{Spec}(f,x) := \{\mathcal{A}(\langle p\rangle) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and we set

$$\mathrm{gap}(f,x) := \min\{|\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle)| :\ \langle p\rangle \neq \langle q\rangle \in \tilde{\mathcal{H}}_{pr}\}.$$

For a chosen interval $]a,b]$ of action values, *filtered primary Floer homology* is denoted by $H_*^{]a,b]}(f,x)$ and *filtered cylinder Floer homology* by $\mathscr{H}_*^{]a,b]}(f,x)$. According to Theorem 1.2, the flux and $\triangle W$ are meaningful quantities for $\mathrm{Spec}(f,x) := \{\mathcal{A}(\langle p\rangle) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$. Therefore everything which is formulated in terms of the symplectic action spectrum can be interpreted in terms of the flux and $\triangle W$. This means that the algebraic notion of homology has a physical interpretation and measures dynamical quantities.

So far, we were considering relations of algebraic or topological nature between the flux, turnstiles, the action functional and homoclinic Floer homology. But actually the flux and the action filtration turn out to be crucial objects when studying homoclinic Floer homology under the viewpoint of dynamics. With 'dynamics', we mean the behaviour of the action, the flux, homoclinic Floer homology etc. under iteration of the underlying symplectomorphism, i.e. for instance the difference (if any) between $H_*(f,x)$ and $H_*(f^n,x)$.

The symplectic action of a homoclinic point is invariant under the underlying symplectomorphism. Thus we were able to define $\mathcal{A}(p) = \mathcal{A}(f(p)) =: \mathcal{A}(\langle p\rangle)$. And the same holds for the grading $\mu(p) = \mu(f(p)) =: \mu(\langle p \rangle)$. The invariance of the action and grading allows us to work on the quotient $\mathcal{H}_{pr}/\mathbb{Z}$, i.e. we 'divided' the system 'by the chaos' and discovered its 'order'. But if we actually want to investigate dynamical properties this turns out to be a big hindrance.

At this point, one should remark that, in classical Floer theory, the action and grading (more precisely the so-called *Maslov index*) are not invariant under iteration. Roughly, if γ is a 1-periodic Hamiltonian solution and if γ^k denotes its kth iteration then the classical action functional $\mathscr{A}$ and the so-called *mean Maslov index* $\mathfrak{m}$ transform (cf. Ginzburg & Gürel [15])

$$\mathscr{A}(\gamma^k) = k\mathscr{A}(\gamma) \qquad \text{and} \qquad \mathfrak{m}(\gamma^k) = k\mathfrak{m}(\gamma),$$

i.e. the classical action and mean Maslov index grow linearly. This phenomenon was sucessfully exploited by several authors. For example, Ginzburg [14] (and others) used the growth behaviour of $\mathscr{A}$ and $\mathfrak{m}$ to prove (versions of) the so-called Conley conjecture ('There are periodic Hamiltonian orbits of arbitrary large period'). Polterovich [28] also used growth properties of a certain action difference in order to establish growth results for symplectomorphisms. In a subsequent work, Polterovich [29] used his former results to formulate (and prove) a Hamiltonian version of the Zimmer program.

Since the (linear) growth of the action and index in classical Floer theory turned out to be an important tool, we are looking for an object which can replace the invariant action $\mathcal{A}$ and grading μ. And this place is taken by the flux: Given a symplectomorphism f with homoclinic point p, the flux transforms

$$\mathcal{F}lux_{f^n}(\langle p \rangle) = n\,\mathcal{F}lux_f(\langle p \rangle).$$

Thus the flux shows the same linear growth as the action and mean index of a 1-periodic Hamiltonian orbit in classical Floer theory. Growth can also be observed for the rank of filtered homoclinic Floer homology groups.

Theorem 1.3. *Let $f \in \mathrm{Symp}(\mathbb{R}^2)$ resp. $f \in \mathrm{Ham}^c(\mathcal{Z})$. Let $b \in \mathrm{Spec}(f,x)$ and $0 < \varepsilon \leq \frac{1}{2}\,\mathrm{gap}(f,x)$. Assume that there are k primary classes with action b. Then we obtain for the homoclinic Floer homology on $\mathbb{R}^2$ resp. $\mathcal{Z}$*

$$H_*^{]b-\varepsilon,b+\varepsilon]}(f,x) \simeq \mathbb{Z}^k \quad \textit{and} \quad H_*^{]b-\varepsilon,b+\varepsilon]}(f^n,x) \simeq (\mathbb{Z}^k)^n,$$
$$\mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(f,x) \simeq \mathbb{Z}^k \quad \textit{and} \quad \mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(f^n,x) \simeq (\mathbb{Z}^k)^n.$$

Thus the rank grows linearly with the number of iterations.

Note that for the non-filtered homoclinic Floer groups a priori only

$$\mathrm{rk}\,H_*(f,x) \leq \mathrm{rk}\,H_*(f^n,x) \quad \text{and} \quad \mathrm{rk}\,\mathscr{H}_*(f,x) \leq \mathrm{rk}\,\mathscr{H}_*(f^n,x)$$

holds (cf. Hohloch [16]) and, in many examples, we obtains equality of rank. Nevertheless, there are versions of homoclinic Floer homologies (cf. Hohloch [16], [17]) which easily show growth of rank for non-filtered Floer groups.

Organization of the paper. Section 2 recalls the concept of primary Floer homology from Hohloch [16]. Section 3 defines cylinder Floer homology, a version of primary Floer homology on the cylinder. Section 4 introduces the action functional and filterd Floer homology. Section 5 recalls certain facts and definitions from MacKay & Meiss & Percival [19] and Mather [20], [21] which are adjusted, interpreted and generalized to fit our setting. In Section 6, everything is put together and results about the growth behaviour of the flux and the rank of the Floer groups are proven. Section 7 calculates (filtered) cylinder Floer homology for the homoclinic tangle of Chirikov's Standard map.

2. **Primary Floer homology.** In this section, we briefly recall the construction of primary Floer homology from Hohloch [16]: We will introduce the Maslov index and homotopy classes for homoclinic points. Then we define certain di-gons, also known as 2-gons, lunes or half-moons (Chekanov [3], de Silva [5], Gautschi & Robbin & Salamon [11], Robbin [30]). They will be crucial for the definition of the boundary operator of the Floer chain complex. Then we will cover the main properties of primary points which will be the generators of the Floer complex. We will quote the main results of the 'cutting and gluing' procedure on which the well-definedness of the boundary operator is based. Signs will be introduced which are needed for the definition of the boundary operator and we will sketch their compatibility with the so-called 'cutting and gluing' procedure. Finally after these preparations, we proceed to the definition of primary Floer homology and its invariance properties.

Maslov index and homotopy class. Let us consider $\mathbb{R}^{2n}$ with the local coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ and symplectic form $\omega_0 := \sum_{i=1}^n dx_i \wedge dy_i$. The *symplectic complement* of a subspace $V \subset \mathbb{R}^{2n}$ w.r.t. ω_0 is given by $V^{\omega_0} := \{\zeta \in V \mid \omega_0(\zeta, \tilde{\zeta}) = 0 \ \ \forall\ \tilde{\zeta} \in V\}$. A subspace $V \subset \mathbb{R}^{2n}$ is called *Lagrangian* if $V = V^{\omega_0}$. Denote by $\mathcal{L}(n) := \{V \subset \mathbb{R}^{2n} \mid V \text{ Lagrangian}\}$ the space of Lagrangian subspaces w.r.t. ω_0. One can associate to $L \in \mathcal{L}(n)$ certain matrices $U := X + iY \in U(n)$ and define $\rho : \mathcal{L}(n) \to S^1$, $\rho(L) := \det(U \circ U)$ (cf. McDuff & Salamon [22]). For a loop of Lagrangian subspaces $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$, define the *Maslov index of loops of Lagrangian subspaces* by $\mu(\Lambda) := \deg(\rho \circ \Lambda)$ where deg denotes the mapping degree of $\rho \circ \Lambda : \mathbb{R}/\mathbb{Z} \to S^1$. If $\alpha : \mathbb{R} \to \mathbb{R}$ is a lift of $\rho \circ \Lambda$, i.e. $\det(X(t) + iY(t)) = e^{i\pi\alpha(t)}$, we obtain $\mu(\Lambda) = \alpha(1) - \alpha(0)$.

Now let (M, ω) be a $2n$-dimensional symplectic manifold and φ a symplectomorphism with hyperbolic fixed point x. Recall that, for symplectomorphisms, the (un)stable manifolds $W^u := W^u(\varphi, x)$ and $W^s := W^s(\varphi, x)$ are Lagrangian submanifolds, i.e. their tangent spaces are Lagrangian subspaces.

Consider the path space $\mathcal{P}(W^u, W^s) := \{\beta : [0,1] \to M \mid \beta(0) \in W^u,\ \beta(1) \in W^s\}$. Given a homoclinic point $p \in \mathcal{H} := W^u \cap W^s$, one can identify it with a constant path in $\mathcal{P}(W^u, W^s)$. For homoclinic points p, q in the same connected component of $\mathcal{P}(W^u, W^s)$, we consider a *path* $v : [0,1] \to \mathcal{P}(W^u, W^s)$ in the path space with $v(0) \equiv p$ and $v(1) \equiv q$ and see it as a map $v : [0,1]^2 \to M$ via $v(s,t) := v(s)(t)$. The square $[0,1]^2$ is contractible and we can find (cf. Floer [8]) a trivialization $\Phi := \Phi_v : v^*TM \to [0,1]^2 \times \mathbb{R}^{2n}$ such that

(a) the symplectic form on the fibers is mapped to the standard form ω_0 on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$,
(b) Φ is constant on $\{0\} \times [0,1]$ and on $\{1\} \times [0,1]$,
(c) $\Phi(T_pW^s) = i\Phi(T_pW^u)$ and $\Phi(T_qW^s) = i\Phi(T_qW^u)$.

Denote by $\partial[0,1]^2$ the boundary of $[0,1]^2$ and define the loop $\Lambda_v : \partial[0,1]^2 \to \mathcal{L}(n)$ starting in $(0,0)$ and running through $(1,0)$, $(1,1)$ and $(0,1)$ back to $(0,0)$ piecewise via

$$(\xi, 0) \mapsto \Phi(T_{v(\xi,0)}W^u), \qquad (\xi, 1) \mapsto \Phi(T_{v(\xi,1)}W^s),$$
$$(1, \eta) \mapsto e^{\frac{i\pi\eta}{2}}\Phi(T_qW^u), \qquad (0, \eta) \mapsto e^{\frac{i\pi(\eta-1)}{2}}\Phi(T_pW^s).$$

Under the above conventions, we define the *relative Maslov index for p, $q \in \mathcal{H}$* via $\mu(p,q) := \mu(\Lambda_v)$. If $\pi_2(M) = 0$, then $c_1|_{\pi_2(M)} = 0$ (where c_1 denotes the first Chern class of M) and the construction is independent from the chosen path v and the trivialization Φ.

We plan to work mainly on $\mathbb{R}^2$ and on closed oriented surfaces with genus $g \geq 1$, where the second homotopy class always vanishes. Thus the well-definedness of the Maslov index will not be an issue. Moreover, in the two-dimensional situation, the Maslov index can be seen as twice the winding number of the unit tangent vector along the boundary of the range of v.

From now on, (M, ω) is either the symplectic plane $(\mathbb{R}^2, \omega)$ or a closed, two-dimensional manifold with genus $g \geq 1$. Let $i \in \{s, u\}$ and fix a parametrizing immersion $\gamma_i : \mathbb{R} \to W^i$. This immersion induces an ordering $<_i$ resp. $\leq_i$ on W^i via

$$\gamma_i(t) <_i \gamma_i(\tilde{t}) \;\Leftrightarrow\; t < \tilde{t} \quad \text{resp.} \quad \gamma_i(t) \leq_i \gamma_i(\tilde{t}) \;\Leftrightarrow\; t \leq \tilde{t}.$$

By abuse of notation, we say that p, $q \in W^i$ induce an ordering on W^i via setting $p <_i q$ resp. $p \leq_i q$. For $i \in \{0,1\}$ consider p, $q \in W^i$ and set $t_i^p = \gamma_i^{-1}(p)$, $t_i^q := \gamma_i^{-1}(q)$, $t_i^- := \min\{t_i^p, t_i^q\}$ and $t_i^+ := \max\{t_i^p, t_i^q\}$. We call

$$[p,q]_u := \gamma_u([t_u^-, t_u^+]) \quad \text{resp.} \quad [p,q]_s := \gamma_s([t_s^-, t_s^+])$$

the *segments* in W^u resp. W^s between p and q. The segments are independent of the chosen immersion and a priori just sets of points, thus $[p,q]_i = [q,p]_i$. Analogously, we define the open and half-open segments $]p,q[_i$ and $[p,q[_i$.

Now we assign to each $p \in \mathcal{H}$ a *homotopy class* in $\pi_0(\mathcal{P}(W^u, W^s)) \simeq \pi_1(M,x)$: Denote by $c_p : [0,1] \to W^u \cup W^s$ a curve with $c_p(0) = x = c_p(1)$ which runs through $[x,p]_u$ to p and through $[p,x]_s$ back to x. Set $[p] := [c_p] \in \pi_1(M,x)$ and $[-p]$ for the path with the inverse parametrization. Then $\mathcal{H}_{[x]} := \{p \in \mathcal{H} \mid [p] = [x]\}$ is the set of *contractible* homoclinic points. $\mathcal{H}_{[x]}$ is invariant under the $\mathbb{Z}$-action $\mathbb{Z} \times \mathcal{H} \to \mathcal{H}$, $(n,p) \mapsto \varphi^n(p)$.

Remark 2.1. For contractible p, $\tilde{p}$, $q \in \mathcal{H}$, we observe:

1. $\mu(q,p) = -\mu(p,q)$ and $\mu(p,q) + \mu(q,\tilde{p}) = \mu(p,\tilde{p})$.
2. $\mu(p,q) = \mu(\varphi^n(p), \varphi^n(q))$ for $n \in \mathbb{Z}$, i.e. the (relative) Maslov index of p and q is invariant under the $\mathbb{Z}$-action of φ on $\mathcal{H}$.
3. $\mu(p, \varphi^n(p)) = 0$ for all $n \in \mathbb{Z}$.
4. $\mu(p,q) = \mu(p, \varphi^n(q))$ for $n \in \mathbb{Z}$.

The (relative) Maslov index yields a *grading* $\mu : \mathcal{H}_{[x]} \to \mathbb{Z}$ via $\mu(p) := \mu(p,x)$ such that for contractible p and q holds $\mu(p,q) = \mu(p,x) + \mu(x,q) = \mu(p,x) - \mu(q,x) = \mu(p) - \mu(q)$.

Immersions, di-gons and hearts. A *di-gon* is the polygon $D \subset \mathbb{R}^2$ with two convex vertices at $(-1,0)$ and $(1,0)$ sketched in Figure 2 (a). Denote its upper boundary by B_s and its lower boundary by B_u.

A *heart* is either the polygon D_b of Figure 2 (b) or the polygon D_c of Figure 2 (c). A heart is characterised by two vertices at $(-1,0)$ and $(1,0)$ where one is convex and one concave. Denote their upper boundaries by B_s and their lower boundaries by B_u.

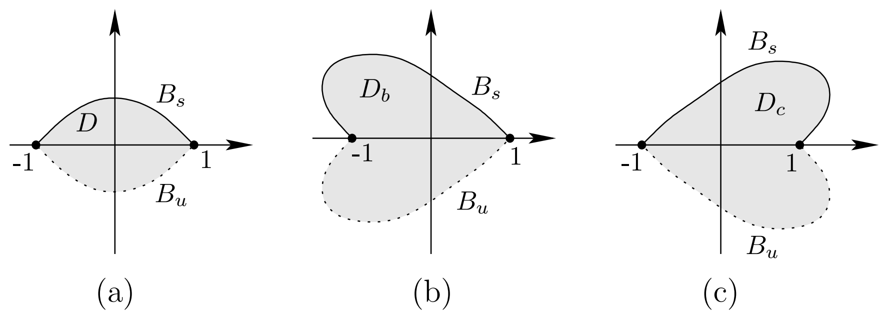


FIGURE 2. Di-gon and heart

We require the immersions below to be immersions also on the boundaries and vertices. Thus the image of a small neighbourhood of a convex resp. concave vertex of a polygon is a wedge-shaped region with angle smaller resp. larger than π.

Let D be the di-gon and p, $q \in \mathcal{H}$ with $\mu(p,q) = 1$. We define $\mathcal{M}(p,q)$ to be the *space of smooth, immersed di-gons* $v : D \to M$ which are orientation preserving and satisfy $v(B_u) \subset W^u$, $v(B_s) \subset W^s$, $v(-1,0) = p$ and $v(1,0) = q$. Denote by $G(D)$ the group of orientation preserving diffeomorphisms of D which preserve the vertices and call $\widehat{\mathcal{M}}(p,q) := \mathcal{M}(p,q)/G(D)$ the space of unparametrized immersed di-gons.

Since there is exactly one segment $[p,q]_i$, $i \in \{s,u\}$, joining p, $q \in \mathcal{H}$ and since $\pi_2(M) = 0$ we deduce $\#\widehat{\mathcal{M}}(p,q) \in \{0,1\}$ for p and q with $\mu(p,q) = 1$.

Now consider the hearts D_b and D_c and p, $r \in \mathcal{H}$ with $\mu(p,r) = 2$. We define $\mathcal{N}_b(p,r)$ resp. $\mathcal{N}_c(p,r)$ to be the *space of smooth immersed hearts* $w : D_b \to M$ resp. $w : D_c \to M$ which are orientation preserving and satisfy $w(B_u) \subset W^u$, $w(B_s) \subset W^s$, $w(-1,0) = p$ and $w(1,0) = r$. We set $\mathcal{N}(p,r) := \mathcal{N}_b(p,r) \dot\cup \mathcal{N}_c(p,r)$. Denote by $G(D_b)$ resp. $G(D_c)$ the group of orientation preserving diffeomorphisms of D_b resp. D_c which preserve the vertices and let $\widehat{\mathcal{N}}_b(p,r) := \mathcal{N}_b(p,r)/G(D_b)$ resp. $\widehat{\mathcal{N}}_c(p,r) := \mathcal{N}_c(p,r)/G(D_c)$ and $\widehat{\mathcal{N}}(p,r) := \widehat{\mathcal{N}}_b(p,r) \dot\cup \widehat{\mathcal{N}}_c(p,r)$ be the spaces of unparametrized immersed hearts.

If we work with the spaces $\mathcal{M}(p,q)$ and $\mathcal{N}(p,r)$ we always implicitly assume p, q, $r \in \mathcal{H}$ with $[p] = [q]$, $[p] = [r]$, $\mu(p,q) = 1$ and $\mu(p,r) = 2$.

Primary points, gluing and cutting. Especially in pictures, we will abbreviate $p^n := \varphi^n(p)$ for $p \in \mathcal{H}$ and $n \in \mathbb{Z}$. Keep in mind that in this notation $p = p^0$. We call the two connected components of $W^s \backslash \{x\}$ resp. $W^u \backslash \{x\}$ the *branches* of the (un)stable manifolds. Let λ and λ^{-1} be the pair of eigenvalues of $D\varphi(x)$. If $\lambda > 0$ then φ is orientation preserving on the stable and unstable manifolds. If $\lambda < 0$ then it is orientation reversing on both. In the first case, we call φ *W-orientation preserving* and in the latter case *W-orientation reversing*.

$p \in \mathcal{H}\backslash\{x\}$ is called *semi-primary* if $]x,p[_u \cap\,]x,p[_s = \emptyset$. $p \in \mathcal{H}_{[x]}\backslash\{x\}$ is *primary* if $]x,p[_u \cap\,]x,p[_s \cap \mathcal{H}_{[x]} = \emptyset$. Nonprimary points are called *secondary*.

On manifolds with vanishing first homotopy class, the notions of semi-primary and primary coincide. Clearly, iterates of a (semi-)primary point are again (semi-) primary. If $W^u \cap W^s \neq \emptyset$ then semi-primary points always exist.

We require $[p] = [x]$ in the definition of primary points, since this condition was already necessary for the invariance of the Maslov index and the homotopy classes under the $\mathbb{Z}$-action of φ. The condition '$\ldots \cap \mathcal{H}_{[x]}$' is necessary for invariance properties of the Floer homology.

We summarize the most important properties of primary points in the following statement.

Lemma 2.2 ([16])**.** *(i) Let φ be W-orientation preserving, $p \in \mathcal{H}$ (semi)primary and denote the branches containing p by W^u_p and W^s_p. Then for every (semi) primary $q \in (W^u_p \cap W^s_p)\backslash\{p^n \mid n \in \mathbb{Z}\}$ there is a unique $n \in \mathbb{Z}$ such that $q^n \in \,]p, \varphi(p)[_u \cap \,]p, \varphi(p)[_s$.*

(ii) For a primary point p holds $\mu(p) := \mu(p, x) \in \{\pm 1, \pm 2, \pm 3\}$.

(iii) Let all primary points $p \in W^u \cap W^s$ be transverse. Then there are modulo $\mathbb{Z}$-action only finitely many primary points.

Now we consider the so-called 'gluing and cutting' procedure on which the well-definedness of Floer homology is based, more precisely the proof of $\partial \circ \partial = 0$. For the following, compare Figure 3. Briefly, *gluing* of two immersed di-gons $v \in \widehat{\mathcal{M}}(p, q)$ and $\hat{v} \in \widehat{\mathcal{M}}(q, r)$ with $\mu(p, q) = 1 = \mu(q, r)$ (and therefore $\mu(p, r) = 2$) is the construction which recognizes the tupel $(v, \hat{v})$ as an element of $\widehat{\mathcal{N}}(p, r)$. *Cutting* is the 'inverse' construction which starts with $w \in \widehat{\mathcal{N}}(p, r)$ and finds two significant points q_u, $q_s \in \mathcal{H}_{pr}$ such that w can be seen either as tupel $(v, \hat{v}) \in \widehat{\mathcal{M}}(p, q_u) \times \widehat{\mathcal{M}}(q_u, r)$ or as tupel $(v', \hat{v}') \in \widehat{\mathcal{M}}(p, q_s) \times \widehat{\mathcal{M}}(q_s, r)$.

Theorem 2.3 ('Gluing', [16])**.** *Let p, q, $r \in \mathcal{H}$ with $[p] = [q] = [r]$ and $\mu(p, q) = 1 = \mu(q, r)$. Let $v \in \widehat{\mathcal{M}}(p, q)$ and $\hat{v} \in \widehat{\mathcal{M}}(q, r)$. Then the* gluing procedure $\#$ *for v and $\hat{v}$ yields an immersed heart $w := \hat{v} \# v \in \widehat{\mathcal{N}}(p, r)$.*

The four possible geometric positions of the three involved points are described in Figure 3. The q which lies on that part of the *un*stable manifold, which crossed the interior of the immersed heart after passing the concave vertex, is called q_u. The other 'cutting point' q_s is named analogously. The gluing construction $\#$ glues $v \in \mathcal{M}(p, q_u)$ and $\hat{v} \in \mathcal{M}(q_u, r)$ along the common boundary segment $[p, q_u]_u$.

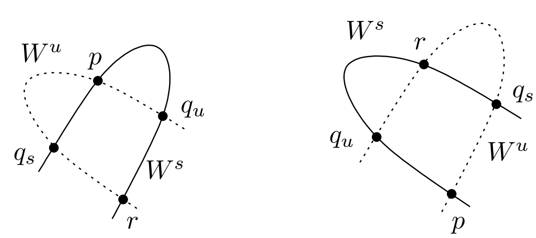


FIGURE 3. Immersions with $\mu(p) = \mu(q) + 1 = \mu(r) + 2$

Theorem 2.4 ('Cutting for primary points', [16])**.** *Let all primary points be transverse and p, $r \in \mathcal{H}_{pr}$ with $\mu(p, r) = 2$ and $w \in \mathcal{N}(p, r)$. Then there are unique* *points q_u and q_s such that either both q_i are primary admitting $v_i \in \mathcal{M}(p, q_i)$ and $\hat{v}_i \in \mathcal{M}(q_i, r)$ with $\hat{v}_i \# v_i = w$ for $i \in \{s, u\}$ or none of them is primary.*

Being 'primary' is a strong geometric condition — we can split $\mu(p, r) = \mu(p) - \mu(r)$ and sketch and draw all possible situations which match $\mu(p, r) = \mu(p) - \mu(r) = 2$ where $\mu(p), \mu(r) \in \{\pm 1, \pm 2, \pm 3\}$ according to Remark 2.2. The possible cutting situations are sketched in Figure 4.

With (slight) assumptions on the intersection behaviour of the branches, the above theorem is also true for p, $r \in \mathcal{H}$ yielding q_s, $q_u \in \mathcal{H}$.

Signs and coherent orientations. For $i \in \{s, u\}$, denote the two branches of W^i by W^i_+ and W^i_-. Associate to each branch its 'jump direction' as orientation sand denote it by $o(W^i_+)$ resp. $o(W^i_-)$. Let p, q be primary with $\mu(p, q) = 1$ and $v \in \mathcal{M}(p, q)$. Associate to $v(B_i) = [p, q]_i$ the orientation induced by the parametrization from p to q and call it o_{pq}. In Hohloch [16], it is shown that $x \notin \,]p, q[_u \cap \,]p, q[_s$. Thus, there is a branch $W_{pq} \in \{W^u_+, W^u_-, W^s_+, W^s_-\}$ containing both p and q. We set

$$m(p, q) := \begin{cases} 1 & \text{if } \mu(p, q) = 1,\ \mathcal{M}(p, q) \neq \emptyset,\ o(W_{pq}) = o_{pq}, \\ -1 & \text{if } \mu(p, q) = 1,\ \mathcal{M}(p, q) \neq \emptyset,\ o(W_{pq}) \neq o_{pq}, \\ 0 & \text{otherwise.} \end{cases}$$

If there are two branches W^u_{pq} and W^s_{pq} containing p and q then p and q are adjacent and $o(W^u_{pq}) = o_{pq} = o(W^s_{pq})$ as shown in Hohloch [16]. Thus $m(p, q)$ is well-defined. We do not need to distinguish the cases W-orientation preserving and reversing since $m(p, q) = m(p^l, q^l)$ for all $l \in \mathbb{Z}$. This definition does *not* generalize to *arbitrary* homoclinic points. However, there is also a way to define signs for arbitrary homoclinic points, see Hohloch [16].

Lemma 2.5 ([16])**.** *Let p and r be primary with $\mu(p, r) = 2$ and $w \in \widehat{\mathcal{N}}(p, r)$. For $i \in \{s, u\}$ assume the existence of q_i with $\mu(p, q_i) = 1 = \mu(q_i, r)$ and $v_i \in \widehat{\mathcal{M}}(p, q_i)$ and $\hat{v}_i \in \widehat{\mathcal{M}}(q_i, r)$ such that $\hat{v}_i \# v_i = w$. Then*

$$m(p, q_u) \cdot m(q_u, r) = -m(p, q_s) \cdot m(q_s, r).$$

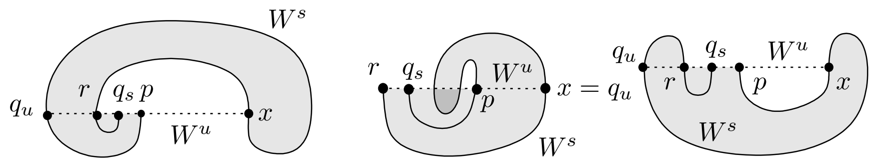
This follows from Figure 4: just check the eight possible $w = \hat{v}_i \# v_i \in \widehat{\mathcal{N}}(p, r)$ sketched in the left and right column. This skew symmetry w.r.t. the cutting and gluing procedure is called 'coherent orientations' in classical Floer theory.

Primary Floer homology. Now we are ready to define the Floer chain complex. We assume for the rest of the section (if not stated otherwise) all homoclinic points to be primary and transverse.

We define an equivalence relation on $\mathcal{H}_{pr} := \{p \in \mathcal{H} \mid p \text{ primary}\}$ via $p \sim q \Leftrightarrow \exists\, n \in \mathbb{Z}$ with $q^n = p$. We set $\tilde{\mathcal{H}}_{pr} := \mathcal{H}_{pr}/_\sim$ and denote by $\langle p \rangle$ the *equivalence class* or *orbit* of p. Note that $\#\tilde{\mathcal{H}}_{pr} < \infty$ according to Remark 2.2. Due to Remark 2.1, we can establish a well-defined homotopy class and a Maslov index via $[\langle p \rangle] := [p]$, $\mu(\langle p \rangle, \langle q \rangle) := \mu(p, q)$ and $\mu(\langle p \rangle) := \mu(p, x)$. For $\langle p \rangle$, $\langle q \rangle \in \tilde{\mathcal{H}}_{pr}$ set $m(\langle p \rangle, \langle q \rangle) :=$

Case $\quad]x,p[_u \cap\,]x,r[_u \neq \emptyset = \,]x,p[_s \cap\,]x,r[_s$

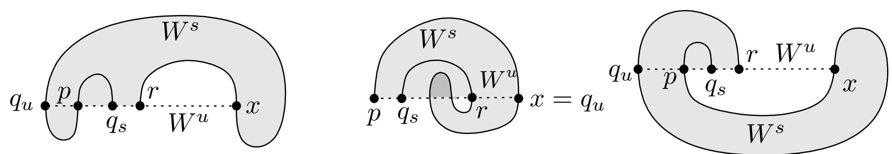
(i) $\quad]x,p[_u \subset\,]x,r[_u$



$(\mu(p,x), \mu(x,r)) = (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_u nor q_s primary!

(ii) $\quad]x,r[_u \subset\,]x,p[_u$

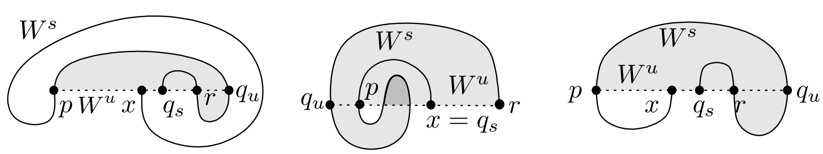


$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_u nor q_s primary!

Case $\quad]x,p[_u \cap\,]x,r[_u = \emptyset \neq \,]x,p[_s \cap\,]x,r[_s$

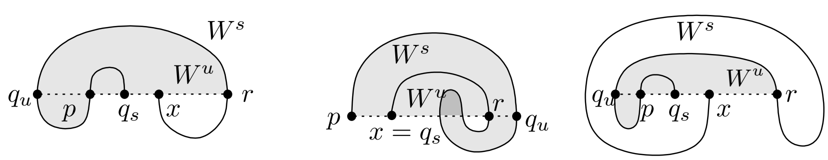
(i) $\quad]x,p[_s \subset\,]x,r[_s$



$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_u nor q_s primary!

(ii) $\quad]x,r[_s \subset\,]x,p[_s$



$= (3,-1)$ $\quad = (1,1)$ $\quad = (-1,3)$

Neither q_u nor q_s primary!

FIGURE 4. Cutting for primary points

$\sum_{n\in\mathbb{Z}} m(p, q^n)$ and define

$$C_m := C_m(\varphi, x; \mathbb{Z}) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle)=m}} \mathbb{Z}\langle p\rangle,$$

$$\partial_m : C_m \to C_{m-1}, \qquad \partial\langle p\rangle := \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle, \langle q\rangle)\langle q\rangle$$

on a generator $\langle p\rangle$ and extend ∂ by linearity. We have $\mathrm{rk}_{\mathbb{Z}}(C_m) = \#\{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \mid \mu(\langle p\rangle) = m\} < \infty$. And due to Remark 2.2, at most $C_{\pm 1}$, $C_{\pm 2}$ and $C_{\pm 3}$ are nonzero.

Theorem 2.6 ([16])**.** *$\partial \circ \partial = 0$, i.e. (C_*, ∂_*) is a chain complex and*

$$H_m := H_m(\varphi, x; \mathbb{Z}) := \frac{\ker \partial_m}{\operatorname{Im} \partial_{m+1}}$$

is called primary Floer homology of φ in x.

Since already the C_m have finite rank over $\mathbb{Z}$ so has H_m and all homology groups H_m with $m \neq \pm 1, \pm 2, \pm 3$ vanish.

The proof of the well-definedness of ∂ and the proof of $\partial \circ \partial = 0$ are based on the cutting and gluing construction (and on the skewsymmetry of the signs) which rely on the classification of $\mathcal{M}(p, q)$ and $\mathcal{N}(p, r)$. Both classifications use the fact that the loops $[p, x]_s \cup [p, x]_u$, $[q, x]_s \cup [q, x]_s$ and $[r, x]_u \cup [r, x]_s$ are contractible. Certain parts of the proofs are of combinatorial nature whereas other parts make use of the iteration behaviour of the (un)stable manifolds and use classical dynamical results like Palis' λ-Lemma [23].

Since $\tilde{\mathcal{H}}_{pr}$ and the sum in the definition of ∂ are finite, primary Floer homology is in fact completely determined by compact segments of the (un)stable manifolds centered around the fixed point.

Invariance. Consider $\varphi \in \mathrm{Diff}^k(M)$ with $k \geq 1$ with $x \in \mathrm{Fix}(\varphi)$ hyperbolic. Let $\psi \in \mathrm{Diff}^k(M)$ be sufficiently C^k-near to φ. Then it is wellknown that ψ has a hyperbolic fixed point y near x. $W^i(\psi, y)$ is C^k-near $W^i(\varphi, x)$ for $i \in \{u, s\}$, at least on compact neighbourhoods of y and x in $W^i(\psi, y)$ and $W^i(\varphi, x)$. y is called the *continuation* of x and the signs of the corresponding eigenvalues coincide.

W^u and W^s are called *strongly intersecting* (w.r.t. x) if each branch of W^u intersects each branch of W^s, i.e. $W^i_+ \cap W^j_+ \neq \emptyset \neq W^i_- \cap W^j_+$ for $i,\ j \in \{0, 1\}$ and $i \neq j$.

Let φ, $\psi \in \mathrm{Symp}(M)$ and assume $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$ to be hyperbolic. An *isotopy between* (φ, x) *and* (ψ, y) is a smooth path $\Phi : [0, 1] \to \mathrm{Symp}(M)$, $\tau \mapsto \Phi(\tau) =: \Phi_\tau$ with $\Phi_0 = \varphi$, $\Phi_1 = \psi$, $x_0 = x$ and $x_1 = y$ and $x_\tau \in \mathrm{Fix}(\Phi_\tau)$ as continuation between x and y for all $\tau \in [0, 1]$. Attaching τ to a symbol associates it to (Φ_τ, x_τ), i.e. $\mathcal{H}^\tau_{pr}$ denotes the set of primary points of (Φ_τ, x_τ) etc. (φ, x) is called *contractibly strongly intersecting (csi)* if W^u and W^s are strongly intersecting and if each pair of branches has contractible homoclinic points. An isotopy Φ is csi if (Φ_τ, x_τ) is csi for all $\tau \in [0, 1]$.

Theorem 2.7 (Invariance [16])**.** *Let (M, ω) be a closed symplectic two-dimensional manifold with genus $g \geq 1$. Let φ, $\psi \in \mathrm{Diff}_\omega(M)$ with hyperbolic fixed points $x \in \mathrm{Fix}(\varphi)$ and $y \in \mathrm{Fix}(\psi)$. Let (φ, x) and (ψ, y) be csi and let all primary points* *of φ and ψ be transverse. Assume there is a csi isotopy Φ from (φ, x) to (ψ, y). Then*

$$H_*(\varphi, x) \simeq H_*(\psi, y).$$

The proof of Theorem 2.7 carries over to symplectomorphisms on $\mathbb{R}^2$ with compact support.

3. **Cylinder Floer homology.** One can define (at least) two different types of primary Floer homology on cylinders:

(i) We can set up primary Floer homology on the cylinder in the very same way as primary Floer homology was defined in Section 2 on $\mathbb{R}^2$ and on closed surfaces. But, by definition, this type of Floer homology is based on *contractible* homoclinic points. Therefore important examples like the perturbed pendulum or Chirikov's Standard map on the cylinder are excluded resp. have trivial homology since there are no contractible homoclinic points.

(ii) If we want to include non-contractible homoclinic points on the cylinder we might identify the cylinder with an annulus in $\mathbb{R}^2$ and 'forget' about the hole of the annulus. In this way, we avoid the nontrivial first homotopy group of the cylinder and we can use large parts of the homology construction from Section 2. Moreover, we can adjust said construction to 'keep in mind' the original homotopy class of a homoclinic point such that we get meaningful homologies e.g. for the perturbed pendulum and Chirikov's Standard map. We will call this type of Floer homology *cylinder Floer homology.*

We will not pursue the first approach any further, but focus entirely on the second one. The rest of the section will be spent on the construction of cylinder Floer homology.

Symplectomorphisms on the cylinder. Whenever we work on the (infinite) symplectic cylinder $(\mathcal{Z}, \Omega)$, we assume the symplectomorphisms to be *compactly supported* if not stated otherwise. We denote by $\mathrm{Symp}_0(\mathcal{Z}) := \mathrm{Symp}_0(\mathcal{Z}, \Omega)$ the group of symplectomorphisms isotopic to the identity.

Analogously to Section 2, we define the homotopy class of a homoclinic point p on the cylinder as $[p] := [c_p] \in \pi_1(\mathcal{Z}, x)$.

Remark 3.1. Let $f \in \mathrm{Symp}_0(\mathcal{Z})$ with hyperbolic fixed point x. Without further assumptions, f can be W-orientation preserving or reversing. But if we require that (at least) one pair of intersecting branches gives rise to non-contractible semiprimary points as in Figure 10 then f must be W-orientation preserving w.r.t. x.

Proof. Let p for instance be as in Figure 10. Then $[p] \neq [f(p)]$ in contradiction to Lemma 3.2. □

A main pillar in the construction of primary Floer homology was the use of *contractible* homoclinic points. The contractibility ensures that the homotopy classes and the Maslov index are invariant under iteration of the symplectomorphism. But we now want to admit non-contractible semi-primary points on the cylinder. Thus we need to investigate if the homotopy classes and Maslov index still may be invariant under iteration of the symplectomorphism. Given $f \in \mathrm{Symp}_0(\mathcal{Z})$, we call $x \in \mathrm{Fix}(f)$ *contractible* if there is a path $[0,1] \to \mathrm{Symp}_0(\mathcal{Z})$, $t \mapsto f_t$ with $f_0 = \mathrm{Id}$ and $f_1 = f$ such that $t \mapsto f_t(x)$ is contractible in $\mathcal{Z}$.

Lemma 3.2. *Let $f \in \mathrm{Symp}_0(\mathcal{Z})$ and $x \in \mathrm{Fix}(f)$. Then x is contractible and $[p] = [f^n(p)]$ for all $p \in \mathcal{H}$ and $n \in \mathbb{Z}$.*

Proof. Polterovich ([28], Example 1.3.B) proved that, on the standard symplectic $2n$-dimensional torus $(\mathbb{T}^{2n}, dp \wedge dq)$, every fixed point $x \in \mathrm{Fix}(f)$ of any $f \in \mathrm{Symp}_0(\mathbb{T}^{2n}, dp \wedge dq)$ is contractible. His proof is also valid in our situation.

Let f_t be a symplectic path with $f_0 = \mathrm{Id}$ and $f_1 = f$ such that x is contractible. Set $\xi(\tau) := f_\tau(x)$ and compute $[p] = [\xi]^{-1} * [f(p)] * [\xi] \in \pi_1(\mathcal{Z}, x)$. Thus $[p] = [f(p)]$ since ξ is contractible.

There is another way to prove the claim: Since $\pi_1(\mathcal{Z}, x) \simeq \mathbb{Z}$ is abelian we directly deduce $[p] = [\xi]^{-1} * [f(p)] * [\xi] \in \pi_1(\mathcal{Z}, x)$. $\square$

Let $0 < R_- < R_+ < \infty$ and denote by $\mathcal{Q} := \mathcal{Q}(R_-, R_+)$ the open annulus in $(\mathbb{R}^2, \omega)$ centered at the origin with radii R_- and R_+. Let $h : \mathcal{Z} \to \mathcal{Q}$ be an orientation preserving diffeomorphism which identifies the cylinder with the annulus. Given $f \in \mathrm{Symp}_0(\mathcal{Z})$, we denote by $F := F_h := h \circ f \circ h^{-1} \in \mathrm{Diff}(\mathcal{Q})$ its conjugate. If $x \in \mathrm{Fix}(f)$ is hyperbolic so is $\mathfrak{x} := h(x) \in \mathrm{Fix}(F)$. Denote by $\mathcal{H}(f, x) := W^s(f, x) \cap W^u(f, x)$ the set of homoclinic points of f w.r.t. x and analogously define $\mathcal{H}(F, \mathfrak{x}) := W^s(F, \mathfrak{x}) \cap W^u(F, \mathfrak{x})$ seen as points in $\mathbb{R}^2$, i.e. all of them are considered contractible. Denote by $\mathcal{H}_{pr}(F, \mathfrak{x}) \subset \mathcal{H}(F, \mathfrak{x})$ the set of primary points of F w.r.t. $\mathfrak{x}$ and define $\mathcal{H}_{pr}(f, x) := h^{-1}(\mathcal{H}_{pr}(F, \mathfrak{x}))$. Images under h of homoclinic points $p \in \mathcal{H}(f, x)$ are abbreviated in Gothic print as $\mathfrak{p} := h(p)$ etc.

The construction of primary Floer homology is purely combinatorial although the invariance and certain applications only make sense for symplectomorphisms. In the following, we will define Floer homology on the cylinder by using the image of the homoclinic tangle on the annulus in the plane. Later on, we will add symplectic information obtained directly from the system on the cylinder.

Signs, gluing and cutting. By Lemma 3.2, $f \in \mathrm{Symp}_0(\mathcal{Z}, ds \wedge dt)$ preserves $[p] = [f^n(p)]$. Thus we can define $[\langle p \rangle] := [p]$. We denote by $\tilde{\mathcal{H}}_{pr}(f, x)$ resp. $\tilde{\mathcal{H}}_{pr}(F, \mathfrak{x})$ the equivalence classes of primary points. Therefore we might consider $h(p) \in \mathbb{R}^2$ to be contractible in $\mathbb{R}^2$ and define the Maslov index of $p \in \mathcal{H}(f, x)$ to be $\mu(p) := \mu_h(p) := \mu(h(p), \mathfrak{x})$. Then we obtain the analogous properties as in Remark 2.1.

The moment we consider $h(p)$ as point in $\mathbb{R}^2$ we are about to loose the information about its homotopy class on the cylinder. We keep track of the original homotopy class $[p] \in \pi_1(\mathcal{Z}, x)$ via the boundary operator. In contrast to the boundary operator of primary Floer homology, we consider the following: Let $\langle p \rangle, \langle q \rangle \in \tilde{\mathcal{H}}_{pr}(f, x)$ and recall the signs $m(\mathfrak{p}, \mathfrak{q})$ for primary points in $\mathbb{R}^2$ from Section 2. We define new signs

$$\nu_h(p, q) := \nu(\mathfrak{p}, \mathfrak{q}) := \begin{cases} m(\mathfrak{p}, \mathfrak{q}) & \text{if} \quad \emptyset \neq \mathcal{M}(\mathfrak{p}, \mathfrak{q}) \ni v, \ 0 \neq \mathrm{Im}(v), \\ 0 & \text{otherwise} \end{cases}$$

and set $\nu_h(\langle p \rangle, \langle q \rangle) := \sum_{n \in \mathbb{Z}} \nu_h(p, q^n)$. Recall that $B_{R_-}(0) \subset \mathbb{R}^2$ corresponds to the hole of the annulus resp. the S^1-direction of the cylinder. The new signs ensure that only immersions between primary points $\mathfrak{p}$, $\mathfrak{q} \in \mathcal{H}_{pr}(F, \mathfrak{x})$ with $[p] = [q] \in \pi_1(\mathcal{Z}, x)$ are counted. Thus we stay with the boundary operator in the same cylinder-homotopy class, i.e. the new Floer complex will split w.r.t. cylinder-homotopy classes and thus preserve the information about the original homotopy classes on the cylinder, see Corollary 3.5.

The ball $B := B_{R_-}(0) \subset \mathbb{R}^2$ is untouched by F and can therefore be considered invariant under iteration. Excluding invariant sets from the range of immersions is 'compatible' with the cutting and gluing procedure, more precisely, we need a result similar to Lemma 2.5 for the new signs ν_h.

Lemma 3.3. *Let $\mathfrak{p}$ and $\mathfrak{r}$ be primary with $\mu(\mathfrak{p}, \mathfrak{r}) = 2$ and $w \in \widehat{\mathcal{N}}(\mathfrak{p}, \mathfrak{r})$. For $i \in \{s, u\}$ assume the existence of $\mathfrak{q}_i$ with $\mu(\mathfrak{p}, \mathfrak{q}_i) = 1 = \mu(\mathfrak{q}_i, r)$ and $v_i \in \widehat{\mathcal{M}}(\mathfrak{p}, \mathfrak{q}_i)$ and $\hat{v}_i \in \widehat{\mathcal{M}}(\mathfrak{q}_i, \mathfrak{r})$ such that $\hat{v}_i \# v_i = w$. Then*

$$\nu(\mathfrak{p}, \mathfrak{q}_u) \cdot \nu(\mathfrak{q}_u, \mathfrak{r}) = -\nu(\mathfrak{p}, \mathfrak{q}_s) \cdot \nu(\mathfrak{q}_s, \mathfrak{r}).$$

Proof. Consider Figure 5 which sketches a standard 'gluing and cutting' situation as described in the hypothesis. Assume B lies in the range of v_s, but not in the range of $\hat{v}_s$ as sketched in Figure 5. Then

$$\nu(\mathfrak{p}, \mathfrak{q}_u) \cdot \nu(\mathfrak{q}_u, \mathfrak{r}) = \nu(\mathfrak{p}, \mathfrak{q}_u) \cdot 0 = 0 = -0 \cdot \nu(\mathfrak{q}_s, \mathfrak{r}) = -\nu(\mathfrak{p}, \mathfrak{q}_s) \cdot \nu(\mathfrak{q}_s, \mathfrak{r}).$$

The proofs for all other possible placements of B proceed similarly. $\square$

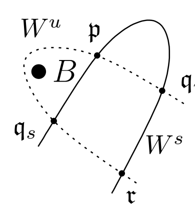


FIGURE 5. Gluing and cutting while 'excluding' B

Cylinder Floer homology. The chain groups are defined analogously to Section 2 via

$$\mathscr{C}_k := \mathscr{C}_k(f, x, h) := \bigoplus_{\substack{\langle p \rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle p \rangle) = k}} \mathbb{Z}\langle p \rangle.$$

For the boundary operator, we apply the new signs and set

$$\mathscr{D} : \mathscr{C}_* \to \mathscr{C}_{*-1}, \qquad \mathscr{D}\langle p \rangle := \sum_{\substack{\langle q \rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle q \rangle) = \mu(\langle p \rangle) - 1}} \nu_h(\langle p \rangle, \langle q \rangle) \langle q \rangle$$

on the generators and extend $\mathscr{D}$ by linearity.

Theorem 3.4. *We have $\mathscr{D} \circ \mathscr{D} = 0$ and $\mathscr{H}_*(f, x, h) := \frac{\ker \mathscr{D}_*}{\operatorname{Im} \mathscr{D}_{*+1}}$ is called* cylinder Floer homology *on $\mathcal{Z}$.*

Proof. Using Lemma 3.3 instead of Lemma 2.5, the proof of $\partial \circ \partial = 0$ (Theorem 2.6) carries over. $\square$

Since the boundary operator only connects points within the same homotopy class on the cylinder we obtain

Corollary 3.5. *$\mathscr{C}_*(f, x, h)$ and $\mathscr{H}_*(f, x, h)$ split into a direct sum w.r.t. the homotopy classes in $\pi_1(\mathcal{Z}, x)$:*

$$\mathscr{C}_*(f, x, h) = \mathscr{C}_*(f, x, h, [\cdot] = 1) \oplus \mathscr{C}_*(f, x, h, [\cdot] = 0) \oplus \mathscr{C}_*(f, x, h, [\cdot] = -1),$$
$$\mathscr{H}_*(f, x, h) = \mathscr{H}_*(f, x, h, [\cdot] = 1) \oplus \mathscr{H}_*(f, x, h, [\cdot] = 0) \oplus \mathscr{H}_*(f, x, h, [\cdot] = -1).$$

Dependency on **h**. Now let us have a look how the chain complex and the homology depend on h. Concatenating h with an orientation preserving diffeomorphism of $\mathcal{Z}$ or $\mathcal{Q}$ does not change anything. The same goes for the concatenation with some orientation preserving diffeomorphism to another annulus $\hat{h} : \mathcal{Q}(R_-, R_+) \to \mathcal{Q}(\hat{R}_-, \hat{R}_+)$.

If we concatenate h with an orientation *reversing* diffeomorphism, the homology changes since the Maslov indices of the generators change. Therefore we always have to keep in mind the orientation of the underlying map h. We denote for orientation preserving resp. reversing h the Floer complex and homology by $\mathscr{C}_*(f, x, +)$ and $\mathscr{H}_*(f, x, +)$ resp. $\mathscr{C}_*(f, x, -)$ and $\mathscr{H}_*(f, x, -)$. We summarize

Theorem 3.6. *Up to the choice of an orientation, there are well-defined Floer homologies $\mathscr{H}_*(f, x, \pm)$ on the cylinder. They are usually not isomorphic.*

Invariance. Assume $f \in \mathrm{Symp}_0(\mathcal{Z})$ to have compact support. Then $F = h \circ f \circ h^{-1}$ has compact support in $\mathcal{Q}$ and therefore can be considered to have compact support in $\mathbb{R}^2$.

In this case, Theorem 2.7 holds true for F: Since the hole of the annulus can be considered as invariant set the invariance properties continue to hold true for the modified boundary operator $\mathscr{D}$. And since the homology of (f, x) is defined as the one of $(F, \mathfrak{x})$ we obtain invariance in the sense of Theorem 2.7 for the homoclinic tangle of (f, x).

4. **Action filtration.** The definitions of primary Floer homology and of cylinder Floer homology are up to now purely combinatorial and do not take any symplectic features of the underlying manifold or symplectomorphism into account. In this section, we will introduce a notion which allows to measure symplectic properties. Since the technique requires an *exact* symplectic manifold (i.e. $\omega = d\alpha$ for some 1-form α) we will consider primary Floer homology only on $\mathbb{R}^2$ and not on closed surfaces. In the following, let $(M, \omega = d\alpha)$ stand for the exact manifolds $(\mathbb{R}^2, \omega)$ resp. $(\mathcal{Z}, \Omega)$ if not stated otherwise. We assume $f \in \mathrm{Symp}(\mathbb{R}^2)$ and, on the cylinder, $f \in \mathrm{Ham}^c(\mathcal{Z})$ with $x \in \mathrm{Fix}(f)$ hyperbolic.

The symplectic action. For $p \in \mathcal{H}$ and $i \in \{s, u\}$, fix a smooth parametrization $\gamma_p^i : [0, 1] \to [x, p]_i$ with $\gamma_p^i(0) = x$ and $\gamma_p^i(1) = p$. We introduce two functions $S : W^s \to \mathbb{R}$ and $U : W^u \to \mathbb{R}$ via

$$S(p) := \int_{\gamma_p^s} \alpha \qquad \text{and} \qquad U(p) := \int_{\gamma_p^u} \alpha$$

which satisfy $dS = \alpha|_{W^s}$ and $dU = \alpha|_{W^u}$. S and U are often called *generating functions.* Using the generating functions, we define the *symplectic action* of $p \in \mathcal{H}$ via

$$\mathcal{A}(p) := (S - U)(p) = \int_{\bar{\gamma}_p^u \# \gamma_p^s} \alpha$$

with $\bar{\gamma}_p^u(\tau) := \gamma_p^u(1 - \tau)$ and where $\#$ stands for the concatenation of paths. If $M = \mathbb{R}^2$, denote by $G(x, p)$ the region enclosed by $\bar{\gamma}_p^u \# \gamma_p^s$ and call it the *resonance domain of* p. Provide $G(x, p)$ with the orientation whose restriction to the boundary coincides with the one of $\bar{\gamma}_p^u \# \gamma_p^s$. In that case, we obtain furthermore

$$\mathcal{A}(p) = \int_{\bar{\gamma}_p^u \# \gamma_p^s} \alpha = \int_{G(x,p)} \omega$$

which is the (signed) symplectic area of the resonance domain of p. Back to $M = \mathbb{R}^2$ or $M = \mathcal{Z}$, the *relative action* of p, $q \in \mathcal{H}$ is given by

$$\mathcal{A}(p,q) := (S-U)(p) - (S-U)(q) = \mathcal{A}(p) - \mathcal{A}(q).$$

Since immersions in $\mathcal{M}(p,q)$ are orientation preserving, Stokes' theorem yields

Lemma 4.1. *Let p, $q \in \mathcal{H}$ (with $[p] = [q] \in \pi_1(\mathcal{Z}, x)$ if $M = \mathcal{Z}$). Assume $\mu(p,q) = 1$ and $v \in \mathcal{M}(p,q) \neq \emptyset$. Then*

$$\mathcal{A}(p,q) = \int_v \omega > 0, \quad \textit{implying} \quad \mathcal{A}(p) > \mathcal{A}(q).$$

In particular, $\mathcal{A}(p,q)$ is the symplectic area *enclosed by $[p,q]_s$ and $[p,q]_u$.*

In Section 2 and Section 3, we worked purely with the *existence* of those orientation preserving immersions $v \in \mathcal{M}(p,q)$, but we did not use the interpretation of $\int_v \omega$ as symplectic area. Now recall

Lemma 4.2 ([22], Prop. 9.19)**.** *$f \in \mathrm{Ham}^c(\mathcal{Z})$ if and only if $f^*\alpha - \alpha = d\tilde{H}$ for a smooth function $\tilde{H} : \mathcal{Z} \to \mathbb{R}$.*

We conclude

Corollary 4.3. *Let $f \in \mathrm{Symp}(\mathbb{R}^2)$ resp. $f \in \mathrm{Ham}^c(\mathcal{Z})$. Then $\mathcal{A}(p) = \mathcal{A}(f^n(p))$ and $\mathcal{A}(p,q) = \mathcal{A}(f^n(p), f^n(q))$ for all $n \in \mathbb{Z}$.*

Proof. In case $f \in \mathrm{Symp}(\mathbb{R}^2)$, the invariance follows from the fact that f is volume preserving and that the simply closed curve c_p associated to a homoclinic point p bounds the compact 'disc' $G(x,p)$.

Now consider the case $f \in \mathrm{Ham}^c(\mathcal{Z})$. By Poincaré duality, the singular homology of $\mathcal{Z}$ is isomorphic to the cohomology with compact support. Therefore a 1-cohomology class is exact if and only if it vanishes on all 1-homology classes. By Lemma 4.2, $f^*\alpha - \alpha$ is exact. Evaluating on the curve c_p associated to a homoclinic point p we get

$$0 = \int_{c_p} (f^*\alpha - \alpha) = \int_{c_{f(p)}} \alpha - \int_c \alpha = \mathcal{A}(f(p)) - \mathcal{A}(p).$$

□

Thus the action is invariant under f and descends to $\tilde{\mathcal{H}}_{pr}$. We define $\mathcal{A}(\langle p \rangle) := \mathcal{A}(p)$ and $\mathcal{A}(\langle p \rangle, \langle q \rangle) := \mathcal{A}(p,q)$. Analogously to classical Floer homology (cf. Schwarz [34], Polterovich [28]), we define the *action spectrum* as $\mathrm{Spec}(f,x) := \{\mathcal{A}(\langle p \rangle) \mid \langle p \rangle \in \tilde{\mathcal{H}}_{pr}\}$. The *width* of the action spectrum is defined as

$$\mathrm{width}(f,x) := \max_{\langle p \rangle, \langle q \rangle \in \tilde{\mathcal{H}}_{pr}} |\mathcal{A}(\langle p \rangle) - \mathcal{A}(\langle q \rangle)|.$$

Let $W^s_{br} \subset W^s$ and $W^u_{br} \subset W^u$ be a branch of the (un)stable manifold and assume $W^s_{br} \cap W^u_{br} \neq \emptyset$. The *minimal gap* in the action spectrum *w.r.t.* $W^s_{br} \cap W^u_{br}$ is denoted by

$$\mathrm{gap}(f,x,W^s_{br},W^u_{br}) := \min_{\substack{\langle p \rangle, \langle q \rangle \in \tilde{\mathcal{H}}_{pr} \\ \langle p \rangle, \langle q \rangle \in W^s_{br} \cap W^u_{br}}} |\mathcal{A}(\langle p \rangle) - \mathcal{A}(\langle q \rangle)|$$

and the *minimal gap* is defined as

$$\mathrm{gap}(f,x) := \min_{\langle p \rangle, \langle q \rangle \in \tilde{\mathcal{H}}_{pr}} |\mathcal{A}(\langle p \rangle) - \mathcal{A}(\langle q \rangle)|.$$

Clearly, we have $\mathrm{gap}(f,x) \leq \mathrm{gap}(f,x,W^s_{br},W^u_{br}) \leq \mathrm{width}(f,x)$.

The action filtration. In classical Floer theory, the symplectic action has been used with success to define and interprete symplectic invariants — one method is the so-called *filtration by the action* (see e.g. Schwarz [34]) which we will apply to our setting. It will make $H_*(f,x)$ resp. $\mathscr{H}_*(f,x,\pm)$ sensitive for symplectic features. The main observation, on which the construction of the action filtration relies, is Lemma 4.1: It holds $\mathcal{A}(p) \geq \mathcal{A}(q)$ if p, $q \in \mathcal{H}_{pr}(f,x)$ with $\mu(p,q) = 1$ and $\widehat{\mathcal{M}}(p,q) \neq \emptyset$. We demonstrate the construction with (positive) cylinder Floer homology $\mathscr{H}_*(f,x,+)$. The constructions for $H_*(f,x)$ and $\mathscr{H}_*(f,x,-)$ are similar.

Let $a \in \mathbb{R}$ and define the *filtered Floer complex* via

$$\mathscr{C}_k^a := \mathscr{C}_k^a(f,x,+) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr}(f,x) \\ \mu(\langle p\rangle) = k \\ \mathcal{A}(\langle p\rangle) \leq a}} \mathbb{Z}\langle p\rangle.$$

Since $\mathcal{A}(p) > \mathcal{A}(q)$ for p, $q \in \mathcal{H}_{pr}(f,x)$ with $\mu(p,q) = 1$ and $\widehat{\mathcal{M}}(p,q) \neq \emptyset$ according to Lemma 4.1, the boundary operator $\mathscr{D}$ restricts to $\mathscr{C}_k^a$. Thus $(\mathscr{C}_*^a, \mathscr{D})$ is a subcomplex of $(\mathscr{C}_*(f,x,+), \mathscr{D})$. For $a < b$, we define $\mathscr{C}_*^{]a,b]} := \mathscr{C}_*^b / \mathscr{C}_*^a$ and there is a short exact sequence of chain complexes

$$0 \to \mathscr{C}_*^a \xrightarrow{i} \mathscr{C}_*^b \xrightarrow{j} \mathscr{C}_*^{]a,b]} \to 0 \quad \text{for } -\infty \leq a < b \leq \infty.$$

We identify $\mathscr{C}_*^\infty = \mathscr{C}_*(f,x,+)$ and $\mathscr{C}_*^{]-\infty,a]} = \mathscr{C}_*^a$ and define $\mathscr{H}_*^{]a,b]} := \mathscr{H}_*^{]a,b]}(f,x,+)$ as the homology of $\mathscr{C}_*^{]a,b]}$. For $-\infty \leq a < b < c \leq \infty$ we obtain the long exact sequence

$$\cdots \to \mathscr{H}_{k+1}^{]b,c]} \to \mathscr{H}_k^{]a,b]} \xrightarrow{i_*} \mathscr{H}_k^{]a,c]} \xrightarrow{j_*} \mathscr{H}_k^{]b,c]} \to \mathscr{H}_{k-1}^{]a,b]} \to \ldots$$

of *filtered Floer homology.*

Remark 4.4. Let $a < \min\{\mathcal{A}(p) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and $b > \max\{\mathcal{A}(p) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and set $a_{min} := \min\{\mathcal{A}(p) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$. Then $\mathscr{H}_*^{]-\infty,a]} = 0$ and $\mathscr{H}_*^{]b,\infty]} = 0$ such that the homology is concentrated in the interval $]a_{min} - \varepsilon, a_{min} + \mathrm{width}(f,x)]$ for $\varepsilon > 0$.

For certain homology classes, we exactly know their critical levels in the action filtration:

Remark 4.5. Set $I := \{n \in \{\pm1, \pm2, \pm3\} \mid \mathscr{C}_{n-1} = 0\}$ and for $k \in I$ consider $c \in \mathscr{C}_k$ with $c = \sum_l c_l \langle p_l\rangle$. Then c represents a homology class and lives in $\mathscr{H}_*^{]a,b]}$ for $a < \min_l \mathcal{A}(\langle p_l\rangle)$ and $\max_l \mathcal{A}(\langle p_l\rangle) \leq b$. In particular $1, -3 \in I$.

5. **Flux and transport.** Let us recall the following definition from MacKay & Meiss & Percival [19].

Definition 5.1. Let c be a simply closed curve in $(\mathbb{R}^2, \omega)$ and let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ be W-orientation preserving. Denote by $\mathrm{Int}(c)$ the interior of c and by $\mathrm{Ext}(c)$ its exterior. We define

$$\mathcal{F}lux_\varphi(c) := \mathrm{vol}_\omega(\varphi(\mathrm{Int}(c)) \cap \mathrm{Ext}(c)) = \mathrm{vol}_\omega(\mathrm{Int}(\varphi(c)) \cap \mathrm{Ext}(c))$$

to be the *absolute flux* of φ through c. If φ is W-orientation reversing then φ^2 is W-orientation preserving and we set $\mathcal{F}lux_\varphi := \mathcal{F}lux_{\varphi^2}$.

$\mathcal{F}lux_\varphi(c)$ is the (symplectic) area of the set of all points mapped by φ from the interior of c to the exterior of c. Since φ is area preserving the absolute flux coincides with the area of the set of all points mapped from the exterior of c to its interior, i.e. $\mathcal{F}lux_\varphi(c) = \mathrm{vol}_\omega(\varphi(\mathrm{Ext}(c)) \cap \mathrm{Int}(c))$. In the following, we usually call the absolute flux briefly *flux*.

MacKay & Meiss & Percival [19] used the flux in order to study the long-term behaviour of symplectomorphisms w.r.t. iteration. More precisely, they were interested in the question how, when and, in particular, *where* points were mapped from one region to another by the symplectomorphism. They called the whole procedure of motion of points under iteration *transport*. For instance, consider the following phenomenon. $\mathcal{F}lux_\varphi(c)$ measures how much of a 'barrier' the curve c is for the transport: For example, if c is invariant under φ, the flux through c is zero. In that case, c is a 'complete barrier' for the transport of points by φ. Later on, we will investigate certain curves which form a 'partial barrier' and where the 'outlet' only happens along a small part of c.

Now consider the (infinite) cylinder $(\mathcal{Z}, \Omega)$ and let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$. Let c be a curve with $[c] \in \{\pm 1\} \subset \pi_1(\mathcal{Z}) \simeq \mathbb{Z}$ without self-intersections. Since φ is isotopic to the identity it holds $[c] = [\varphi(c)]$. The range of c cuts the cylinder into two connected components. Denote one of them by $\mathcal{Z}_c$ (in explicit examples $\mathcal{Z}$ is the component towards $-\infty$). On the cylinder, the flux is defined via

Definition 5.2. Let c be a curve on $(\mathcal{Z}, \Omega)$ with $[c] \in \{0, \pm 1\}$ without self-intersections and let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$. If c is not contractible define the *absolute flux* through c as

$$\mathcal{F}lux_\varphi(c) := \mathrm{vol}_\omega(\mathcal{Z}_{\varphi(c)} \backslash \mathcal{Z}_c).$$

If c is contractible, define $\mathcal{F}lux_\varphi(c)$ as in Definition 5.1.

Again, we usually call the absolute flux briefly *flux*. $\mathcal{F}lux_\varphi(c)$ is well-defined since one deduces $\mathrm{vol}_\omega(\mathcal{Z}_{\varphi(c)} \backslash \mathcal{Z}_c) = \mathrm{vol}_\omega(\mathcal{Z}_c \backslash \mathcal{Z}_{\varphi(c)})$ from Lemma 4.2 analogously to the proof of Corollary 4.3. Note that 'flux' is used for different objects in the literature:

Remark 5.3. a) The definition of $\mathcal{F}lux_\varphi(c)$ in Definition 5.1 and Definition 5.2 differs from the flux homomorphisms flux and Flux used in symplectic geometry as e.g. in McDuff & Salamon [22] and Polterovich [27]: $\mathcal{F}lux_\varphi(c)$ only measures how much is mapped out of $\mathrm{Int}(c)$ whereas the flux in [22] and [27] also takes into account how much is mapped into $\mathrm{Int}(c)$. In [27] and [22], Hamiltonian diffeomorphisms therefore are characterized by vanishing flux.

b) MacKay & Meiss & Percival [19] consider area preserving maps in the plane and on the infinite cylinder. They define the absolute flux, but call it just flux. We renamed it 'absolute flux' in order to avoid confusion with Polterovich [27] and McDuff & Salamon [22]. Moreover, MacKay & Meiss & Percival [19] set

$$\mathrm{Cal}_\varphi(c) := \mathrm{vol}_\omega(\mathcal{Z}_{\varphi(c)} \backslash \mathcal{Z}_c) - \mathrm{vol}_\omega(\mathcal{Z}_c \backslash \mathcal{Z}_{\varphi(c)})$$

and call it the *net flux* or *Calabi invariant*. It corresponds to the flux in [22] and [27]. For applications, they require the area preserving maps to have zero net flux, i.e. in our notation, to be Hamiltonian diffeomorphisms.

Turnstiles. Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ with hyperbolic fixed point x. We need φ to be W-orientation preserving since we want the branches of the (un)stable manifolds to be invariant under iteration. Therefore replace in the following φ by φ^2 if φ is W-orientation reversing.

Let p be a homoclinic point and consider the (un)stable segments $[x,p]_u$ and $[x,p]_s$. For the segments $[x,\varphi(p)]_u$ and $[x,\varphi(p)]_s$ holds $[x,\varphi(p)]_s \subset [x,p]_s$ and $[x,\varphi(p)]_u \supset [x,p]_u$. Denote by c_p a curve which runs from x via $[x,p]_u$ to p and via $[x,p]_s$ back to x. Then the ranges of the curves c_p and $c_{\varphi(p)}$ coincide except in the segments $[p,\varphi(p)]_u$ and $[p,\varphi(p)]_s$.

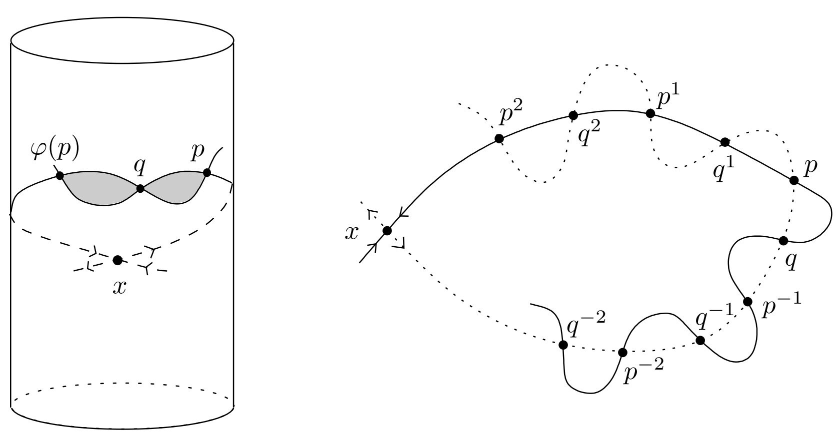
Definition 5.4. Given a primary orbit $\langle p\rangle$ on the cylinder resp. $\mathbb{R}^2$, we set

$$\mathcal{F}lux_\varphi(\langle p\rangle) := \mathcal{F}lux_\varphi(c_p).$$

Apart from homoclinic orbits, MacKay & Meiss & Percival [19] also consider (quasi)periodic and heteroclinic orbits and define the flux through those orbits as the flux through suitable curves joining the points in the orbit. But MacKay & Meiss & Percival [19] are actually mainly interested in the flux through cantori. For that, they close the gaps of the cantorus with a suitable curve and define the flux through the cantorus as the flux through the associated curve. Which kind of curves minimizes the flux through cantori is treated in Polterovich [26].

Recall from Remark 2.2 how and where primary points appear in the tangle.

Definition 5.5. φ is called *x-simple* if each pair of intersecting branches contains exactly two primary orbits (an example is sketched in Figure 6).



(a) (b)

FIGURE 6. Turnstiles: (a) on the cylinder and (b) on $\mathbb{R}^2$

Definition 5.6. Let φ be x-simple and $\langle p\rangle$ and $\langle q\rangle$ the primary points in a chosen pair of intersecting branches. Assume $\{q\} =]p,\varphi(p)[_s \cap]p,\varphi(p)[_u$. The resulting picture is called a *true turnstile with pivot q and frame p and $\varphi(p)$*. The regions enclosed by $[p,\varphi(p)]_s \cup [p,\varphi(p)]_u$ are called the *wings* of the turnstile.

The basic idea of turnstiles goes back to MacKay & Meiss & Percival [19], but we are refining the notion. An example of a true turnstile (with shaded wings) is sketched in Figure 6 (a). We observe that the shaded region between $\varphi(p)$ and q is swept from $\mathcal{Z}_{c_p}$ to $\mathcal{Z}\backslash\mathcal{Z}_{c_{\varphi(p)}}$ whereas at the same time the shaded region between p and q is swept from $\mathcal{Z}\backslash\mathcal{Z}_{c_p}$ to $\mathcal{Z}_{c_{\varphi(p)}}$. All points which are mapped by φ from $\mathcal{Z}_{c_p}$ to $\mathcal{Z}\backslash\mathcal{Z}_{c_p}$ lie in the region enclosed by $[p,q]_u \cup [p,q]_s$. And analogously, all points moving from $\mathcal{Z}\backslash\mathcal{Z}_{c_p}$ to $\mathcal{Z}_{c_p}$ lie in the region enclosed by $[q,\varphi(p)]_u \cup [q,\varphi(p)]_s$. A similar observation clearly holds for turnstiles in $\mathbb{R}^2$.

Remark 5.7. There is no 'turnstile-like' picture between primary points of different branches.

The following statement explains the absolute flux of a primary point in terms of the related turnstile.

Lemma 5.8. *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ and assume φ to be x-simple. Let p be a primary point and pivot of a true turnstile with frame q and $\varphi(q)$. Denote by c_{pq} a curve which runs from p through $[p,q]_u$ to q and then through $[p,q]_s$ back to p. The wing enclosed by $c_{p,q}$ is called $G(p,q)$. Then we have*

$$\mathcal{F}lux_\varphi(\langle q\rangle) = \left|\int_{c_{pq}} \alpha\right| = \left|\int_{G(p,q)} \omega\right| = \left|\int_{G(p,\varphi(q))} \omega\right| = \left|\int_{c_{p\varphi(q)}} \alpha\right|$$

and in particular

$$\mathcal{F}lux_\varphi(\langle p\rangle) = \mathcal{F}lux_\varphi(\langle q\rangle).$$

Thus the flux through $\langle p\rangle$ resp. $\langle q\rangle$ equals the symplectic volume of one wing of the associated turnstile.

Proof. As in the proof of Corollary 4.3, this follows from Lemma 4.2. □

MacKay & Meiss & Percival [19] are not interested in a more general definition of turnstiles since it is not relevant for their theory. But, for us, the case with *several* pivots is also interesting. 'Several pivots' can mean nonprimary orbits 'between' the primary ones or more than two primary orbits in a pair of intersecting branches:

Definition 5.9.
1. Let φ be x-simple with primary orbits $\langle p\rangle$ and $\langle q\rangle$, but assume $\#(]p,\varphi(p)[_s \cap]p,\varphi(p)[_u) = 3$. The resulting picture is called an *overtwisted turnstile* with frame p and $\varphi(p)$ and pivot q. An example is sketched in Figure 7
2. Now assume that a pair of intersecting branches has k primary orbits given by $\langle p_1\rangle,\dots,\langle p_k\rangle$ and that they satisfy $]p_1,\varphi(p_1)[_s \cap]p_1,\varphi(p_1)[_u = \{p_2,\dots,p_k\}$. We call this picture a *k-generalized turnstile* with frame p_1 and $\varphi(p_1)$ and pivots $p_2,\dots,p_k$. Note that the *wings* between p_i and p_{i+1} not always have the same symplectic volume for $1 \le i \le k$ with $p_{k+1} := \varphi(p_1)$. An example is sketched in Figure 8.

Remark 5.10.
1. Overtwisted turnstiles with frame p and $\varphi(p)$ and pivot q always look schematically like the one in Figure 7 (a). The proof is similar to the one of Remark 2.2. Moreover, there is always an associated true turnstile with frame q and $\varphi(q)$ and pivot $\varphi(p)$ as in Figure 7 (b).
2. Overtwisted turnstiles correspond to so-called 'mixed moves with primary-secondary flips' in Hohloch [16], k-generalized turnstiles correspond to so-called 'primary moves' in Hohloch [16].

Analogously to Lemma 5.8 we obtain

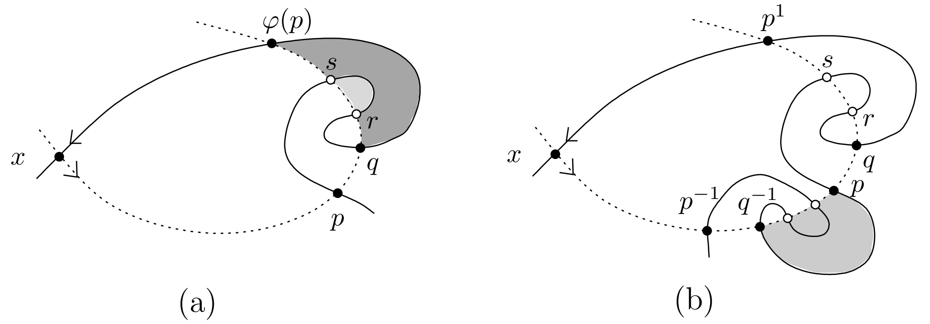


FIGURE 7. (a) Overtwisted turnstile with frame p and $\varphi(p)$ and pivot q; (b) True turnstile with frame $\varphi^{-1}(q)$ and q and pivot p

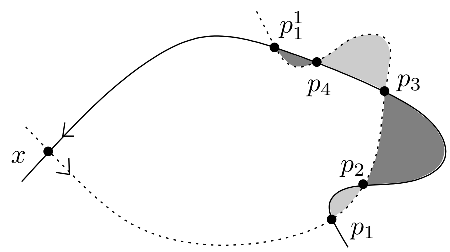


FIGURE 8. Different relative actions

Lemma 5.11. *For a k-generalized turnstile with frame p_1 and $\varphi(p_1)$ and pivots $p_2, \ldots, p_k$ holds*

$$\mathcal{F}lux_{\varphi}(\langle p_1 \rangle) = \sum_{i=1}^{\frac{k}{2}} \mid \mathcal{A}(\langle p_{2i-1} \rangle, \langle p_{2i} \rangle) \mid = \sum_{i=1}^{\frac{k}{2}} \mid \mathcal{A}(\langle p_{2i} \rangle, \langle p_{2i+1} \rangle) \mid .$$

For overtwisted turnstiles with frame p and $\varphi(p)$ and pivot q holds

$$\mid \mathcal{A}(\langle p \rangle, \langle q \rangle) \mid > \mathcal{F}lux_{\varphi}(\langle p \rangle).$$

There are also combinations of generalized and overtwisted turnstiles, but we are mainly interested in a special case of generalized turnstiles. Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple having a true turnstile with frame p and $\varphi(p)$ and pivot q. Now consider the iterate φ^n for $n \in \mathbb{N}$. We have $W^s(\varphi, x) = W^s(\varphi^n, x)$ and $W^u(\varphi, x) = W^u(\varphi^n, x)$ as sets. But the two primary orbits $\langle p \rangle$ and $\langle q \rangle$ split into $2n$ classes $\langle p^0 \rangle, \ldots, \langle p^{n-1} \rangle$ and $\langle q^0 \rangle, \ldots, \langle q^{n-1} \rangle$. In particular, we have a $2n$-generalized turnstile with frame p^0 and p^n and pivots $q^0, p^1, \ldots, q^{n-1}$ as sketched in Figure 6 (b). Lemma 5.11 and Lemma 5.8 imply

Corollary 5.12. *Under the above assumptions holds for $0 \le i \le n-1$*

$$\mathcal{F}lux_{\varphi^n}(\langle p^i \rangle) = \sum_{l=0}^{n-1} \mid \mathcal{A}(\langle p^l \rangle, \langle q^{l+1} \rangle) \mid = \sum_{l=0}^{n-1} \mid \mathcal{A}(\langle q^l \rangle, \langle p^l \rangle) \mid$$

and in particular

$$\mathcal{F}lux_{\varphi^n}(\langle p^i \rangle) = n\, \mathcal{F}lux_{\varphi}(\langle p \rangle).$$

We conclude for the growth behaviour

Remark 5.13. A homoclinic orbits forms a partial barrier for the transport of φ where the only in- and outlet is the associated turnstile. For true and generalized turnstiles, the relative action and the flux coincide, but not for overtwisted ones. Moreover, the flux grows linearly in n if n is the number of iterations of φ.

Variational principle and discrete action functional. In order to describe the flux analytically, MacKay & Meiss & Percival [19] resort to a variational principle with an action functional and refer to Mather [20] for details. Mather uses a minimax principle to proof a criterion for the existence of invariant circles of certain area-preserving diffeomorphisms of the annulus or cylinder. Mather's technique was inspired by Birkhoff's work on the billiard problem. In the following, we recall the main idea.

MacKay & Meiss & Percival [19] write the symplectic cylinder $(\mathcal{Z},\Omega)$ as $(\mathbb{R}\times S^1, ds\wedge dt)$ with coordinates $s\in\mathbb{R}$ and $t\in S^1$. The corresponding annulus also has radial coordinate s and circle coordinate t. An area preserving map f with $f(s,t) =: (\tilde{s},\tilde{t})$ on the annulus is called a *monotone twist map* if $\frac{\partial\tilde{t}}{\partial s}>0$ for all s and t. For a monotone twist map f, there exists a function F with

$$s=-\frac{\partial F(t,\tilde{t})}{\partial t}\qquad\text{and}\qquad \tilde{s}=\frac{\partial F(t,\tilde{t})}{\partial\tilde{t}}$$

called *generating function.* It implies $\frac{\partial^2 F}{\partial t\partial\tilde{t}}<0$. F uniquely determines f, and f determines F up to a constant. Set $\partial_1 F := \partial F/\partial t$ and $\partial_2 F := \partial F/\partial\tilde{t}$. A bi-infinite sequence $\tau=(\tau_i)_{i\in\mathbb{Z}}$ is called an *equilibrium sequence* if

$$\partial_2 F(\tau_{i-1},\tau_i)+\partial_1 F(\tau_i,\tau_{i+1})=0$$

for all $i\in\mathbb{Z}$. For $\sigma_i := \partial_1 F(\tau_i,\tau_{i+1})$, this can be reformulated to $f(\sigma_i,\tau_i) = (\sigma_{i+1},\tau_{i+1})$. Thus we obtain the following relation between equilibrium sequences and orbits of the monotone twist map: $(\tau_i)_i$ is an equilibrium sequence if and only if $(\sigma_i,\tau_i)_i$ is an orbit of f.

More generally, Mather introduces for sequences $\tau=(\tau_i)_{i\in\mathbb{Z}}$ with $\tau_{i+n}=\tau_i+m$ the *action functional*

$$W(\tau)=\sum_{i=0}^{n-1}F(\tau_i,\tau_{i+1}).$$

He shows that any such sequence, which maximizes W, is an equilibrium sequence. The corresponding orbit is called *Birkhoff max orbit of type* (m,n). Given such a maximizing sequence $\tau=(\tau_i)_i$, there is another maximizing sequence called τ^+ defined via $\tau_i^+ := \tau_{i+i_0}+j_0$ where mi_0+nj_0 is the minimal positive element of $\{mi+nj \mid i,j\in\mathbb{Z}\}$. Now consider such two maximizing sequences τ and τ^+. They satisfy $W(\tau)=W(\tau^+)=: W^{max}_{m,n}$. The minimax principle yields a saddle between τ and τ^+ and Mather associates to the saddle its minimax value $W^{minimax}_{\tau,m,n}$. Sequences $\tau' := (\tau_i')_{i\in\mathbb{Z}}$ with $\tau'_{i+n}=\tau'_i+m$ and $W(\tau')=W^{minimax}_{\tau,m,n}$ are shown to be equilibrium sequences and the corresponding orbits are called *Birkhoff minimax orbits of type* (m,n). Moreover, Mather shows that for two maximizing sequences τ and $\hat{\tau}$ the minimax values coincide, i.e.

$$W^{minimax}_{\tau,m,n}=W^{minimax}_{\hat{\tau},m,n}=: W^{minimax}_{m,n}.$$

With this in mind, the *difference in action* is defined as

$$\triangle W_{m,n} := W_{m,n}^{max} - W_{m,n}^{minimax}.$$

For irrational $r \in \mathbb{R}\backslash\mathbb{Q}$, Mather sets $\triangle W_r$ as the limit $\triangle W_r := \lim_{\frac{m}{n}\to r} \triangle W_{m,n}$.

Flux and difference in action. Whereas Mather [20] focuses on periodic orbits, MacKay & Meiss & Percival [19] introduce an analogous action and minimax principle for (quasi)periodic, homoclinic and heteroclinic orbits on the cylinder $(\mathbb{R} \times S^1, ds \wedge dt)$. They obtain an interesting relation between Mather's difference in action $\triangle W$ and the flux. Their intuition is given by the following calculation.

Fix m, $n \in \mathbb{N}$ and let τ be an equilibrium sequence which gives rise to a Birkhoff max orbit of type (m, n). Denote by τ' the associated minimax sequence. Choose a curve c joining (σ_0, τ_0) to (σ_1, τ_1) and passing through (σ_0', τ_0'). Concatenating c, $f \circ c$, ..., $f^{n-1} \circ c$ yields a closed curve γ. Then the range of $f \circ \gamma$ coincides with the range of γ except for the segment between (σ_0, τ_0) to (σ_1, τ_1). Denote by $\tilde{c}$ the segment between (σ_0, τ_0) and (σ_0', τ_0'). Then the flux through γ is given by

$$\begin{aligned}
\mathcal{F}lux_f(\gamma) &= \int_{f\circ\gamma} sdt - \int_\gamma sdt = \int_{f^n\circ\tilde{c}} sdt - \int_{\tilde{c}} sdt \\
&= \int \partial_2 F(\tau_{n-1}, t)dt + \int \partial_1 F(t, \tau_1)dt \\
&= \int \partial_2 F(\tau_{n-1}, t) + \partial_1 F(t, \tau_1)\, dt + \sum_{i=1}^{n-1} \int \partial_2 F(\tau_{i-1}, t) + \partial_1 F(t, \tau_{i+1})\, dt \\
&= \sum_{i=0}^{n-1} F(\tau_i, \tau_{i+1}) - \sum_{i=0}^{n-1} F(\tau_i', \tau_{i+1}') \\
&= \triangle W_{m,n}.
\end{aligned}$$

More generally, they obtain

Theorem 5.14 ([19])**.** *Let f be a Hamiltonian diffeomorphism on the cylinder which is in addition also a monotone twist map. Then holds for the periodic, quasiperiodic and heteroclinic orbits of f: The difference in action $\triangle W$ between a maximizing orbit and the associated minimax orbit coincides with the area of one wing of the turnstile, i.e. the flux through the associated curve.*

Peierl's energy barrier. Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple and let W_{br}^s and W_{br}^u be a pair of intersecting branches with primary orbits $\langle p \rangle$ and $\langle q \rangle$, both inducing true turnstiles. By Lemma 5.8, we have

$$\mathcal{F}lux_\varphi(\langle p \rangle) = \mathcal{F}lux_\varphi(\langle q \rangle) = \mathrm{gap}(\varphi, x, W_{br}^s, W_{br}^u).$$

Lemma 5.11 implies that the flux is an invariant of the pair of intersecting branches and contains the same information as the (minimal) gap of the spectrum.

If φ is not x-simple, but has primary orbits $\langle p_1 \rangle, \ldots, \langle p_k \rangle$ in $W_{br}^s \cap W_{br}^u$ inducing a k-generalized turnstile associated to $\langle p_1 \rangle$, the situation changes to

$$\mathcal{F}lux_\varphi(\langle p_1 \rangle) > \mathrm{gap}(\varphi, x, W_{br}^s, W_{br}^u).$$

If one wants to filter the homology by the action as described in Section 4, one is interested in the difference of critical action levels. For not x-simple symplectomorphisms, the flux is therefore too coarse and cannot replace the information of the action spectrum. But there are many important dynamical systems (like e.g. the Standard map in Section 7) which are x-simple with true turnstiles.

The (minimal) gap also appears in Mather's setting [20]. He calls it the *Peierls' energy barrier* and he shows it to be a lower bound of $\triangle W$. In [21], he proves a modulus of continuity of the Peierls energy barrier. The Peierl's energy barrier goes back to a physics paper by Aubry & Le Daeron & André [1]. For Lagrangian systems, it has been studied by Fathi [6], [7].

6. **Filtered Floer homology, turnstiles, flux and growth.** In this section, we point out the relation between action, turnstiles, $\triangle W$ and flux. We study how the rank of certain filtered homoclinic Floer groups depends on the number of iterations of the symplectomorphism and deduce that it grows linearly.

Turnstiles, flux and $\triangle W$ in Floer homology. First, we study x-simple symplectomorphisms and the effect of turnstiles on the boundary operator.

Proposition 6.1. *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ or $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple. Let W^s_{br} and W^u_{br} be a pair of intersecting branches with primary classes $\langle p\rangle$ and $\langle q\rangle$. Assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle)+1$ and $p \in\,]q,\varphi(q)[_s \cap\,]q,\varphi(q)[_u$. Then $\langle p\rangle$ and $\langle q\rangle$ give rise to two distinct (families of) turnstiles, more precisely p is the pivot of a turnstile with frame q and $\varphi(q)$ and q is the pivot of a turnstile with frame $\varphi^{-1}(p)$ and p. The first turnstile enters the boundary operator via*

$$\partial\langle p\rangle = \langle q\rangle - \langle q\rangle + \sum_{\substack{\langle q\rangle\neq\langle\tilde q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle\tilde q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle\tilde q\rangle)\langle\tilde q\rangle = \sum_{\substack{\langle q\rangle\neq\langle\tilde q\rangle\in\tilde{\mathcal{H}}_{pr}\\ \mu(\langle\tilde q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle\tilde q\rangle)\langle\tilde q\rangle$$

Proof. The existence of the two (families of) turnstiles follows from the very definition. But it depends on the relative Maslov index of $\langle p\rangle$ and $\langle q\rangle$ which of the turnstiles will appear in the boundary operator: If (w.l.o.g.) $\mu(\langle p\rangle) > \mu(\langle q\rangle)$ we have automatically $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $\mathcal{M}(p^j, q^j) \neq \emptyset \neq \mathcal{M}(p^j, q^{1+j})$ for $j \in \mathbb{Z}$ and the ranges of the immersions form the wings of the turnstiles with pivots p^j and frames q^j and q^{j+1}. Since $\mathcal{M}(q^j, p^{j-1}) = \emptyset = \mathcal{M}(q^j, p^j)$ the second (family of) turnstiles is ignored by the boundary operator. The situation is reversed if we assume $\mu(\langle q\rangle) = \mu(\langle p\rangle) + 1$. □

We conclude

Corollary 6.2. *In case of x-simple symplectomorphisms, turnstiles are annihilated by the boundary operator. If $\mathcal{M}(p,\tilde q) = \emptyset$ for all $\langle q\rangle \neq \langle\tilde q\rangle \in \tilde{\mathcal{H}}_{pr}$ the turnstile with pivot p lies in the kernel of the boundary operator, i.e. the pivot is a cycle.*

If we consider a non x-simple symplectomorphism with classes $\langle p_1\rangle, \dots, \langle p_k\rangle$ for (even) $k \geq 4$ and a generalized turnstile with frame p_1 and $\varphi(p_1)$ and pivots $p_2, \dots, p_k$, then at least one of moduli spaces $\mathcal{M}(p_i, p_1)$ and $\mathcal{M}(p_i, \varphi(p_1))$ is empty for all $2 \leq i \leq k$. Thus there is no analog for Proposition 6.1 for generalized turnstiles.

The following statement unites the notions of orientation preserving immersions, (relative) symplectic action of homoclinic points, wings of turnstiles, flux and Mather's difference in action $\triangle W$.

Theorem 6.3. *Let $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$ be x-simple and a monotone twist map. Let W^s_{br} and W^u_{br} be a pair of intersecting branches having a true turnstile with frame p and*

$\varphi(p)$ and pivot q and assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $v \in \mathcal{M}(p,q) \neq \emptyset$ and

$$\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle p\rangle, \langle q\rangle) = \int_v \omega = \mathcal{F}lux_\varphi(\langle p\rangle) = \triangle W_{p,q}.$$

Proof. In Lemma 4.1, we showed that the integral over an orientation preserving immersion $v \in \mathcal{M}(p,q)$ coincides with the relative action of p and q and with the symplectic area of the region enclosed by $[p,q]_s \cup [p,q]_u$. Lemma 5.8 yields the relation to the flux through $\langle p\rangle$. Eventually, Theorem 5.14 states that the flux through $\langle p\rangle$ equals Mather's difference in action $\triangle W_{p,q}$. □

Analogously we prove

Corollary 6.4. *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ be x-simple. Let W^s_{br} and W^u_{br} be a pair of intersecting branches having a true turnstile with frame p and $\varphi(p)$ and pivot q and assume w.l.o.g. $\mu(\langle p\rangle) = \mu(\langle q\rangle) + 1$. Then $v \in \mathcal{M}(p,q) \neq \emptyset$ and*

$$\mathcal{A}(\langle p\rangle) - \mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle p\rangle, \langle q\rangle) = \int_v \omega = \mathcal{F}lux_\varphi(\langle p\rangle).$$

Therefore the flux and $\triangle W$ are meaningful quantities for the action spectrum of the Floer homology. Theorem 6.3 and Corollary 6.4 imply that everything which is formulated in terms of the symplectic action spectrum can be interpreted in terms of the flux and $\triangle W$. This means that the algebraic notion of homology has a dynamical interpretation and measures dynamical quantities.

In particular, we observe the following. Let $a := \min\{\mathcal{A}(\langle p\rangle) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and $b := \max\{\mathcal{A}(\langle p\rangle) \mid \langle p\rangle \in \tilde{\mathcal{H}}_{pr}\}$ and consider the action interval $]a-\varepsilon, b]$ with $\varepsilon > 0$ small. Now let $\beta(t) := (a-\varepsilon)t + b(1-t)$ and observe $\mathscr{H}_*^{]a-\varepsilon,\beta(t)]}$. At $t = 0$, we have $\mathscr{H}_*^{]a-\varepsilon,\beta(t)]} = \mathscr{H}_*$. As soon as $t > 0$, the upper action level drops under the maximal action value with the effect that we loose at least one generator. The minimal gap resp. for x-simple symplectomorphisms the flux now indicate how much we have to increase t at least to obtain the next change in homology.

Filtered Floer homology, dynamics and growth. A large part of the study of dynamical systems deals either with telling a map φ apart from its iterates φ^n or studying the sequence of systems associated to φ^n for $n \in \mathbb{Z}$.

If one wants to study dynamical aspects of homoclinic Floer homology, then the iteration of the underlying symplectomorphism φ resp. the $\mathbb{Z}$-action of the symplectomorphism on the set of homoclinic points is a natural candidate.

Nevertheless, if one compares $H_*(\varphi, x)$ with $H_*(\varphi^n, x)$, one notices two competitive schemes. On the one hand, the number of generators multiplies by n since $\langle p\rangle$ splits up into the n classes $\langle p\rangle, \dots, \langle \varphi^{n-1}(p)\rangle$. On the other hand, a priori, we have only $\mathrm{rk}\, H_*(\varphi, x) \leq \mathrm{rk}\, H_*(\varphi^n, x)$ (cf. Hohloch [16]) with equality for all known examples.

One glance at the construction of homoclinic Floer homology tells us that we actually divided by the $\mathbb{Z}$-action and considered homoclinic *equivalence classes* $\langle p\rangle \in \tilde{\mathcal{H}}$ instead of homoclinic points $p \in \mathcal{H}$. (One can of also define a homoclinic Floer homology without dividing by the action and the construction is $\mathbb{Z}$-equivariant, cf. Hohloch [16].) And during the construction of homoclinic Floer homology, we used that the action and Maslov index are invariant under the action, i.e. $\mathcal{A}(p) = \mathcal{A}(\varphi^n(p))$ and $\mu(p) = \mu(\varphi^n(p))$ for all $n \in \mathbb{Z}$ (cf. Corollary 4.3 and Remark 2.1).

In classical Floer theory, the situation is different: Let γ be a 1-periodic Hamiltonian orbit and denote by γ^k the orbit given by iterating γ k times. In classical Floer theory, the symplectic action $\mathscr{A}$ and mean index $\mathfrak{m}$ of a 1-periodic Hamiltonian orbit γ grow linearly with the number of iterations, more precisely

$$\mathscr{A}(\gamma^k) = k\mathscr{A}(\gamma) \qquad \text{and} \qquad \mathfrak{m}(\gamma^k) = k\mathfrak{m}(\gamma).$$

This phenomenon was exploited for example by Ginzburg [14], Ginzburg & Gürel [15] and others on the way to a proof of (generalizations of) the so-called Conley Conjecture ('There are infinitely many periodic Hamiltonian orbits on certain symplectic manifolds'). The growth behaviour of the classical action functional is also an ingredient in Polterovich's [28] study of the growth of the uniform norm of the differential of a symplectomorphism and of the word length (if the symplectomorphism lies in a finitely generated subgroup). In a subsequent paper, Polterovich [29] uses growth results in order to outline and to prove a Hamiltonian version of the Zimmer program.

As mentioned above, the action $\mathcal{A}$ and the Maslov index μ are invariant under iteration of φ. But in Corollary 5.12, we showed linear growth for the absolute flux

$$\mathcal{F}lux_{\varphi^n}(\langle p\rangle) = n\mathcal{F}lux_{\varphi}(\langle p\rangle).$$

Thus the flux seems to take over the dynamical role which the action plays in the classical Floer setting. Corollary 5.12 suggests the absolute flux as a mean to study the iteration behaviour in our setting and to distinguish a symplectomorphism from its iterates. Since the flux corresponds by Theorem 6.3 and Corollary 6.4 to the spectrum of an x-simple symplectomorphisms one should also be able to observe growth for *filtered* Floer groups.

Theorem 6.5. *Let $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$ resp. $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$. Let $b \in \mathrm{Spec}(\varphi, x)$ and $0 < \varepsilon \leq \frac{1}{2}\,\mathrm{gap}(\varphi, x)$. Denote by $\langle p_1\rangle \ldots \langle p_k\rangle \in \tilde{\mathcal{H}}_{pr}$ the primary orbits with action b. Then we obtain in case $\varphi \in \mathrm{Symp}(\mathbb{R}^2)$*

$$H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x) = \mathrm{Span}_{\mathbb{Z}}\{\langle p_1\rangle, \ldots, \langle p_k\rangle\} \simeq \mathbb{Z}^k,$$
$$H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = \mathrm{Span}_{\mathbb{Z}}\{\langle p_1^0\rangle, \ldots, \langle p_1^{n-1}\rangle, \ldots, \langle p_k^0\rangle, \ldots, \langle p_k^{n-1}\rangle\} \simeq (\mathbb{Z}^k)^n$$

and in case $\varphi \in \mathrm{Ham}^c(\mathcal{Z})$

$$\mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x) = \mathrm{Span}_{\mathbb{Z}}\{\langle p_1\rangle, \ldots, \langle p_k\rangle\} \simeq \mathbb{Z}^k,$$
$$\mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = \mathrm{Span}_{\mathbb{Z}}\{\langle p_1^0\rangle, \ldots, \langle p_1^{n-1}\rangle, \ldots, \langle p_k^0\rangle, \ldots, \langle p_k^{n-1}\rangle\} \simeq (\mathbb{Z}^k)^n.$$

Proof. Consider the case of primary Floer homology first. By assumption, all points $\langle p_1\rangle, \ldots, \langle p_k\rangle$ have the same action. The action strictly decreases along the boundary operator. Thus we have $\partial\langle p_1\rangle = \cdots = \partial\langle p_k\rangle = 0$. Thus we have $H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x) = C_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x) = \mathrm{Span}_{\mathbb{Z}}\{\langle p_1\rangle, \ldots, \langle p_k\rangle\}$. The argument for $H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x)$ is the same. For cylinder Floer homology proceed analogously. □

Therefore the ranks of the filtered homology groups grow linearly:

Corollary 6.6. *The filtered homology distinguishes between φ and its iterate φ^n. Under the assumptions of Theorem 6.5 we obtain linear growth*

$$\mathrm{rk}\, H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = n\,\mathrm{rk}\, H_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x),$$
$$\mathrm{rk}\, \mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi^n, x) = n\,\mathrm{rk}\, \mathscr{H}_*^{]b-\varepsilon,b+\varepsilon]}(\varphi, x).$$

7. **An example: Chirikov's Standard map.** In this section, we compute the filtered Floer homology for the homoclinic tangle given by Chirikov's Standard map

$$f_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (s,t) \mapsto (s + t + \varepsilon \sin s, t + \varepsilon \sin s)$$

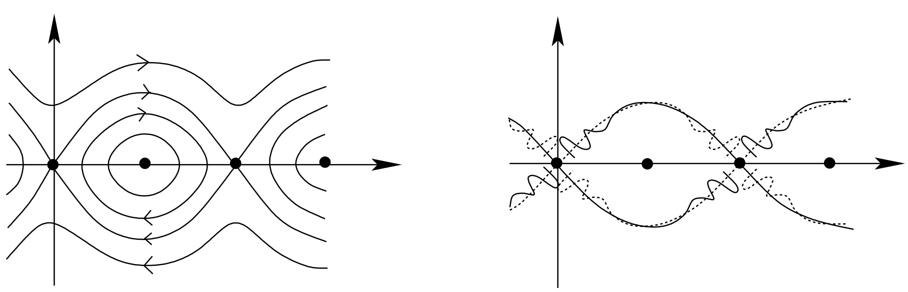
with $\varepsilon > 0$. The dynamics of the Standard map have been studied by many mathematicians and physicists, but the estimates in Melnikov's method are to coarse to predict intersection points for the (un)stable manifolds. Eventually Lazutkin [18] came up with a symplectic invariant $\mathscr{L}$ which measures the area of the parallelogram spanned by two (suitably normalized) tangent vectors taken at an intersection point of the stable and unstable manifold. Lazutkin's invariant can be used to study the intersection behaviour of the (un)stable manifolds and to compute the relative symplectic action between two primary points. Gelfreich [12] made this approach rigorous. The Standard map can be generalized to

$$f_{g,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (s,t) \mapsto (s + t + \varepsilon g(s), t + \varepsilon g(s))$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function. Gelfreich & Simo [13] studied (mainly numerically) the arising tangle if g is a polynomial, a trigonometrical polynomial or a meromorphic or rational function.

Properties of the Standard map. The f_ε has several symmetries and thus can also be seen as map on the cylinder, the torus or the square. We consider it as map on the cylinder with coordinates $(s,t) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$.

For $\varepsilon = 0$ the phase space consists of invariant circles with constant t-coordinate. For $\varepsilon > 0$ those invariant lines break up and we obtain hyperbolic fixed points at $(2\pi k, 0)$ and elliptic fixed points at $((2k+1)\pi, 0)$ for $k \in \mathbb{Z}$. The (un)stable manifolds of the hyperbolic fixed points give rise to a homoclinic tangle on the cylinder as sketched in Figure 9 (b). The Standard map corresponds to the pendulum equation $u'' = \sin u$ whose phase portrait is sketched in Figure 9 (a). Computer plots can be



(a) (b)

FIGURE 9. Phase portrait of the pendulum (a) and Standard map (b)

found in Gelfreich [12], Gelfreich & Simó [13] and MacKay & Meiss & Percival [19]. The symmetry of the Standard map implies the existence of a primary homoclinic point p at $s = \pi$. It turns out that f_ε is x-simple for all its hyperbolic fixed points x.

On the cylinder $\mathcal{Z} := (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R})$, we consider the exact symplectic form $\omega := ds \wedge dt = d(-tds) =: d\alpha$. Then $f_\varepsilon^* \alpha - \alpha = d\tilde{f}_\varepsilon$ with $\tilde{f}_\varepsilon(s,t) := -\frac{1}{2}t^2 -$

$\varepsilon t \sin(s) + \varepsilon \cos(s) - \frac{\varepsilon^2}{2} \sin^2(s)$. Thus the Standard map is Hamiltonian. Moreover f_ε is a monotone twist map. Therefore it has a generating function and an action principle in the sense of Mather (for details see [19]).

Cylinder Floer homology of the Standard map. Let $0 < R_- < R_+ < \infty$ and let $r : \mathbb{R} \to]R_-, R_+[$ be a smooth, strictly increasing function with $\lim_{s\to\pm\infty} r(s) = R_\pm$. We identify the cylinder with the annulus $\mathcal{Q} := \mathcal{Q}(R_-, R_+)$ via

$$h : S^1 \times \mathbb{R} \to \mathcal{Q}, \qquad (s,t) \mapsto (r(s)\cos t, r(s)\sin t).$$

We calculate $\det Dh(s,t) = -r'(s)r(s) < 0$, thus h is orientation reversing. For sake of readability, we will drop the minus in $\mathscr{C}_*(f_\varepsilon, x, -)$ and $\mathscr{H}_*(f_\varepsilon, x, -)$ in the following.

The homoclinic tangle is mapped from the cylinder to $\mathcal{Q} \subset \mathbb{R}^2$ as displayed in Figure 10. A larger part of the tangle is sketched in Figure 11. On the annulus,

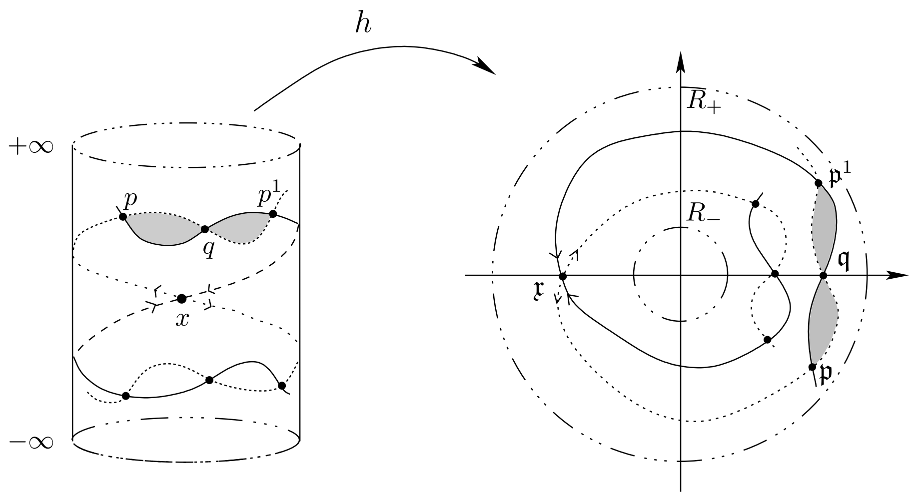


FIGURE 10. Identification of the cylinder and the annulus in $\mathbb{R}^2$

there are eight primary equivalence classes $\langle\mathfrak{p}\rangle$, $\langle\mathfrak{q}\rangle$, $\langle\tilde{\mathfrak{p}}\rangle$, $\langle\tilde{\mathfrak{q}}\rangle$, $\langle\mathfrak{r}\rangle$, $\langle\mathfrak{s}\rangle$, $\langle\tilde{\mathfrak{r}}\rangle$ and $\langle\tilde{\mathfrak{s}}\rangle$ with $\mu(\langle\tilde{\mathfrak{p}}\rangle) = -3$, $\mu(\langle\tilde{\mathfrak{q}}\rangle) = \mu(\langle\mathfrak{s}\rangle) = \mu(\langle\tilde{\mathfrak{s}}\rangle) = -2$, $\mu(\langle\mathfrak{r}\rangle) = \mu(\langle\tilde{\mathfrak{r}}\rangle) = -1$, $\mu(\langle\mathfrak{p}\rangle) = 1$ and $\mu(\langle\mathfrak{q}\rangle) = 2$. The corresponding classes on the cylinder are $\langle p\rangle$, $\langle q\rangle$, $\langle\tilde{p}\rangle$, $\langle\tilde{q}\rangle$, $\langle r\rangle$, $\langle s\rangle$, $\langle\tilde{r}\rangle$ and $\langle\tilde{s}\rangle$ with

$$\mu(\langle\tilde{p}\rangle) = -3, \quad \mu(\langle\tilde{q}\rangle) = \mu(\langle s\rangle) = \mu(\langle\tilde{s}\rangle) = -2, \quad \mu(\langle r\rangle) = \mu(\langle\tilde{r}\rangle) = -1,$$
$$\mu(\langle p\rangle) = 1, \quad \mu(\langle q\rangle) = 2.$$

Their homoptoy classes in $\pi_1(S^1 \times \mathbb{R}, x)$ are

$$[\langle p\rangle] = [\langle q\rangle] = 1 \in \mathbb{Z},$$
$$[\langle\tilde{p}\rangle] = [\langle\tilde{q}\rangle] = -1 \in \mathbb{Z},$$
$$[\langle r\rangle] = [\langle s\rangle] = [\langle\tilde{r}\rangle] = [\langle\tilde{s}\rangle] = [x] = 0 \in \mathbb{Z}.$$

We obtain as chain groups

$$\mathscr{C}_2(f_\varepsilon, x, [\cdot] = 1) = \mathbb{Z}\langle q\rangle \quad \text{and} \quad \mathscr{C}_1(f_\varepsilon, x, [\cdot] = 1) = \mathbb{Z}\langle p\rangle$$

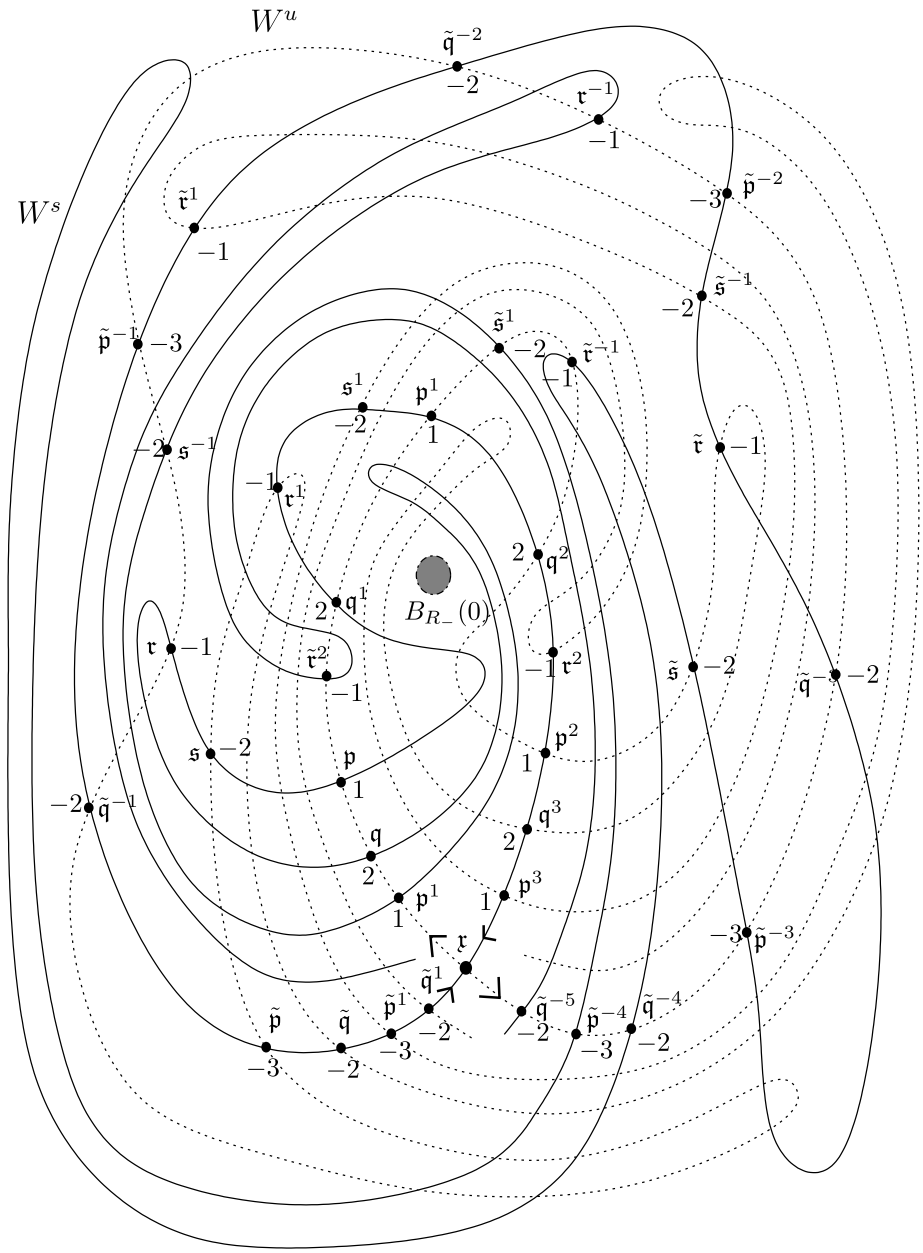


FIGURE 11. Cylinder Floer homology of the Standard map

with $\partial\langle q\rangle = \langle p\rangle - \langle p^1\rangle = 0$ and $\partial\langle p\rangle = 0$ and thus $\mathscr{H}_*(f_\varepsilon, x, [\cdot] = 1) \simeq \mathscr{C}_*(f_\varepsilon, x, [\cdot] = 1)$ with

$$\mathscr{H}_2(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z} \quad \text{and} \quad \mathscr{H}_1(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z}.$$

Moreover, we compute

$$\mathscr{C}_{-1}(f_\varepsilon, x, [\cdot] = 0) = \mathbb{Z}\langle r\rangle \oplus \mathbb{Z}\langle \tilde{r}\rangle \quad \text{and} \quad \mathscr{C}_{-2}(f_\varepsilon, x, [\cdot] = 1) = \mathbb{Z}\langle s\rangle \oplus \mathbb{Z}\langle \tilde{s}\rangle$$

with $\partial\langle r\rangle = \langle s\rangle - \langle s^{-1}\rangle = 0$ and $\partial\langle\tilde r\rangle = -\langle\tilde s\rangle + \langle\tilde s^1\rangle = 0$ and $\partial\langle s\rangle = 0$ and $\partial\langle\tilde s\rangle = 0$. Thus we have $\mathscr{H}_*(f_\varepsilon, x, [\cdot] = 0) \simeq \mathscr{C}_*(f_\varepsilon, x, [\cdot] = 0)$ and therefore

$$\mathscr{H}_{-1}(f_\varepsilon, x, [\cdot] = 0) \simeq \mathbb{Z}\oplus\mathbb{Z} \quad \text{and} \quad \mathscr{H}_{-2}(f_\varepsilon, x, [\cdot] = 0) \simeq \mathbb{Z}\oplus\mathbb{Z}.$$

It remains to calculate

$$\mathscr{C}_{-2}(f_\varepsilon, x, [\cdot] = -1) = \mathbb{Z}\langle\tilde q\rangle \quad \text{and} \quad \mathscr{C}_{-3}(f_\varepsilon, x, [\cdot] = -1) = \mathbb{Z}\langle\tilde p\rangle$$

with $\partial\langle\tilde q\rangle = \langle\tilde p^1\rangle - \langle\tilde p\rangle$ and $\partial\langle\tilde p\rangle = 0$. Thus we obtain $\mathscr{H}_*(f_\varepsilon, x, [\cdot] = -1) \simeq \mathscr{C}_*(f_\varepsilon, x, [\cdot] = -1)$ with

$$\mathscr{H}_{-2}(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z} \quad \text{and} \quad \mathscr{H}_{-3}(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z}.$$

Due to symmetry, we have $\mathcal{A}(\langle p\rangle) = \mathcal{A}(\langle\tilde p\rangle)$ and $\mathcal{A}(\langle q\rangle) = \mathcal{A}(\langle\tilde q\rangle)$ and $\mathcal{A}(\langle r\rangle) = \mathcal{A}(\langle\tilde r\rangle)$ and $\mathcal{A}(\langle s\rangle) = \mathcal{A}(\langle\tilde s\rangle)$. There is an asymptotic formula in Gelfreich [12] for the relative action given by

$$\mathcal{A}(q,p) \overset{as}{=} \frac{2}{\pi} e^{-\frac{\pi^2}{h}} \left(\sum_{n=0}^{\infty} \kappa^{2n} \mathscr{L}_n \right)$$

where $\varepsilon = 4\sinh^2\frac{\kappa}{2}$ which can be computed numerically. By Theorem 6.3, it coincides with $\mathcal{F}lux_{f_\varepsilon}(\langle p\rangle) = \mathcal{F}lux_{f_\varepsilon}(\langle q\rangle)$

Filtered Floer homology. We demonstrate the effect of the action filtration on the homology $\mathscr{H}_*(f_\varepsilon, x, [\cdot] = 1)$. If $[\mathcal{A}(\langle p\rangle), \mathcal{A}(\langle q\rangle)] \subset\ ]a,b]$ then $\mathscr{H}_*^{]a,b]}(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z}\oplus\mathbb{Z}$. For $\mathcal{A}(\langle p\rangle) < b < \mathcal{A}(\langle q\rangle)$ we obtain $\mathscr{H}_*^{]a,b]}(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z}$ since $\mathscr{C}_*^{]a,b]}(f_\varepsilon, x, [\cdot] = 1) \simeq \mathbb{Z}\langle p\rangle$. And if $b < \mathcal{A}(\langle p\rangle)$ then $\mathscr{H}_*^{]a,b]}(f_\varepsilon, x, [\cdot] = 1) = 0$.

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