Pleading for a functorial approach of Delzant correspondence

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Preliminary comments

- This is a *work in progress* with Damien Lejay : it is possible that despite our research in the literature, our resultsare already known, or that the question get a disappointing answer. One goal for this talk is to check that with the audience.
- Category theory / functorial approach is essentially a language: *"Its merit is that it exists"*. It can help identify the core of a given problem, not make it disappear !
- Classification of almost-toric systems *"à la Delzant"*is the definition and the study of its moduli space. The only definition of a moduli space comes from algebraic geometry, and is written in the language of category. Defining what is "the" category of almost-toric systems, starting with "the" category of toric systems shall help us understand the meaning of "à la Delzant" beyond some heuristic approach.

Plan

Joint work with D. Lejay.

1. Reminder of the language of categories
   - Definition
   - Functors and equivalence of categories
2. The existing categorie(s) of toric systems and Delzant polytopes
   - On hamiltonian torus action
   - Toric integrable systems
   - The Delzant correspondence
   - A category of toric systems
3. Can we do better ?
   - Polarized toric varieties

Definition

A (small) category $\mathcal{C}$ is defined as

- a set of objects $Ob(\mathcal{C})$,
- For each pair of objects A and B of $\mathcal{C}$, a set $\mathcal{C}(A,B)$ called the set of "morphisms", or arrows : $f : A \to B$.
- For each triplet A, B, C of objects in $\mathcal{C}$, a binary operation

$$\begin{aligned} \circ : \mathcal{C}(A,B) \times \mathcal{C}(B,C) &\to \mathcal{C}(A,C) \\ (f,g) &\mapsto \circ(f,g) =: g \circ f \end{aligned}$$

that is associative : $(f \circ g) \circ h = f \circ (g \circ h)$, and with an identity morphism $id_A \in \mathcal{C}(A,A)$ for each object in $\mathcal{C}$: $f \circ id_A = f$, $id_A \circ g = g$.

Example

- $\mathbf{Set}$: $Ob(\mathbf{Set}) = \{$all sets$\}$, and $\mathbf{Set}(X, Y) := \{$maps from X to $Y\}$
- $\mathbf{Top}$: $Ob(\mathbf{Top}) = \{$topological spaces$\}$, and $\mathbf{Top}((X,\tau),(Y,\tau')) := \{$continuous maps from (X,τ) to $(Y,\tau)\}$
- $\mathbf{Grp}$: $Ob(\mathbf{Grp} = \{$groups$\}$ and $\mathbf{Grp}(G, H) := \{$group homomorphisms from G to $H\}$
- A groupoid is a category for which every arrow as an inverse.

Example (Rel)

- $Ob(\mathbf{Rel}) = \{$all sets$\}$
- $\mathbf{Rel}(A, B) :=$ $\{$all binary relations between A and Bi.e. : subsets of $A \times B\}$,
- for $\mathcal{R} : A \to B$ and $\mathcal{R}' : B \to C$, $\mathcal{R}' \circ \mathcal{R}$ is defined by : $z(\mathcal{R}' \circ \mathcal{R})x$ if there exists a $y \in B$ such that $y\mathcal{R}x$ and $z\mathcal{R}'y$.

Definition

A functor $F : \mathcal{C} \to \mathcal{C}'$ associates objects of $\mathcal{C}$ to objects of $\mathcal{C}'$, $A \to F(A)$, and to each morphism $f \in \mathcal{C}(A, B)$, associates a morphism $F(f) \in \mathcal{C}'(F(A), F(B))$ such that :

$$F(f \circ_{\mathcal{C}} g) = F(f) \circ_{\mathcal{C}'} F(g) \text{ and } F(id_A) = id_{F(A)}$$

.

Example

- $\pi_1 : \mathbf{Top}_p \to \mathbf{Grp}$ which associates to a pointed topological set (X, p) its fundamental group $\pi_1(X, p)$
- $Spec : \mathbf{ComRing} \to \mathbf{LocRngSp}$, which associates to a commutative ring its spectrum of prime ideals. An important property in algebraic geometry is that it defines an **equivalence of category** between $\mathbf{ComRing}^{op}$ and the “image” of $Spec$, which are called *affine schemes*.

Some “mantras” in category theory

- *“Morphisms are everything”*: more than the objects, they give the “shape” of the category $\Rightarrow$ Adjusting the number of morphisms is like having enough open sets in a topology
- *“Do NOT make choices”*: the power of category language is to formulate construction as “best/universal solution to a problem” which is formulated by a diagram.
  Each non-canonical choice hinders the power, and thus the purpose of using category, so we try to avoid them as much as possible.

Definition

A torus is here a compact connected abelian Lie group of finite dimension, here denoted $\mathbf{T}$. $\ker(\exp)$ defines a canonical lattice Λ on $\mathfrak{t}$, and by duality Λ^* on $\mathfrak{t}^*$.

Let (M^{2n}, ω) be a symplectic manifold.

Definition

An Lie group action $\rho : G \to Diff(M)$ is called Hamiltonian if there exists a Lie algebra homomorphism $\eta : \to \mathcal{C}^\infty(M, \mathbb{R})$ called the *comoment map* such that

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\rho"] \arrow[dr, dotted, "\eta"'] & \Gamma(M, TM)^G \\ & \mathcal{C}^\infty(M) \arrow[u, "ad"'] \end{tikzcd}$$

The moment map μ is defined by duality. Both are uniquely defined up to a constant.

First definition of an Integrable Hamiltonian systems

Definition

An **I**ntegrable **H**amiltonian **S**ystem (IHS) will be a triplet (M^{2n}, ω, F) with (M, ω) a symplectic manifold, and $F := (f_1, \ldots, f_n) \in \mathcal{C}^\infty(M^{2n} \to \mathbb{R}^n)$, such that:

- $\forall i, j = 1..n,\ \{f_i, f_j\}_\omega = 0$,
- $\mathrm{rk}(dF)$ is maximal almost everywhere.

F is the moment map for a Poisson $\mathbb{R}^n$-action,
$\mathcal{F} := \{\text{c.c. of } F^{-1}(c) \mid c \in F(M)\}$ its associated (singular) Lagrangian foliation with projection $\pi_\mathcal{F} : M \to B$ so that $\mathcal{F} = \{\Lambda_b\}_{b \in B}$.

For us, M will always be compact connected.
We set $B_r := \{b \in B \mid \forall m \in \Lambda_b,\ \mathrm{rk}(dF_m) = n\}$

Action-Angles coordinates

Theorem (Liouville-Arnold-Mineur theorem)

Let $b \in B_r$ the set of regular value of $\pi_{\mathcal{F}}$, then there exists an open set $\mathcal{U}$ of Λ_b, and an n-dimensional torus $\mathbf{T}$ such that $\mathcal{F}|_{\mathcal{U}}$ is symplectically isomorphic to a fibration by Lagrangian tori in a saturated neighborhood of the zero section in $T^*\mathbf{T}$.

$\pi_{\mathcal{F}}$ defines a *singular* Lagrangian torus fibration.
A byproduct : At each $b \in B_r$, the integral covectors define *canonical* lattice A_b on $T^*_b B$: $(B_r, \mathcal{A} := (A_b)_{b \in B_r})$ is an open integral affine manifold: $GL_n(\mathbb{Z}) \rtimes \mathbb{R}^n$.

B and $F(M)$ are related by:

$$\begin{tikzcd} (M^{2n}, \omega) \arrow[r, "\pi_{\mathcal{F}}"] \arrow[dr, "F"'] & B \supseteq B_r \arrow[d, "\tilde{F}", "\simeq"'] \\ & F(M) \end{tikzcd}$$

On B_r (at least) the map $\tilde{F} : (B, \mathbf{C}) \to (F(M), \mathcal{C}^\infty(,\mathbb{R}))$ is a local diffeomorphism.

To chose or not to chose ?

Definition

An *intrinsic*, or *geometric* integrable system is a singular Lagrangian torus fibration $p : M \to B$.
An intrinsic integrable system is called "genuine" if there exists an F such that $p = \pi_{\mathcal{F}}$; an *immersed* integrable system is an intrinsic integrable system together with a choice of an F.

Definition

A genuine system $(M, \omega, \pi_{\mathcal{F}})$ is called toric if there exists an (effective) action of a torus whose moment map is a choice of F.
An "*immersed*" toric system is a genuine toric system together with a choice of such an action.

The connectedness of the fiber in [AGS] ensures that in the toric case, the immersion is actually an embedding.

Theorem (Atiyah – Guillemin & Sternberg,1982)

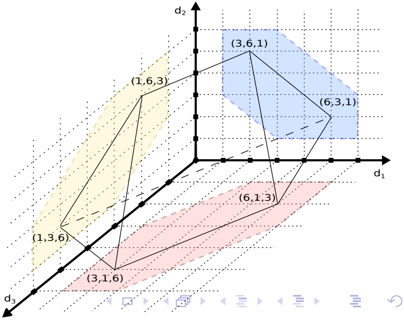
Let $\mathbf{T} \circlearrowleft (M^{2n}, \omega)$ *be an Hamiltonian action of a torus of dimension* d*, with moment map* $\mu : M^{2n} \to \mathfrak{t}^*$*. We have :*

- *the fibers of* μ *are connected,*
- $\mu(M) =: \Delta$ *is an intrinsic polytope i.e. : an integral affine* **convex** *manifold with corners with a finite number of extremal points, that is genuine, ie. there exists a global section for* $\mathcal{A}_\Delta$*.*

Example

The Gelfand-Cetlin system with $\Lambda = (1, 3, 6)$: the set of isospectral 3×3-Hermitian matrices have diagonals (d_1, d_2, d_3) contained in the polytope

$$\Delta := Conv((1, 3, 6), (1, 6, 3), \ldots)$$



Theorem (Delzant '87)

- *Let $(M, \omega, \pi_{\mathcal{F}})$ be an intrinsic toric integrable system. Then: $Delta$ is moreover normal and smooth*
- *Given two "embedded" toric integrable system manifolds,* **with the same torus** T*. If $\mu_1(M_1) = \mu_2(M_2)$, there exists a T-equivariant symplectomorphism such that the following diagram commutes*

$$\begin{tikzcd} M_1 \arrow[r, "\Phi"] \arrow[d, "\mu_1"'] & M_2 \arrow[d, "\mu_2"] \\ \mu_1(M_1) \arrow[r, equal] & \mu_2(M_2) \end{tikzcd}$$

  Such an symplectomorphism is uniquely defined up to automorphisms.
- *Given a Delzant polytope Δ, there is an explicit construction for a canonical genuine toric integrable system. If Δ is given together with a choice of an embedding, on can reconstruct an "embedded" toric integrable system, which is unique up to a T-equivariant symplectomorphism.*

From that, one can give definition for the category of genuine and embedded toric systems, and the category of genuine and embedded Delzant polytopes

Definition

We set

- $\mathbf{gSysTor}_0$ the trivial category whose object are genuine toric systems, and with identity morphisms only,
- $\mathbf{eSysTor}_0$ the category whose object are embedded toric systems, and the morphisms are the T-equivariant symplectomorphisms,
- $\mathbf{gDel}_0$ the trivial category whose object are genuine Delzant polytopes, and with the identity morphisms only,
- $\mathbf{eDel}_0$ the trivial category whose object are embedded Delzant polytopes, and with the identity morphisms only,

Not satisfying from a categorical viewpoint: **very poor categories !**
$\mathbf{eSysTor}_0$ is a groupoid; the Delzant classification defines an equivalence of categories, but it is just a bijection between their π_0's.

We need more maps, that are not isomophisms !

A hint: Arnol'd-Liouville -Mineur theorem

$\Longrightarrow$ the integral affine structure is the crucial structure ! $\mathbf{gDel}_0$ is a (non-full !) subcategory of the category of integral affine manifolds.

$\Longrightarrow$ A possible $\mathbf{gDel}_1$ with

$gDel_1(\Delta, \Delta') = \{a : \Delta \to \Delta' | a(\Lambda)$ a sublattice of $\Lambda'\}$?

I. e. : $\mathbf{eDel}_1$ with $eDel_1(\Delta, \Delta') = \{A \in GL_n(\mathbb{Z}) \rtimes \mathbb{R}^n | A(\Delta) \subseteq \Delta'\}$?

The underlying toric variety

There exists a forgetful functor to the category of toric varieties.

Theorem

There are no non-constant algebraic map from $\mathbb{CP}^n$ *to any algebraic variety of smaller dimension.*

Proof : Bézout theorem !

Morphisms of fans

Definition

A morphism $D : \sigma \to \sigma'$ is a integral linear map such that every cone $\tau \subseteq \sigma$ is sent to a cone $\tau' \subseteq \sigma'$

Definition

A morphism of toric varieties $V \to W$ is the same as a morphism of fan from $\sigma_V \to \sigma_W$

$$\begin{tikzcd} M \arrow[rr, "\Phi"] \arrow[d, "\mu_1"'] & & \arrow[d, "\mu_2"] \\ \mu_1(M_1) \arrow[rr, equal] & & \mu_2(M_2) \end{tikzcd}$$

Let $(M^{2m}, \omega_M, \mathbf{T}, \rho_M, \mu_M)$ and $(N^{2n}, \omega_N, \mathbf{S}, \rho_N, \mu_N)$ two embedded toric systems.

Definition

Let $\varphi \in \mathcal{C}^\infty(M, N)$ and ψ be such that $\varphi^*\omega_N = \omega_M$, and such that it is (S, T)-equivariant, i.e : the following diagram commutes

$$\begin{tikzcd} T \times M \arrow[r, "id_T \times \varphi"] \arrow[d, "\rho_M^T"'] & S \times N \arrow[d, "\rho_N^S \circ (\psi \times id_M)"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Thank you !