Commutativity in Lagrangian and Hamiltonian mechanics

In Hamiltonian mechanics, integrability is commutativity (recall Liouville–Arnold).

Let $H_1, H_2 : T^*M \to \mathbb{R}$ be two Hamilton functions, we say they Poisson commute if $\{H_1, H_2\} = 0$ (stronger than commutativity of their flows $F_1^{t_1}, F_2^{t_2} : T^*M \to T^*M$, which is equivalent to $\{H_1, H_2\} = \text{const}$).
Suppose they come from non-degenerate Lagrange functions $L_1, L_2 : TM \to \mathbb{R}$ (with locally invertible Legendre transforms). What is the counterpart of $\{H_1, H_2\} = 0$ on the Lagrangian side? EL eqs are 2^{nd} order, so commutativity notion is non-obvious.

The answer is best expressed in terms of principal action functions. Recall: $S_1(q_0, q_1, t) = \int_0^t L_1(q(t), \dot q(t))\,dt$ where $q : [0,t] \to M$ is the solution of EL eqs for L_1 with boundary conditions $q(0) = q_0$, $q(t) = q_1$ (exists and is unique if (q_0, q_1) close to diagonal of $M \times M$ and t sufficiently small). Recall: $S_1 : M \times M \times \mathbb{R} \to \mathbb{R}$ satisfies the Hamilton–Jacobi eqs

$$\frac{\partial S_1}{\partial q_0} = -p(0) = -\frac{\partial L_1}{\partial \dot q}(q_0, \dot q_0), \quad \frac{\partial S_1}{\partial q_1} = p(t) = \frac{\partial L_1}{\partial \dot q}(q_1, \dot q_1)$$

$$\frac{\partial S_1}{\partial t} = -H_1,$$

where $\dot q_0$ and $\dot q_1$ are velocities at the endpoints of the minimizing path

Composing principal actions: for $q_0, q_{12} \in M$ and $t_1, t_2 > 0$, set

$$S_{12}(q_0, q_{12}, t_1, t_2) = \min_{q_1 \in M} \left(S_1(q_0, q_1, t_1) + S_2(q_1, q_{12}, t_2) \right)$$

$$= \min_{\substack{q : [0, t_1 + t_2] \to M,\ \text{continuous} \\ q(0) = q_0,\ q(t_1 + t_2) = q_{12}}} \left(\int_0^{t_1} L_1(q(t), \dot q(t))\,dt + \int_{t_1}^{t_1 + t_2} L_2(q(t), \dot q(t))\,dt \right)$$

Similarly,

$$S_{21}(q_0, q_{12}, t_2, t_1) = \min_{q_2 \in M} \left(S_2(q_0, q_2, t_2) + S_1(q_2, q_{12}, t_1) \right)$$

$$= \min_{\substack{q:[0,t_1+t_2]\to M \text{ continuous} \\ q(0)=q_0,\ q(t_1+t_2)=q_{12}}} \left(\int_0^{t_2} L_2(q(t),\dot q(t))\,dt + \int_{t_2}^{t_1+t_2} L_1(q(t),\dot q(t))\,dt \right)$$

Theorem 1.

$$S_{12}(q_0, q_{12}, t_1, t_2) = S_{21}(q_0, q_{12}, t_2, t_2) \Longrightarrow \{H_1, H_2\} = 0.$$

(for all (q_0, q_{12}) from some neighborhood of diagonal in $M\times M$ and sufficiently small $t_1, t_2 > 0$).

Will give a proof, but first a discrete counterpart. Recall that a discrete time Lagrange function is $\Lambda: M\times M \to \mathbb{R}$, a discrete action is $S[q] = \sum_{n=a}^{b-1} \Lambda(q(n), q(n+1))$, minimisers satisfy $\delta S[q]=0$, or, with $q = q(n)$, $q_1 = q(n+1)$, $q_{-1} = q(n-1)$, dEL:

$$\frac{\partial \Lambda(q,q_1)}{\partial q} + \frac{\partial \Lambda(q_{-1},q)}{\partial q} = 0 \quad - \ 2^{nd} \text{ order difference eq.}$$

Discrete Legendre transformation

$$M\times M \ni (q_{-1}, q) \mapsto (q,p) \in T^*M$$

$$p = \frac{\partial \Lambda(q_{-1},q)}{\partial q} \in T^*_q M$$

Symplectic map (under non-degeneracy condition $\det\left(\frac{\partial^2 \Lambda}{\partial q_{-1}\partial q}\right) \neq 0$) : $F: (q,p) \mapsto (q_1, p_1)$

$$p = -\frac{\partial \Lambda(q,q_1)}{\partial q}, \quad p_1 = \frac{\partial \Lambda(q,q_1)}{\partial q_1}$$

Theorem 2. If two discrete Lagrange functions $\Lambda_1, \Lambda_2: M\times M \to \mathbb{R}$ satisfy

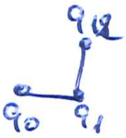
$$S_{12}(q, q_{12}) \equiv S_{21}(q, q_{12}),$$

where

$$S_{12} = \min_{q_1 \in M} \left(\Lambda_1(q_0, q_1) + \Lambda_2(q_1, q_{12}) \right) =$$

$= \Lambda_1(q_0, q_1) + \Lambda_2(q_1, q_{12})$, where q_1 is defined from

$$\frac{\partial \Lambda_1(q_0,q_1)}{\partial q_1} + \frac{\partial \Lambda_2(q_1,q_{12})}{\partial q_1} = 0, \qquad (E_1)$$

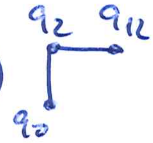


and

$$S_{21} = \min_{q_2 \in M} \left(\Lambda_2(q_0, q_2) + \Lambda_1(q_2, q_{12}) \right)$$

$= \Lambda_2(q_0, q_2) + \Lambda_1(q_2, q_{12})$, where q_2 is defined from

$$\frac{\partial \Lambda_2(q_0,q_2)}{\partial q_2} + \frac{\partial \Lambda_1(q_2,q_{12})}{\partial q_2} = 0, \qquad (E_2)$$

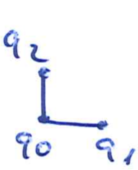


then the symplectic maps $F_1, F_2 : T^*M \to T^*M$ commute.

<u>Proof of theorem 2</u>.

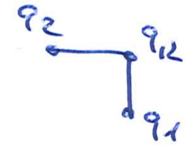
- Let $S_{12}(q_0, q_{12}) \equiv S_{21}(q_0, q_{12})$. If, for given q_0, q_{12}, q_1 and q_2 satisfy (E_1), (E_2), then also the following two corner eqs are satisfied:

$$\frac{\partial \Lambda_1(q_0,q_1)}{\partial q_0} - \frac{\partial \Lambda_2(q_0,q_2)}{\partial q_0} = 0 \qquad (E_0)$$



and

$$\frac{\partial \Lambda_1(q_2,q_{12})}{\partial q_{12}} - \frac{\partial \Lambda_2(q_1,q_{12})}{\partial q_{12}} = 0 \qquad (E_{12})$$



Indeed, by differentiating $S_{12} = \Lambda_1(q_0,q_1) + \Lambda_2(q_1,q_{12})$ and because of (E_1),

$$\frac{\partial S_{12}}{\partial q_0} = \frac{\partial \Lambda_1(q_0,q_1)}{\partial q_0}, \qquad \frac{\partial S_{12}}{\partial q_{12}} = \frac{\partial \Lambda_2(q_1,q_{12})}{\partial q_{12}}$$

and similarly

$$\frac{\partial S_{21}}{\partial q_0} = \frac{\partial \Lambda_2(q_0,q_2)}{\partial q_0}, \qquad \frac{\partial S_{21}}{\partial q_{12}} = \frac{\partial \Lambda_1(q_2,q_{12})}{\partial q_{12}}$$

• Now take an arbitrary $(q_0,p_0)\in T^*M$ and set $(q_1,p_1)=F_1(q_0,p_0)$, $(q_{12},p_{12})=F_2(q_1,p_1)$. This means

$$p_0=\frac{\partial\Lambda_1(q_0,q_1)}{\partial q_0},\quad p_1=-\frac{\partial\Lambda_1(q_0,q_1)}{\partial q_1}$$

$$p_1=\frac{\partial\Lambda_2(q_1,q_{12})}{\partial q_1},\quad p_{12}=-\frac{\partial\Lambda_2(q_1,q_{12})}{\partial q_{12}}.$$

$\to$ (E_1) is satisfied.

For the so defined q_{12}, solve (E_2) for q_2, and then set

$$p_2=-\frac{\partial\Lambda_2(q_0,q_2)}{\partial q_2}\overset{(E_2)}{=}\frac{\partial\Lambda_1(q_2,q_{12})}{\partial q_2}$$

$(E_1),(E_2)\Rightarrow(E_0),(E_{12})$. Therefore,

$$p_0=\frac{\partial\Lambda_2(q_0,q_2)}{\partial q_0}\longrightarrow(q_2,p_2)=F_2(q_0,p_0)$$

and

$$p_{12}=-\frac{\partial\Lambda_1(q_2,q_{12})}{\partial q_{12}}\longrightarrow(q_{12},p_{12})=F_1(q_2,p_2).\ \blacksquare$$

<u>Proof of Theorem 1.</u>

<u>Lemma 1.</u> Let $q^*:[0,t_1+t_2]\to M$ be a continuous critical curve of

$$\int_0^{t_1}L_1(q(t),\dot q(t))\,dt+\int_{t_1}^{t_1+t_2}L_2(q(t),\dot q(t))\,dt$$

(could be non-differentiable at $t=t_1$). However, the conjugate momentum p^* is continuous at $t=t_1$.

<u>Proof.</u> Let q_1^* be the minimiser of $S_{12}(q_0,q_{12},t_1,t_2)$
$=S_1(q_0,q_1^*,t_1)+S_2(q_1^*,q_{12},t_2)$.

Due to criticality,

$$\frac{\partial S_1}{\partial q_1}(q_0,q_1^*,t_1)+\frac{\partial S_2}{\partial q_1}(q_1^*,q_{12},t_2)=0\qquad(*)\quad\text{(analog of } E_1\text{)}.$$

By Hamilton-Jacobi,

$$\lim_{t\to t_1-0} \frac{\partial L_1}{\partial \dot q}(q^*(t), \dot q^*(t)) - \lim_{t\to t_1+0} \frac{\partial L_2}{\partial \dot q}(q^*(t), \dot q^*(t)) = 0. \blacksquare$$

$$\lim_{t\to t_1-0} p^*(t) = \lim_{t\to t_1+0} p^*(t).$$

<u>Lemma 2</u>. Partial derivatives of $S_{12}: M\times M\times \mathbb{R}_{>0}\times\mathbb{R}_{>0}\to R$ are given by

$$\frac{\partial S_{12}}{\partial q_0} = -p^*(0) = -\frac{\partial L_1}{\partial \dot q}(q_0, \dot q^*(0))$$

$$\frac{\partial S_{12}}{\partial q_{12}} = p^*(t_1+t_2) = \frac{\partial L_2}{\partial \dot q}(q_{12}, \dot q^*(t_1+t_2))$$

$$\frac{\partial S_{12}}{\partial t_1} = -H_1(\underset{q_0}{\underset{\parallel}{q^*(0)}}, p^*(0))$$

$$\frac{\partial S_{12}}{\partial t_2} = -H_2(\underset{q_{12}}{\underset{\parallel}{q^*(t_1+t_2)}}, p^*(t_1+t_2))$$

<u>Proof</u>. Differentiate $S_{12}(q_0, q_{12}, t_1, t_2) = S_1(q_0, q_1^*, t_1) + S_2(q_1^*, q_{12}, t_2)$ This involves differentiation of q_1^* w.r.t. any of the four variables (chain rule). But by (*) this vanishes $\blacksquare$

<u>Lemma 3</u>. Let $q^*: [0, t_1+t_2]\to M$ be the minimizer of S_{12}, $q^{**}: [0, t_1+t_2]\to M$ the minimizer of S_{21}, and p^*, p^{**} the corresp. conjugate momenta (continuous by Lemma 1). If $S_{12}\equiv S_{21}$, then

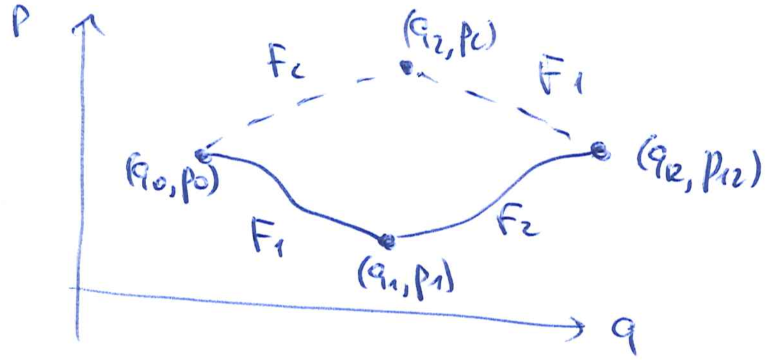
$$p^*(0) = p^{**}(0), \quad p^*(t_1+t_2) = p^{**}(t_1+t_2) \qquad (**)$$

(analogs of (E_0), (E_{12}))

<u>Proof</u>. Follows from Lemma 2, since these momenta are partial derivatives of S_{12}, resp. S_{21}.

Lemma 4. If $S_{12} \equiv S_{21}$, then $F_1^{t_1}, F_2^{t_2}$ commute.

Proof. Take an arbitrary $(q_0, p_0) \in T^*M$, set
$(q_1, p_1) = F_1^{t_1}(q_0, p_0)$, $\quad (q_{12}, p_{12}) = F_2^{t_2}(q_1, p_1)$.



With this q_{12}, consider minimizer q^{**} of $S_{21}(q_0, q_{12}, t_2, t_1)$ and its lift (q^{**}, p^{**}). By Lemma 3, $p^{**}(0) = p^*(0) = p_0$ and $p^{**}(t_1+t_2) = p^*(t_1+t_2) = p_{12}$. Set $(q_2, p_2) = F_2^{t_2}(q_0, p_0)$, we see that the diagram commutes. □

Lemma 5. $S_{12} \equiv S_{21} \Rightarrow$

$$H_1(q_0, p_0) = H_1(q_{12}, p_{12}) \qquad \left(\frac{\partial S_{12}}{\partial t_1} = \frac{\partial S_{21}}{\partial t_1}\right)$$

$$H_2(q_0, p_0) = H_2(q_{12}, p_{12}) \qquad \left(\frac{\partial S_{21}}{\partial t_2} = \frac{\partial S_{21}}{\partial t_1}\right)$$

Proof. Follows from Lemma 2 □

Proof of Theorem

$$H_1(q_1, p_1) = H_1(q_0, p_0), \quad H_2(q_{12}, p_{12}) = H_2(q_1, p_1)$$

$$H_2(q_2, p_2) = H_2(q_0, p_0), \quad H_1(q_{12}, p_{12}) = H_1(q_2, p_2)$$

$$\Downarrow$$

$H_1(q_1, p_1) = H_1(q_0, p_0) = H_1(q_{12}, p_{12})$ (H₁ invariant on integral curve of F_2)

$H_2(q_2, p_2) = H_2(q_0, p_0) = H_2(q_{12}, p_{12})$ (H₂ invariant on integral curve of F_1)

$H_2(q_0, p_0) = H_2(q_{12}, p_{12}) = H_2(q_1, p_1)$ (H₂ invariant on integral curve of F_1)

$H_1(q_0, p_0) = H_1(q_{12}, p_{12}) = H_1(q_2, p_2)$ (H₁ invariant on integral curve of F_2)