Floer homology groups in hyperkähler geometry

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1 The hypersymplectic action functional

In this paper we outline the construction of a hyperkähler analogue of symplectic Floer homology [6, 7, 8]. The theory is a based on the gradient flow of the hypersymplectic action functional on the space of maps from a suitable 3-manifold M to a hyperkähler manifold X. The gradient flow lines satisfy a nonlinear analogue of the Dirac equation and can also be viewed as hyperkähler analogues of holomorphic curves. If the target manifold X is flat the analysis in symplectic Floer theory can be adapted to the hyperkähler setting as explained below. The full details are given in [16]. To extend the theory to the non-flat case one must deal with new bubbling and compactness phenomena.

We assume throughout that X is a hyperkähler manifold with complex structures I, J, K and symplectic forms $\omega_1, \omega_2, \omega_3$. We also assume that M is a compact oriented 3-manifold equipped with a volume form $\sigma \in \Omega^3(M)$ and a positive frame $v_1, v_2, v_3 \in \mathrm{Vect}(M)$ of the tangent bundle. Associated to these data is a natural 1-form on the space $\mathscr{F} := C^\infty(M, X)$ of smooth functions $f : M \to X$ defined by

$$\hat{f} \mapsto \int_M \Big(\omega_1(\partial_{v_1} f, \hat{f}) + \omega_2(\partial_{v_2} f, \hat{f}) + \omega_3(\partial_{v_3} f, \hat{f})\Big)\,\sigma \qquad (1)$$

for $\hat{f} \in T_f\mathscr{F} = \Omega^0(M, f^*TX)$. This 1-form is closed if and only if the vector fields v_i are volume preserving, i.e. $\mathcal{L}_{v_i}\sigma = 0$. Since every closed oriented 3-manifold is parallelizable it admits a volume preserving frame (Gromov [14, Section 2.4.3]). Our main examples are the 3-torus with the coordinate vector fields and the 3-sphere with the standard hypercontact structure.

The zeros of the 1-form (1) are the solutions $f : M \to X$ of the nonlinear elliptic first order partial differential equation

$$\bar{\partial}(f) := I\partial_{v_1} f + J\partial_{v_2} f + K\partial_{v_3} f = 0. \tag{2}$$

This is a nonlinear analogue of the Dirac equation that was first introduced by Taubes [22]. In his paper Taubes also considered natural extensions of (2) for more general 3-manifolds where the target manifold X is equipped with a suitable group action of SO(3) or S^1. This leads to interesting analogues of the Seiberg–Witten equations.

Obviously, the constant functions are solutions of (2). When $M = S^3$ other solutions arise from the composition of rational curves with suitable Hopf fibrations (see Example 3.1 below). When $M = \mathbb{T}^3$ solutions can be obtained from elliptic curves. In the case $M = \Gamma \backslash \mathrm{G}$ solutions arise from the composition of K-holomorphic curves $\Sigma \to X$ with $\pi : M \to \Sigma$. These examples are homologically trivial, even though Hopf-fibrations over holomorphic spheres in the K3-surface do represent nontrivial homotopy classes in π_3. A homologically nontrivial example with target manifold $X := \mathbb{H}/\mathbb{Z}^4$ with its standard hyperkähler structure and domain $M := \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ with vector fields $v_i = \partial/\partial t_i$ is given by $f(t) := t_1 + it_2 + (1 + j)t_3$.

In Taubes' setting one of the motivating ideas was to obtain nonlinear analogues of the Seiberg–Witten equations and possibly new invariants of smooth three and four manifolds. Our motivation was to develop a new type of Floer theory for hyperkähler manifolds in analogy with symplectic Floer theory. This led us to study the L^2 gradient flow equation of the 1-form (1) (see (6) below), which is also known as the Cauchy–Riemann–Fueter equation and plays a similar role in quaternionic geometry as the Cauchy–Riemann equation does in complex geometry (see [15]).

Hypercontact structures

A *hypercontact structure* on a 3-manifold M is a triple of contact forms $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Omega^1(M, \mathbb{R}^3)$ such that

$$\alpha_1 \wedge d\alpha_1 = \alpha_2 \wedge d\alpha_2 = \alpha_3 \wedge d\alpha_3 =: \sigma$$

and $\alpha_i \wedge d\alpha_j + \alpha_j \wedge d\alpha_i = 0$ for $i \neq j$. The Reeb vector fields v_1, v_2, v_3 are pointwise linearly independent and preserve the volume form σ. The hypercontact structure is called *positive* if they form a positive frame of the tangent bundle. In this setting the 1-form (1) is the differential of the action functional $\mathscr{A} : \mathscr{F} \to \mathbb{R}$ defined by

$$\mathscr{A}(f) := -\int_M \Big(\alpha_1 \wedge f^*\omega_1 + \alpha_2 \wedge f^*\omega_2 + \alpha_3 \wedge f^*\omega_3\Big). \tag{3}$$

A positive hypercontact structure is called a *Cartan structure* if the α_i form a dual frame of the cotangent bundle, i.e. $\alpha_i(v_j) = \delta_{ij}$. In the Cartan case

$$\kappa := d\alpha_1(v_2, v_3) = d\alpha_2(v_3, v_1) = d\alpha_3(v_1, v_2)$$

is constant and $d\alpha_i = \kappa\alpha_j \wedge \alpha_k$ and $[v_i, v_j] = \kappa v_k$ for every cyclic permutation i, j, k of $1, 2, 3$. (We use the sign convention of [19] for the Lie bracket.)

The archetypal example is the 3-sphere $M = S^3$, understood as the unit quaternions, with

$$v_1(y) = \mathbf{i}y, \qquad v_2(y) = \mathbf{j}y, \qquad v_3(y) = \mathbf{k}y.$$

Hypercontact structures were introduced by Geiges–Gonzalo [10, 11, 12]. They use the term taut contact sphere for what we call a hypercontact structure. They proved that every Cartan hypercontact 3-manifold is diffeomorphic to a quotient of the 3-sphere by the right action of a finite subgroup of Sp(1).

Tori

Let $M = \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the standard 3-torus equipped with the standard volume form $\sigma = dt_1 \wedge dt_2 \wedge dt_3$ and $v_i = \sum_{j=1}^3 a_{ij}\partial_j$ where $A = (a_{ij})_{i,j=1}^3$ is a nonsingular real 3×3 matrix. In this case the lift of the 1-form (1) to the universal cover $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ is the differential of the function

$$\mathscr{A} = \sum_{i,j=1}^{3} a_{ij}\mathscr{A}_{ij} : \widetilde{\mathscr{F}} \to \mathbb{R} \tag{4}$$

where $\mathscr{A}_{ij}(f)$ denotes the ω_i-symplectic action of the loop $t_j \mapsto f(t)$, averaged over the remaining two variables t_k, t_ℓ with $k, \ell \neq j$. If X is flat and $\mathscr{F}_0 \subset \mathscr{F}$ denotes the space of contractible maps $f : \mathbb{T}^3 \to X$ then $\mathscr{A}$ descends to $\mathscr{F}_0$. Explicitly, we have

$$\mathscr{A}_{ij}(f) := -\int_0^1 \int_0^1 \int_{\mathbb{D}} u_{t_k,t_\ell}^* \omega_i \, dt_k \, dt_\ell$$

for $f \in \mathscr{F}_0$, where $u_{t_k,t_\ell} : \mathbb{D} \to X$ is a smooth family of maps satisfying $u_{t_k,t_\ell}(e^{2\pi i t_j}) = f(t_1, t_2, t_3)$.

Hyperbolic spaces

A third class of examples arises from unit tangent bundles of higher genus surfaces or equivalently from quotients of the group $\mathrm{G} := \mathrm{PSL}(2;\mathbb{R})$. Let $\mathcal{H} \subset \mathbb{C}$ denote the upper half plane and $\mathcal{P} := \{(z,\zeta) \in \mathbb{C}^2 \,|\, \mathrm{Im}(z) = |\zeta|\}$ the unit tangent bundle of $\mathcal{H}$. The group G acts freely and transitively on $\mathcal{P}$ by

$$g_*(z,\zeta) := \left(\frac{az+b}{cz+d}, \frac{\zeta}{(cz+d)^2}\right), \qquad g =: \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}(2;\mathbb{R})..$$

Now let $\Gamma \subset \mathrm{PSL}(2;\mathbb{R})$ be a discrete subgroup acting freely on $\mathcal{H}$ such that the quotient $\Sigma := \Gamma\backslash\mathcal{H}$ is a closed Riemann surface. Then the 3-manifold

$$M := \Gamma\backslash\mathrm{G}$$

is diffeomorphic to the unit tangent bundle $T_1\Sigma = \Gamma\backslash\mathcal{P}$ via $[g] \mapsto [g_*(i,1)]$. The group G carries a natural bi-invariant volume form $\sigma \in \Omega^3(\mathrm{G})$ given by

$$\sigma(g\xi, g\eta, g\zeta) := \frac{1}{2}\mathrm{trace}([\xi,\eta]\zeta)$$

for $\xi,\eta,\zeta \in \mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = \mathfrak{sl}(2;\mathbb{R})$. This volume form descends to M and is invariant under the right action of G. Now consider the traceless matrices

$$\xi_1 := \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), \qquad \xi_2 := \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \qquad \xi_3 := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right).$$

The resulting vector fields $v_i(g) := g\xi_i$ on G are Γ-equivarient and preserve the volume form σ. Hence they descend to volume preserving vector fields on M (still denoted by v_i) and so the 1-form (1) is closed in this setting.

Note that $\sigma(v_1,v_2,v_3) = 2$ and $d\pi(v_3) = 0$, $d\pi(v_1) = id\pi(v_2)$. The Lie brackets of the vector fields v_i are given by

$$[v_2,v_3] = -2v_1, \qquad [v_3,v_1] = -2v_2, \qquad [v_1,v_2] = 2v_3$$

(because the ξ_i act on G on the right). Hence, if $\alpha_i \in \Omega^1(M)$ denote the 1-forms dual to the vector fields v_i, we have

$$d\alpha_1 = -2\alpha_2\wedge\alpha_3, \qquad d\alpha_2 = -2\alpha_3\wedge\alpha_1, \qquad d\alpha_3 = 2\alpha_1\wedge\alpha_2.$$

This implies that the 1-form (1) is the differential of the action functional

$$\mathscr{A}(f) := \int_M \left(\alpha_1\wedge f^*\omega_1 + \alpha_2\wedge f^*\omega_2 - \alpha_3\wedge f^*\omega_3\right).$$

However, in this setting the energy identity (7) discussed below does not help in the compactness proof. This is the reason why we do not include the higher genus case in our discussion in the main part of this paper.

2 Floer theory

In [16] we prove an existence result for the solutions of the perturbed nonlinear Dirac equation

$$\partial\!\!\!/_H(f) := I\partial_{v_1}f + J\partial_{v_2}f + K\partial_{v_3}f - \nabla H(f) = 0. \tag{5}$$

Here $H : X \times M \to \mathbb{R}$ is a smooth function and we denote by $\nabla H(f)$ the gradient with respect to the first argument. The linearized operator for this equation is self adjoint and we call a solution $f : M \to X$ of (5) *nondegenerate* if the linearized operator is bijective. In the nondegenerate case, and when X is flat, one can count the solutions with signs, however, it turns out that this count gives zero. Nevertheless we have the following hyperkähler analogue of the Conley-Zehnder theorem confirming the Arnold conjecture for the torus [2]. In fact, in the torus case with $v_1 = \partial/\partial t_1$ the solutions of (5) can be interpreted as the periodic orbits of a suitable infinite dimensional Hamiltonian system.

Theorem 2.1. [16] *Let M be either a compact Cartan hypercontact 3-manifold (with Reeb vector fields v_i) or the 3-torus (with a constant frame v_i). Let X be a compact flat hyperkähler manifold. Then the space of solutions of (5) is compact. Moreover, if the contractible solutions are all nondegenerate, then their number is bounded below by the sum of the $\mathbb{Z}_2$-Betti numbers of X. In particular, equation (5) has a contractible solution for every H.*

The proof of Theorem 2.1 is based on the observation that the solutions of (5) are the critical points of the perturbed *hypersymplectic action functional* $\mathscr{A}_H(f) := \mathscr{A}(f) - \int_M H(f)\sigma$. As in symplectic Floer theory, this functional is unbounded above and below, and the Hessian has infinitely many positive and negative eigenvalues. Thus the standard techniques of Morse theory are not available for the study of the critical points. However, with appropriate modifications, the familiar techniques of Floer homology carry over to the present case, at least when X is flat, and thus give rise to natural Floer homology groups for a pair (M, X).

The Floer chain complex is generated by the solutions of (5). The boundary operator is determined by the finite energy solutions $u : \mathbb{R} \times M \to X$ of the negative gradient flow equation

$$\partial_s u + I\partial_{v_1}u + J\partial_{v_2}u + K\partial_{v_3}u = \nabla H(u). \tag{6}$$

The Fredholm theory for these equations is standard. The index is given by the spectral flow and depends only on the endpoints. Transversality can be established along the lines of [9]. A key ingredient in the compactness proof is the energy identity

$$\mathscr{E}(f) = \frac{1}{2}\int_M |df|^2 = \frac{1}{2}\int |I\partial_{v_1} f + J\partial_{v_2} f + K\partial_{v_3} f|^2 - \int_M \sum_{i=1}^{3} \varepsilon_i \wedge f^*\omega_i \quad (7)$$

for $f : M \to X$, where the $\varepsilon_i \in \Omega^1(M)$ are dual to the vector fields v_i. In the torus case these forms are closed and thus the last term in (7) is a topological invariant. In the Cartan hypercontact case this term is the hypersymplectic action $\mathscr{A}(f)$.

To compute the Floer homology groups we choose a Morse–Smale function $H : X \to \mathbb{R}$ and study the equation

$$\partial_s u + \varepsilon^{-1}\left(I\partial_{v_1} u + J\partial_{v_2} u + K\partial_{v_3} u\right) = \nabla H(u) \quad (8)$$

for small values of ε. The gradient lines of H are solutions of this equation and in [16] we prove that, for $\varepsilon > 0$ sufficiently small, there are no other contractible solutions. This implies that our Floer homology groups $\mathrm{HF}_*(M, X)$ are isomorphic to the singular homology $H_*(X; \mathbb{Z}_2)$.

Theorem 2.2. **[16]** *Let M be either a compact Cartan hypercontact 3-manifold (with Reeb vector fields v_i) or the 3-torus (with a constant frame v_i). Let X be a compact flat hyperkähler manifold and fix a class $\tau \in \pi_0(\mathscr{F})$. Then, for a generic perturbation $H : X \times M \to \mathbb{R}$, there is a natural Floer homology group $\mathrm{HF}_*(M, X, \tau; H)$ associated to a chain complex generated by the solutions of (5) where the boundary operator is defined by counting the solutions of (6). The Floer homology groups associated to different choices of H are naturally isomorphic. Moreover, for the component τ_0 of the constant maps there is a natural isomorphism $\mathrm{HF}_*(M, X, \tau_0; H) \cong H_*(X; \mathbb{Z}_2)$.*

Remark 2.3. An important technical ingredient in the compactness proof is the estimate

$$\mathscr{L} e_u + r_u \geq -A - B(e_u)^{3/2} \quad (9)$$

for the energy density

$$e_u := \frac{1}{2}|\partial_s u|^2 + \frac{1}{2}\sum_{i=1}^{3} |\partial_{v_i} u|^2$$

and the scalar curvature $r_u : \mathbb{R} \times M \to \mathbb{R}$ along u given by

$$r_u := 2\sum_{j=1}^{3} \langle R(\partial_s u, \partial_{v_j} u)\partial_{v_j} u, \partial_s u\rangle + \sum_{i,j=1}^{3} \langle R(\partial_{v_i} u, \partial_{v_j} u)\partial_{v_j} u, \partial_{v_i} u\rangle.$$

In the flat case we have $r_u = 0$. Since the exponent $3/2$ is equal to the critical exponent $(n+2)/n$ for $n = 4$ (with regard to compactness questions and mean value estimates), the estimate (9) then gives rise to a crucial mean value inequality (see [23]), and hence to the relevant compactness result for the solutions of (6) (see [16]).

Remark 2.4. For general hyperkähler manifolds the estimate (9) implies

$$\mathscr{L} e \geq -c(1 + e^2).$$

In dimensions $n = 3, 4$ the exponent 2 is larger than the critical exponent $(n + 2)/n$. For the critical points f of $\mathscr{A}_H$ this means that the energy

$$\mathscr{E}(f) = \frac{1}{2} \int_M |df|^2 \ \mathrm{dvol}_M$$

does not control the sup norm of $|df|$ even if we assume that there is no energy concentration near points. This is related to noncompactness phenomena that can be easily observed in examples. Namely, composing a holomorphic sphere $u : S^2 \to X$ (for $J_\lambda = \lambda_1 I + \lambda_2 J + \lambda_3 K$) with a suitable Hopf fibration $h : S^3 \to S^2$ gives rise to a solution $f := u \circ h : S^3 \to X$ of (2) (see Example 3.1 below). Thus, if $u_\nu : S^2 \to X$ is a sequence of J_λ-holomorphic curves whose derivatives blow up near z_0, then $f_\nu := u_\nu \circ h$ is a sequence of solutions of (2) whose derivatives blow up along the Hopf circle $h^{-1}(z_0)$, while the energy remains bounded.

Remark 2.5. The precise condition we need for extending the standard techniques of Floer theory to our setting is that X has nonpositive sectional curvature. As every hyperkähler manifold has vanishing Ricci tensor, nonpositive sectional curvature implies that X is flat and hence is a quotient of a hyperkähler torus by a finite group. An example is the quotient of the standard 12-torus $\mathbb{H}^3/\mathbb{Z}^{12}$ by the $\mathbb{Z}_2$-action determined by $(x, y, z) \mapsto (y, x, z + 1/2)$.

3 A more general setting

There is conjecturally a much richer theory which provides Floer homological invariants for all triples (M, X, τ), consisting of a Cartan hypercontact 3-manifold M, a compact hyperkähler manifold X, and a homotopy class τ of maps from M to X. One basic observation is that every holomorphic sphere in a hyperkähler manifold gives rise to a solution of (2) on $M = S^3$. Another point is that $\pi_3(X)$ can be a very rich group. For example, the third homotopy group of the K3-surface has 253 generators (see [1, Appendix]).

Example 3.1. Think of the 3-sphere as the unit sphere in the quaternions $\mathbb{H} \cong \mathbb{R}^4$ and of the 2-sphere as the unit sphere in the imaginary quaternions $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. For $\lambda = \lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k} \in S^2$ define $h_\lambda : S^3 \to S^2$ by $h_\lambda(y) := -\bar{y}\lambda y$ and denote

$$J_\lambda := \lambda_1 I + \lambda_2 J + \lambda_3 K, \qquad \omega_\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3.$$

If $u : S^2 \to X$ is a J_λ-holomorphic sphere then

$$f := u \circ h_\lambda : S^3 \to X$$

is a critical point of $\mathscr{A}$ and

$$E(u) = \frac{1}{2}\int_{S^2} |du|^2 = \int_{S^2} u^*\omega_\lambda = \frac{1}{2\pi}\mathscr{A}(u \circ h_\lambda).$$

To see this, assume $\lambda = \mathbf{i}$ and write

$$h_1(y) := -\bar{y}\mathbf{i}y, \qquad h_2(y) := -\bar{y}\mathbf{j}y, \qquad h_3(y) := -\bar{y}\mathbf{k}y.$$

These functions satisfy $\partial_{v_i} h_i = 0$ and $\partial_{v_j} h_i = -\partial_{v_i} h_j = 2h_k$ for every cyclic permutation i, j, k of $1, 2, 3$. Hence $h_1 \wedge \partial_{v_3} h_1 = \partial_{v_2} h_1$. If $u : S^2 \to X$ is an I-holomorphic sphere it follows that the function $f := u \circ h_1$ satisfies $\partial_{v_1} f = 0$ and $I\partial_{v_3} f = \partial_{v_2} f$ and hence is a solution of (2). Moreover, for every $\sigma \in \Omega^2(S^2)$, we have

$$2\pi \int_{S^2} \sigma = -\int_{S^3} \alpha_1 \wedge h_1^*\sigma$$

(When σ is exact both sides are zero. Since $-\alpha_1 \wedge h_1^*\mathrm{dvol}_{S^2} = 4\mathrm{dvol}_{S^3}$ the value of the factor follows from $\mathrm{Vol}(S^2) = 4\pi$ and $\mathrm{Vol}(S^3) = 2\pi^2$.) With $\sigma = u^*\omega_1$ this implies $2\pi \int_{S^2} u^*\omega_1 = -\int_{S^2} \alpha_1 \wedge h_1^* u^*\omega_1 = \mathscr{A}(u \circ h_1)$. Here the last equation follows from the fact that $u^*\omega_2 = u^*\omega_3 = 0$ for every I-holomorphic curve u.

The main technical difficulty in setting up the Floer theory for general hyperkähler manifolds is to establish a suitable compactness theorem. In contrast to the familiar theory the derivatives for a sequence of solutions of (5) or (6) will not just blow up at isolated points but along codimension-2 subsets as explained in Remark 2.4 above. This phenomenon is analogous to the codimension 4 bubbling in Donaldson–Thomas theory [4].

4 Additional structures

Floer–Donaldson theory

Let Σ be a hyperkähler 4-manifold with complex structures $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and symplectic forms $\sigma_1, \sigma_2, \sigma_3$. Consider the elliptic partial differential equation

$$du - Idu\mathbf{i} - Jdu\mathbf{j} - Kdu\mathbf{k} = 0 \tag{10}$$

for smooth maps $u : \Sigma \to X$. This is sometimes called the Cauchy–Riemann–Fueter equation and it has been widely studied (see [22], [15, Chapter 3] and references). For $\Sigma = \mathbb{R} \times M$ with its standard hyperkähler structure (see below) equation (10) is equivalent to (6) with $H = 0$. The solutions of (10) satisfy the energy identity

$$E(u) = \frac{1}{8}\int_\Sigma |du - Idu\mathbf{i} - Jdu\mathbf{j} - Kdu\mathbf{k}|^2 \,\mathrm{dvol}_\Sigma - \int_\Sigma \sum_{i=1}^3 \sigma_i \wedge u^*\omega_i. \tag{11}$$

The linearized operator $\mathscr{D}_u : \Omega^0(\Sigma, u^*TX) \to \Omega^1_{\mathbb{H}}(\Sigma, u^*TX)$ takes values in the space of 1-forms on Σ with values in u^*TX that are complex linear with respect to I, J, and K. When Σ is closed this operator is Fredholm between appropriate Sobolev completions and its index is

$$\mathrm{ind}(\mathscr{D}_u) = -\big\langle c_2(TX), u_*[\Sigma]\big\rangle + \frac{\chi(\Sigma)}{24}\dim^{\mathbb{R}} X, \tag{12}$$

where $\chi(\Sigma)$ is the Euler characteristic. Equation (12) continues to hold in the case $\Sigma = S^1 \times M$ with its natural quaternionic structure. We sketch a proof below. Conjecturally, there should be Gromov–Witten type invariants obtained from intersection theory on the moduli space of solutions of (10).

One can also consider hyperkähler 4-manifolds Σ with cylindrical ends $\iota^\pm : \mathbb{R}^\pm \times M^\pm \to \Sigma$. Here we assume that $M^\pm$ is either a Cartan hypercontact 3-manifold or a 3-torus. Then $\mathbb{R}^\pm \times M^\pm$ has a natural flat hyperkähler structure [3, 11]. In the hypercontact case the symplectic forms are $\omega_i = \kappa^{-1} d(e^{-\kappa s}\alpha_i) = e^{-\kappa s}\big(-ds \wedge \alpha_i + \alpha_j \wedge \alpha_k\big)$ and in the torus case they are $\omega_i = -ds \wedge \alpha_i + \alpha_j \wedge \alpha_k$ for every cyclic permutation i, j, k of $1, 2, 3$. In both cases the complex structure $\mathbf{i}$ is given by $\partial_s \mapsto -v_1$, $v_1 \mapsto \partial_s$, $v_2 \mapsto v_3$, $v_3 \mapsto -v_2$ and similarly for $\mathbf{j}$ and $\mathbf{k}$. We assume that the embeddings $\iota^\pm$ are hyperkähler isomorphisms onto their images and that the complement $\Sigma \setminus (\mathrm{im}\,\iota^+ \cup \mathrm{im}\,\iota^-)$ has a compact closure. Alternatively, it might also be interesting to consider hyperkähler 4-manifolds with asymptotically cylindrical ends as in [17, 18]. One can then (conjecturally) use the solutions of equation (10) with Hamiltonian perturbations on the cylindrical ends to obtain a homomorphism $\mathrm{HF}_*(M^-,X)\to\mathrm{HF}_*(M^+,X)$ respectively $\mathrm{HF}^*(M^+,X)\to\mathrm{HF}^*(M^-,X)$.

Proof of the index formula. We relate $\mathscr{D}_u$ to a Dirac operator on Σ associated to a spin^c structure. On Σ we have a Hermitian vector bundle $W=W^+\oplus W^-$ where

$$W^+:=u^*TX\oplus u^*TX,\quad W^-:=\mathrm{Hom}_{\mathbb{H}}(T\Sigma,u^*TX)\oplus\mathrm{Hom}_I(T\Sigma,u^*TX).$$

Here $\mathrm{Hom}_{\mathbb{H}}(T\Sigma,u^*TX)$ denotes the bundle of quaternionic homomorphisms and $\mathrm{Hom}_I(T\Sigma,u^*TX)$ denotes the bundle of homomorphisms that are complex linear with respect to I and complex anti-linear with respect to J and K. The complex structures on W^+ and W^- are given by $(\xi_1,\xi_2)\mapsto(I\xi_2,I\xi_1)$. The spin^c structure $\Gamma:T\Sigma\to\mathrm{End}(W)$ has the form

$$\Gamma(v):=\begin{pmatrix}0 & -\gamma(v)^*\\ \gamma(v) & 0\end{pmatrix}$$

for $v\in T_z\Sigma$ where $\gamma(v):W_z^+\to\mathbb{W}_z^-$ is given by

$$\gamma(v)(\xi_1,\xi_2):=(\pi_{\mathbb{H}}(\langle v,\cdot\rangle\xi_1),\pi_I(\langle v,\cdot\rangle\xi_2)).$$

Here $\pi_{\mathbb{H}},\pi_I:\mathrm{Hom}_{\mathbb{R}}(T\Sigma,u^*TX)\to\mathrm{Hom}_{\mathbb{R}}(T\Sigma.u^*TX)$ denote the projections

$$\pi_{\mathbb{H}}(A):=A-IA\mathbf{i}-JA\mathbf{j}-KA\mathbf{k},\quad \pi_I(A):=A-IA\mathbf{i}+JA\mathbf{j}+KA\mathbf{k}.$$

The Dirac operator $D:\Omega^0(\Sigma,W^+)\to\Omega^0(\Sigma,W^-)$ is the direct sum of $\mathscr{D}_u$ and $\widetilde{\mathscr{D}}_u:\Omega^0(\Sigma,u^*TX)\to\Omega^1_I(\Sigma,u^*TX)$ given by $\widetilde{\mathscr{D}}_u\xi:=\pi_I(\nabla\xi)$. These operators have the same index and hence

$$2\mathrm{ind}^{\mathbb{R}}(\mathscr{D}_u)=\mathrm{ind}^{\mathbb{R}}(D)=\frac{\mathrm{rank}^{\mathbb{R}}(W^+)}{24}\chi(\Sigma)+\frac{1}{2}\langle c_1(W^+)^2-2c_2(W^+),[\Sigma]\rangle.$$

The last equation follows from the Atiyah–Singer index theorem (see [20]). Alternatively, one can identify $\Omega^0(\Sigma,W^+)$ with $\Omega^{0,0}(\Sigma,u^*TX)\oplus\Omega^{2,0}(u^*TX)$ via $(\xi_1,\xi_2)\mapsto(\xi_1+\xi_2,J(\xi_2-\xi_1)\omega_{\mathbf{j}}+K(\xi_2-\xi_1)\omega_{\mathbf{k}})$ and the space $\Omega^0(\Sigma,W^-)$ with $\Omega^{1,0}(\Sigma,u^*TX)$ via $(\alpha_1,\alpha_2)\to\alpha_1+\alpha_2$. Under these identifications the Dirac operator D corresponds to the twisted Cauchy–Riemann operator $\partial+\partial^*:\Omega^{\mathrm{ev},0}(\Sigma,u^*TX)\to\Omega^{\mathrm{odd},0}(\Sigma,u^*TX)$. Since I is homotopic to $-I$, the complex Fredholm index of D is the holomorphic Euler characteristic of the bundle $u^*TX\to\Sigma$ and, by the Hirzebruch–Riemann–Roch formula,

$$\mathrm{ind}^{\mathbb{R}}(\mathscr{D}_u)=\mathrm{index}^{\mathbb{C}}(D)=\int_\Sigma\mathrm{ch}(u^*TX)\mathrm{td}(T\Sigma).$$

With $\mathrm{ch}=\mathrm{rank}^{\mathbb{C}}+c_1+\frac{1}{2}(c_1^2-2c_2)$ and $\mathrm{td}=1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)$ this gives again the above formula, and (12) follows because $c_1(TX)=c_1(T\Sigma)=0$.

Ring structure

As an example of this construction we obtain (conjecturally) a ring structure on $\mathrm{HF}^*(S^3, X)$. Take $\Sigma := \mathbb{H} \setminus \{-\frac{1}{2}, \frac{1}{2}\}$ and define $\iota^- : (-\infty, 0] \times S^3 \to \mathbb{H}$ by

$$\iota^-(s, y) := e^{-s} y.$$

The image of this map is the complement of the open unit ball in $\mathbb{H}$. The embedding $\iota^+ : [0, \infty) \times (S^3 \sqcup S^3) \to \mathbb{H}$ is the disjoint union of the embeddings $(s, y) \mapsto e^{-1-s} y \pm \frac{1}{2}$. The resulting *quaternionic pair of pants product*

$$\mathrm{HF}^*(S^3, X) \otimes \mathrm{HF}^*(S^3, X) \to \mathrm{HF}^*(S^3, X)$$

should be independent of the choice of the embeddings and the Hamiltonian perturbations used to define it. Moreover, counting the solutions of (10) on the punctured cylinder $\mathbb{R} \times M \setminus \{\mathrm{pt}\}$, will lead to a module structure of $\mathrm{HF}^*(M, X)$ over $\mathrm{HF}^*(S^3, X)$ for every M.

The compactness and transversality results in the present paper show that this construction is perfectly rigorous and gives rise to an associative product on $\mathrm{HF}^*(S^3, X)$ whenever X is flat. Moreover, in this case it agrees with the usual cup product under our isomorphism

$$\mathrm{HF}^*(S^3, X) \cong H^*(X; \mathbb{Z}_2).$$

Relations with Donaldson–Thomas theory

In [4] Donaldson and Thomas outline the construction of Donaldson type invariants of 8-dimensional Spin(7)-manifolds Z and Floer homological invariants of 7-dimensional G_2-manifolds Y. In the case $Z = \Sigma \times S$, where Σ and S are hyperkähler surfaces, they explain that solutions of their equation on $\Sigma \times S$ correspond, in the adiabatic limit where the metric on S degenerates to zero, to solutions $u : \Sigma \to \mathscr{M}(S)$ of (10) with values in a suitable moduli space $X = \mathscr{M}(S)$ of bundles over S. In a similar vein there is a conjectural correspondence between the Donaldson-Thomas-Floer theory of

$$Y = M \times S$$

with the Floer homology groups $\mathrm{HF}_*(M, \mathscr{M}(S))$ discussed above whenever M is either a Cartan hypercontact 3-manifold or a flat 3-torus. Namely, the solutions of the Floer equation in Donaldson–Thomas theory on $\mathbb{R} \times Y$ with $Y = M \times S$ correspond, in the adiabatic limit, formally to the solutions of (6) on $\mathbb{R} \times M$ with values in $\mathscr{M}(S)$.

Boundary value problems

If M is Cartan hypercontact 3-manifold with boundary ∂M and Reeb vector fields v_1, v_2, v_3 then there is a unique map $\lambda : \partial M \to S^2$ such that

$$\nu := \sum_i \lambda_i v_i : \partial M \to TM$$

is the outward pointing unit normal vector field. In this case the 1-form (1) is not closed. Its differential is given by the formula

$$T_f\mathscr{F} \times T_f\mathscr{F} \to \mathbb{R} : (\hat{f}_1, \hat{f}_2) \mapsto \int_{\partial M} \omega_\lambda(\hat{f}_1, \hat{f}_2) \mathrm{dvol}_{\partial M}.$$

This is a symplectic form on the space of maps $\partial M \to X$. Thus it seems natural to impose the Lagrangian boundary condition

$$f(y) \in L_y, \qquad y \in \partial M,$$

where

$$L := \bigsqcup_{y \in \partial M} L_y$$

is a smooth submanifold of $\partial M \times X$ such that L_y is Lagrangian with respect to $\omega_{\lambda(y)}$ for every $y \in \partial M$. We conjecture that this is an elliptic boundary condition for equation (5).

5 Concluding remarks

The details of the theory outlined here are worked out in [16] for compact flat target manifolds X. The extension to general hyperkähler manifolds X requires a careful understanding of the codimension 2 bubbing phenomenon for the solutions of equation (6). In this general setting there should be a rather rich class of examples. Once this is understood one can approach the Donaldson–Thomas analogue of the Atiyah–Floer conjecture [5] for suitable product manifolds $M \times S$ as suggested in [4] and described above.

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