w/ Milena Pabiniak



- $M^{2n}$ compact manifold
- $\omega$ symplectic
- $(S^1)^n \circlearrowright M$ (effectively)
- $\phi : M \to \mathbb{R}^n$ moment map

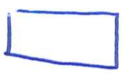
$\Big\}$ symplectic toric manifold

$\Longrightarrow \Delta = \phi(M)$ is a polytope

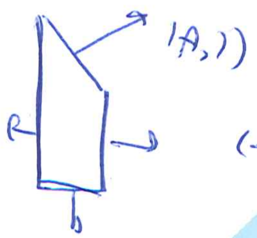
Thm (Delzant)

$\tilde{M}$ is equivariantly symplectomorphic to $M$
$\Longleftrightarrow \tilde{\Delta} = \Delta + c$

Question: When is $\tilde{M}$ symplectomorphic to $M$?
$(\tilde{M} \simeq M)$

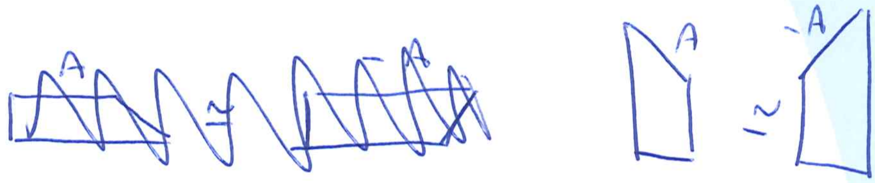
~~[illegible]~~,
Ex:  $\leftrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{array}{ccc} & & M \\ & & \text{is} \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(A)) \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$



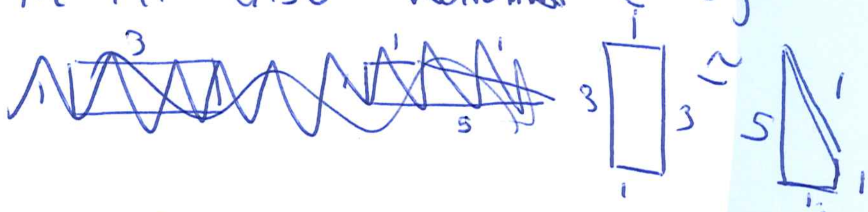
$\leftrightarrow M$

Clearly, if $\tilde{\Delta} = L\Delta$, $L \in Sl(n,\mathbb{Z})$,
then $\tilde{m} \simeq m$

$\Rightarrow$ 

In fact, $\tilde{m}$ diffeomorphic to $m$
$\iff \tilde{A} = A \mod 2$

& $\tilde{m} \simeq m$ iff also area & height agree.



Ex Let $\Delta = \{ y \in \mathbb{R}^n \mid y_j \geq 0 \ \forall j \ \& \ y_j + \sum_{i<j} A^i_j y_i \leq \lambda_j \ \forall j \}$,
where $A^i_j, \lambda_j \in \mathbb{Z}$.

Assume $\{$faces of $\Delta\} \leftrightarrow \{$faces of the hypercube $[0,1]^n\}$
(So opposite facets don't intersect, but otherwise all facets do.)

$\Delta \leftrightarrow m =$ "Bott manifold"

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] & F_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & F_1 \arrow[r] \arrow[d] & m \arrow[d] \\ & \mathbb{P}^1 & & \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

Claim: $H^*(m) \simeq \mathbb{Z}[x_1, \dots, x_n] / (x_j^2 + \sum_{j>i} A^i_j x_j x_i)$

$x_j$ is the Poincaré dual to the preimage of
$\{ y \in \Delta \mid y_j = 0 \}$

Let $M$ & $\tilde{M}$ be toric.

<u>Conjecture</u> (Rigidity):

Given $\varphi : H^*(M) \longrightarrow H^*(\tilde{M})$,

$\exists$ diffeomorphism $f : \tilde{M} \to M$ s.t. $f^* = \varphi$

<u>Thm</u> Rigidity holds for Bott manifolds if

- $\dim M \leq 8$ — Choi-Masuda-Suh, Choi
- $H^*(M, \mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n, \mathbb{Q})$ — Masuda-Panov, Choi-Masuda
- $M$ has only one "twist" — Choi-Suh

+ some other cases (Park, Ishida, Oum)

<u>Conjecture</u> (Symplectic Rigidity)

Given $\varphi : H^*(M) \to H^*(\tilde{M})$ s.t. $[\omega] \to [\tilde{\omega}]$,

$\exists$ symplectomorphism $f : \tilde{M} \to M$ s.t. $f^* = \varphi$

<u>Thm</u> (McDuff)

Symplectic rigidity holds for

$$\begin{array}{ccc} \mathbb{P}^k & \to & M \\ & & \downarrow \\ & & \mathbb{P}^j \end{array}$$

**Thm** (Pabiniak - T)

Symplectic rigidity holds if $M$ is a Bott manifold & $H^*(M;\mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Q})$

**Corollary**

If $M$ is toric & $H^*(M,\mathbb{Z}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Z})$, then $M \simeq (\mathbb{P}^1)^n$

**Proof** $M$ is a Bott manifold.

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# Free Integrable Systems*

Fix smooth variety $Y^n \subseteq \mathbb{P}^N$ &

smooth subvarieties $\{pt\} = X_n \subset X_{n-1} \subset \cdots \subset X_1 \subset X_0 = X$
codim $X_j = j$.

$\rightsquigarrow$ Graded semi-group $S = \bigoplus_j S^j$, $S^j \subset \mathbb{Z}^n$

__Assume__ $S$ finitely generated

$\Rightarrow \exists$ toric variety $X_S \subset \mathbb{P}^M$

__Thm__ (Harada-Kaveh)

$\exists$ continuous $f: Y \to X_S$

& open dense $U \subset X$

s.t. $f|_U$ is a symplectomorphism

$\Rightarrow$ $f^*(\Phi_{\mathbb{P}^m})$ generates an $(S^1)^n$ action on $U$,

where $\Phi_{\mathbb{P}^m} : \mathbb{P}^m \longrightarrow \mathbb{R}^m$

Ex: $Y = Fl(\mathbb{C}^n)$

$X \mapsto$ Gelfand Cetlin polytope

## Step 1

Assume $S^m = \mathbb{Z}^n \cap (m\tilde{\Delta})$ $\forall m$,

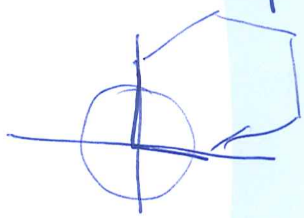
where $\tilde{\Delta} = \tilde{\phi}(\tilde{M})$, $\tilde{M}$ toric manifold.

Then $Y$ is symplectomorphic to $\tilde{M}$

## Step 2

Assume $Y$ is toric & let $\Delta = \phi(Y)$.

WLOG $\Delta = \mathbb{R}^n_{\geq 0}$ near $0$



Fix $c \in \mathbb{Z}$ & $1 \leq k < l \leq n$.

For suitable $Y_n \subset Y_{n-1} \subset \dots \subset Y$,

get $S^m$ from $m\Delta \cap \mathbb{Z}^n$ by sliding dots in $\mathbb{Z}^n_{\geq 0}$ along $-e_k + ce_l$

## Proposition

Assume $M$ & $\tilde{M}$ are Bott manifolds.
Assume $\exists\ \gamma \in \mathbb{Z}$ & $1 \le k < l \le n$ s.t. the homo

$$\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[\tilde{x}_1, \dots, \tilde{x}_n]$$
$$x_k \longmapsto \tilde{x}_k - \gamma \tilde{x}_l$$
$$x_i \longmapsto \tilde{x}_i \quad \forall\ i \neq k$$

descends to an isomorphism $H^*(M) \to H^*(\tilde{M})$
that takes $[\omega] \mapsto [\tilde{\omega}]$.
Then $M \simeq \tilde{M}$.

pf Can get $M\tilde{\Delta} \cap \mathbb{Z}^n$ from $M\Delta \cap \mathbb{Z}^n$
by sliding dots in $\mathbb{Z}^n_{\ge 0}$ along $-e_k + (A^k_l - \gamma) e_l$.
(Or vica versa)

pf of thm

Apply proposition over & over &
use "obvious" symplectomorphisms.