Stability analysis for generalized free rigid body dynamics on a real semi-simple Lie algebra with respect to an arbitrary Cartan subalgebra

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Based on a joint work with Tudor S. Ratiu (Shanghai Jiao Tong University)

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§1. Introduction

$SO(3)$ case 1 – Euler equation

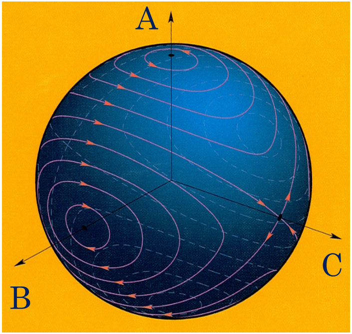
Free rigid body = Left-inv. geodesic flow on $SO(3)$.
Formulated as a Hamiltonian dynamics on $T^*SO(3)$
Described by Euler equation on $\mathfrak{so}(3)$:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left[X, \mathcal{J}^{-1}(X)\right]. \tag{1}$$

$X \in \mathfrak{so}(3)$: angular momentum,
$\mathcal{J} : \mathfrak{so}(3) \ni X \mapsto \mathrm{J}X + X\mathrm{J} \in \mathfrak{so}(3)$: inertia tensor (positive-def. symmetric linear operator), J:diagonal.

- The system can be restricted to an adjoint orbit.
- (1) is completely integrable in the sense of Liouville w.r.t. the orbit symplectic form.
- The stability of the equilibria of (1) is well known.

$SO(3)$ case 2 – equilibria



From *Introduction to Mechanics and Symmetry* by J. E. Marsden and T. S. Ratiu (1999).

Observation. The system on a generic adjoint orbit has six equilibria. Four of them on the A- and B-axes are elliptic, while the other two on the C-axis are hyperbolic.

History of generalization 1

– Complete integrability (on generic orbits)

- $\mathfrak{so}(n)$: Mishchenko, Dikii, Manakov, Ratiu (1970's, 80's)
- semi-simple Lie algebra: Mishchenko-Fomenko (1978)
- $\mathfrak{u}(n)$: Iwai (2004), Ratiu-T 2015.
- Bloch-Iserles system: Li-Tomei (2006), Bloch-Brînzănescu-Iserles-Marsden-Ratiu (2009).

History of generalization 2

–Stability analysis

- $\mathfrak{so}(n)$: Spiegler (2004), Birtea-Casu-Ratiu-Turhan ($n = 4$, 2012, cf. Fehér-Marshall (2003)), Izosimov (2014)
- $\mathfrak{u}(n)$: Ratiu-T (2015).
- normal (split) and compact real forms of complex semi-simple Lie algebra: Ratiu-T.
- Bloch-Iserles system: Ratiu-T.
- real semi-simple of type A: Izosimov (2015, 2016).

$\rightsquigarrow$ These systems are either on special real semi-simple Lie algebras or concerning a fixed Cartan subalgebra.

Task of this talk

Analyze the stability of equilibria for the system on any real semi-simple Lie algebra for an arbitrary Cartan subalgebra.

Basic (technical) concepts

Completely integrable systems on symplectic mfd

(M, ω): symplectic mfd (ω: non-degenerate closed 2-form).
For $H \in \mathcal{C}^\infty(M)$: Hamiltonian, the Hamiltonian vector field Ξ_H is defined through

$$\iota_{\Xi_H}\omega = -\mathrm{d}H.$$

Assume $\dim M = 2n$. The Hamiltonian system is called *completely integrable* in the sense of Liouville, if there exist n functionally independent functions $F_1, \cdots, F_{n-1}, F_n (= H)$ which Poisson commute:

$$\{F_i, F_j\} = 0, \qquad (i, j = 1, \cdots, n).$$

Here, $\{F, G\} = \omega(\Xi_F, \Xi_G) = \Xi_F(G)$.

Equilibrium of Hamiltonian system on sympl mfd

For a Hamiltonian system (M, ω, H) on a symplectic manifold, consider an isolated equilibrium $x_0 \in M$, where $\Xi_H(x_0) = 0$.

When $\mathrm{Hess}[H](x_0)$ is non-degenerate., the eigenvalues of the linearization for Ξ_H at x_0 consist of the three types as follows:

- pair of purely imaginary eigenvalues $\pm\sqrt{-1}a$ (*elliptic*).
- pair of real eigenvalues $\pm a$ (*hyperbolic*).
- four complex eigenvalues $\pm a \pm \sqrt{-1}b$ (*focus-focus*).

The numbers (n_e, n_h, n_f) of elliptic, hyperbolic, and focus-focus components characterize the equilibrium x_0 (*Williamson type*).

Birkhoff normal form

Around the above equilibrium point x_0 of type (n_e, n_h, n_f), we can take a (formal) canonical coordinates $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ with which the Hamiltonian H is written as a (formal) power series $\mathcal{H}$ in n variables:

$$\frac{p_i^2 + q_i^2}{2}, \quad i = 1, \ldots, n_e; \qquad p_j q_j, \quad j = n_e + 1, \ldots, n_e + n_h;$$
$$p_k q_{k+1} - q_k p_{k+1}, p_k q_k + p_{k+1} q_{k+1}, \quad k = n_e + n_h + 1, \ldots, n.$$

(Birkhoff normal form). N.B. $n = n_e + n_h + 2n_f$.
Siegel (1954) showed that these canonical coordinates are in general divergent (only formally determined), but there are possibilities of convergence for completely integrable systems.

Birkhoff normal form for integrable systems

An isolated equilibrium $x_0 \in M$ of a completely integrable system (M, ω, H) with constants $F_1, \ldots, F_{n-1}, F_n = H$ of motion is called non-degenerate, if the linearization of the Hamiltonian vector fields $\Xi_{F_1}, \cdots, \Xi_{F_n}$ at x_0 generate a Cartan subalgebra in $\mathfrak{sp}\left(T_{x_0}M, \omega\left(x_0\right)\right)$.

Theorem (Vey 1978)

Around a non-degenerate isolated equilibrium $x_0 \in M$ for a real-analytic completely integrable system, we can take a convergent canonical coordinates which put the Hamiltonian into Birkhoff normal form.

Generalizations: Ito (1989), Eliasson (1990), Zung (2004).

Relation to the stability

To analyze the stability of an isolated equilibrium $x_0 \in M$ for a Hamiltonian system (M, ω, H) on a sympl mfd, we consider the linearization of the Hamiltonian vector field Ξ_H at x_0:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \omega(x_0)^{-1}\,\mathrm{Hess}\,[H]\,(x_0) \cdot P, \quad P \in T_{x_0}M.$$

The linear stability can be analyzed by the eigenvalue problem of the linearization matrix $\omega(x_0)^{-1}\mathrm{Hess}\,[H]\,(x_0)$.

In general, the linear stability does not imply the nonlinear (Lyapunov) stability. However, if the system is completely integrable and if one can take the Birkhoff normal form through convergent canonical transformation, then the linear stability implies the ellipticity (and hence Lyapunov stability) of the equilibrium.

§2. Geometric settings

Geodesic flow on Lie groups w.r.t. left-inv. metric

G: (real) semi-simple Lie group, $\mathfrak{g} = Lie(G)$,
$\kappa(X, Y) = \mathrm{Tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$: Killing form, $X, Y \in \mathfrak{g}$.

The geodesic flow on G w.r.t. a left-inv. metric can be formulated as the Hamiltonian system (T^*G, Ω, H).
Ω: canonical symplectic form on $T^*G \cong G \times \mathfrak{g}^* \cong G \times \mathfrak{g}$,
$H(X) = \frac{1}{2}\kappa\,(X, \varphi(X)))$: Hamiltonian, $X \in \mathfrak{g}$,
$\varphi : \mathfrak{g} \to \mathfrak{g}$: symmetric operator.

Because of the left-invariance w.r.t. G (Lie-Poisson reduction), the system can be described by Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X = [X, \varphi(X)], \qquad X \in \mathfrak{g}.$$

Euler equation

Euler equation is Hamilton's equation for the Hamiltonian H w.r.t. Lie-Poisson bracket

$$\{F,G\}(X) = \kappa(X,[\mathrm{d}F(X),\mathrm{d}G(X)]), \qquad F,G \in \mathcal{C}^\infty(\mathfrak{g}).$$

(Hamiltonian vector field Ξ_F is defined by $\Xi_F(G) = \{F,G\}$.)

Further, the system can be restricted to an adjoint orbit $\mathcal{O} = \{\mathrm{Ad}_g X_0 \mid g \in G\} \subset \mathfrak{g}$. The restriction is a Hamiltonian system w.r.t. the orbit symplectic form $\omega_{\mathcal{O}}$, where

$$\omega_{\mathcal{O}}(\mathrm{ad}_Y X, \mathrm{ad}_Z X) = \kappa(X,[Y,Z]), \qquad \mathrm{ad}_Y X, \mathrm{ad}_Z X \in T_X\mathcal{O}.$$

We consider the complete integrability and the stability of the (isolated) equilibria on a generic adjoint orbit $\mathcal{O}$.

Lie algebraic preliminary 1

To define Mishchenko-Fomenko integrable geodesic flow, we consider a (Lie algebra isomorphic) involution $\theta : \mathfrak{g} \to \mathfrak{g}$: $\theta^2 = \mathrm{id}_{\mathfrak{g}}$ and set $\kappa_\theta(X, Y) := -\kappa(X, \theta Y)$ for $X, Y \in \mathfrak{g}$. If κ_θ is positive-definite, then θ is called *Cartan involution*. (Cartan involution is known to exist and it is unique up to inner automorphism.)

With respect the Cartan involution θ, we have the orthogonal eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\theta|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{k}}$, $\theta|_{\mathfrak{p}} = -\mathrm{id}_{\mathfrak{p}}$.

Lie algebraic preliminary 2

Let $\mathfrak{h} \subset \mathfrak{g}$: ($\theta$-stable) Cartan subalgebra (maximal Abelian subalgebra consisting of semi-simple elements),
$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$: orthogonal decomposition w.r.t. κ.
$\langle \cdot, \cdot \rangle : (\mathfrak{h}^{\mathbb{C}})^* \times \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ be the dual pairing.

For $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$, set
$\mathfrak{g}_\alpha^{\mathbb{C}} := \left\{ X \in \mathfrak{g}^{\mathbb{C}} \,\middle|\, \mathrm{ad}_Y X = \langle \alpha, Y \rangle X, \forall Y \in \mathfrak{h}^{\mathbb{C}} \right\}$: root space,
$\Delta := \left\{ \alpha \in \left(\mathfrak{h}^{\mathbb{C}}\right)^* \,\middle|\, \alpha \neq 0, \mathfrak{g}_\alpha^{\mathbb{C}} \neq 0 \right\}$: root system.

Then, $\mathrm{Span}_{\mathbb{C}} \Delta = \left(\mathfrak{h}^{\mathbb{C}}\right)^*$,
$\dim_{\mathbb{C}} \mathfrak{g}_\alpha^{\mathbb{C}} = 1$ for $\alpha \in \Delta$,
$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha^{\mathbb{C}}$: root decomposition.

Lie algebraic preliminary 3

A subset $\Pi = \{\alpha_1, \cdots, \alpha_r\} \subset \Delta$ is called a base, when Π is a basis of the vector space $\left(\mathfrak{h}^{\mathbb{C}}\right)^*$ and when all $\alpha \in \Delta$ writes

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i, \qquad m_i \in \mathbb{Z},$$

s.t. either $m_1, \cdots, m_r \geq 0$ or $m_1, \cdots, m_r \leq 0$. The elements in Π are called simple roots.

We have $\Delta = \Delta_+ \sqcup \Delta_-$, setting

$$\Delta_\pm = \left\{ \alpha = \sum_{i=1}^{r} m_i \alpha_i \in \Delta \,\middle|\, m_1, \cdots, m_r \geq 0 (\text{resp. } \leq 0) \right\}.$$

Sectional operator by Mishchenko-Fomenko

For a given Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we take $a, b \in \mathfrak{h}$, where $\langle a, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, and a symmetric operator $D : \mathfrak{h} \to \mathfrak{h}$.

For $X = X_{\mathfrak{h}} + X_{\mathfrak{m}} \in \mathfrak{g}$, where $X_{\mathfrak{h}} \in \mathfrak{h}$, $X_{\mathfrak{m}} \in \mathfrak{m}$, we set $\varphi_{a,b}(X_{\mathfrak{m}}) = (\mathrm{ad}_a^{-1} \circ \mathrm{ad}_b)(X_{\mathfrak{m}})$. The operator $\varphi_{a,b,D} : \mathfrak{g} \to \mathfrak{g}$ defined through

$$\varphi_{a,b,D}(X) = D(X_{\mathfrak{h}}) + \varphi_{a,b}(X_{\mathfrak{m}}), \qquad X \in \mathfrak{g}.$$

is called the *sectional operator*.
We consider Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X = [X, \varphi_{a,b,D}(X)], \qquad X \in \mathfrak{g}.$$

Bi-Hamiltonian structure of Euler equation

Euler equation has the *bi-Hamiltonian structure* as follows: Besides the Lie-Poisson bracket $\{\cdot,\cdot\}$, Euler equation is Hamilton's equation w.r.t. Poisson brackets

$$\{F,G\}_{a,\lambda}(X) = \kappa\left(X+\lambda a, [\mathrm{d}F(X), \mathrm{d}G(X)]\right), \quad F,G \in \mathcal{C}^{\infty}(\mathfrak{g}).$$

for all $\lambda \in \mathbb{R}$.
N. B. $\{\cdot,\cdot\}_{a,\lambda} = \{\cdot,\cdot\} + \lambda\{\cdot,\cdot\}_a$, $\{\cdot,\cdot\}_{a,0} = \{\cdot,\cdot\}$.
This can be checked through the following Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(X+\lambda a) = \left[X+\lambda a, \varphi_{a,b,D}(X) + \lambda b\right],$$

equivalent to Euler equation.

Methods by Bolsinov-Oshemkov 1

Bolsinov-Oshemkov (2009) developed systematic methods to deal with the complete integrability and the equilibria for bi-Hamiltonian systems (cf. Bolsinov-Izosimov (2014)).

Theorem (Mishchenko-Fomenko, Bolsinov-Oshemkov)

Let $\mathcal{F}_a$ be the ring of functions generated by Casimir functions for all $\{\cdot,\cdot\}_{a,\lambda}$, $\lambda \in \mathbb{R}$. ($F \in \mathcal{C}^\infty(\mathfrak{g})$ is Casimir w.r.t. $\{\cdot,\cdot\}_{a,\lambda} \Longleftrightarrow \{F,\cdot\}_{a,\lambda} = 0$.) Then, $\mathcal{F}_a$ is complete, i.e.

$$\mathrm{d}\mathcal{F}_a(X) := \{\mathrm{d}F(X) \mid F \in \mathcal{F}_a\} \subset \mathfrak{g}$$

is a maximal isotropic subspace for X in an open dense subset of $\mathfrak{g}$. Consequently, on a generic adjoint orbit $\mathcal{O} \subset \mathfrak{g}$, the restriction of Euler equation is completely integrable in the sense of Liouville.

Methods by Bolsinov-Oshemkov 2

Theorem (Bolsinov-Oshemkov)

On a generic adjoint orbit $\mathcal{O} \subset \mathfrak{g}$, the set of the isolated equilibria is given as

$$\mathfrak{h} \cap \mathcal{O}.$$

Theorem (Bolsinov-Oshemkov)

The equilibrium $X \in \mathfrak{h} \cap \mathcal{O}$ of the restriction of Euler equation to $\mathcal{O}$ is non-degenerate, if the following numbers are distinct:

$$\frac{\alpha(X)}{\alpha(a)}, \qquad \text{where } \alpha \in \Delta_+ \text{ are arbitrary positive roots.}$$

§3. Stability analysis

Lyapunov stability through linear stability

By Bolsinov-Oshemkov Theorem, the isolated equilibrium $X \in \mathfrak{h} \cap \mathcal{O}$ is Lyapunov stable if

$$\frac{\alpha(X)}{\alpha(a)}, \qquad \text{where } \alpha \in \Delta_+ \text{ are arbitrary positive roots}$$

are distinct and if X is linearly stable.

We compute the linearization matrix of the Hamilton's equation with respect to a basis of $T_X\mathcal{O} = \mathfrak{m} = \mathfrak{h}^{\perp_\kappa}$.

Linear stability 1

We use the following classification of roots:
A root $\alpha \in \Delta$ is called *real*, if $\alpha(\mathfrak{h}) \subset \mathbb{R}$.
$\beta \in \Delta$ is *(purely) imaginary*, if $\beta(\mathfrak{h}) \subset \sqrt{-1}\mathbb{R}$.
Otherwise, a root $\gamma \in \Delta$ is called *complex*.

For a real root $\alpha \in \Delta_+$, we can take the root vector

$$u_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha \cap \mathfrak{g}, \qquad \theta u_\alpha \in \mathfrak{g}^{\mathbb{C}}_{-\alpha} \cap \mathfrak{g}.$$

For a (purely) imaginary root $\beta \in \Delta_+$, we can take the root vectors $v_\beta \in \mathfrak{g}^{\mathbb{C}}_\beta$ and $\overline{v_\beta} \in \mathfrak{g}^{\mathbb{C}}_{-\beta}$. We consider the elements

$$v^r_\beta := \frac{v_\alpha + \overline{v_\alpha}}{2}, v^i_\beta := \frac{v_\alpha - \overline{v_\alpha}}{2\sqrt{-1}} \in \mathfrak{g}.$$

Linear stability 2

For a complex root $\gamma \in \Delta_+$, we take the root vectors $w_{\pm\gamma} \in \mathfrak{g}_{\pm\gamma}$, $\overline{w_{\pm\gamma}} \in \mathfrak{g}_{\pm\overline{\gamma}}$. N. B. $\overline{\gamma} \neq \pm\gamma$ We then consider the elements

$$w^r_\gamma = \frac{w_\gamma + \overline{w_\gamma}}{2}, \qquad w^i_\gamma = \frac{w_\gamma - \overline{w_\gamma}}{2\sqrt{-1}},$$
$$w^r_{-\gamma} = \frac{w_{-\gamma} + \overline{w_{-\gamma}}}{2}, \quad w^i_{-\gamma} = \frac{w_{-\gamma} - \overline{w_{-\gamma}}}{2\sqrt{-1}}.$$

The elements

$$u_\alpha, \theta u_\alpha, v^r_\beta, v^i_\beta, w^r_{\pm\gamma}, w^i_{\pm\gamma} \tag{2}$$

where $\alpha \in \Delta_+$: real, $\beta \in \Delta_+$: purely imaginary, $\gamma \in \Delta$: complex, form a basis of the complement $\mathfrak{m}$.
W.r.t. $\mathrm{ad}_Y X$ where Y is one of (2), the linearization matrix is decomposed into the direct sum of the following blocks.

Linear stability 3

For a real root $\alpha \in \Delta_+$, we have the 2×2 block

$$\left\{ \frac{\alpha(b)}{\alpha(a)} \alpha(X) - \alpha(D(X)) \right\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for $u_\alpha, \theta u_\alpha$, which corresponds to the hyperbolic component of Williamson type.

For a purely imaginary root $\beta \in \Delta_+$, we have the 2×2 block

$$\sqrt{-1} \left\{ \frac{\beta(b)}{\beta(a)} \beta(X) - \beta(D(X)) \right\} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for v_β^r, v_β^i, which corresponds to the elliptic component of Williamson type.

Linear stability 4

For a complex root $\gamma \in \Delta_+$, we have the 4×4 block

$$\begin{pmatrix} M_\gamma & 0 \\ 0 & -M_\gamma \end{pmatrix}, \quad M_\gamma = \begin{pmatrix} \rho_\gamma & \iota_\gamma \\ -\iota_\gamma & \rho_\gamma \end{pmatrix},$$

where

$$\rho_\gamma + \sqrt{-1}\iota_\gamma = \gamma(X)\frac{\gamma(b)}{\gamma(a)} - \gamma(D(X)),$$

for $w^r_{\pm\gamma}, w^i_{\pm\gamma}$. This 4×4 block corresponds to the focus-focus component of Williamson type.

Main results

Theorem

The Williamson type of the isolated equilibrium $X \in \mathfrak{h} \cap \mathcal{O}$ is given as (n_e, n_h, n_f), where

$$n_e = \# \left\{ \beta \in \Delta_+ : \textit{purely imaginary} \right\},$$
$$n_h = \# \left\{ \alpha \in \Delta_+ : \textit{real} \right\},$$
$$n_f = \frac{1}{2} \cdot \# \left\{ \gamma \in \Delta_+ : \textit{complex} \right\}.$$

Theorem

The isolated equilibrium $X \in \mathfrak{h} \cap \mathcal{O}$ is linearly (and hence Lyapunov) stable if and only if Δ_+ only contains purely imaginary roots.