On the rigidity of Lagrangian products

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Statement of problem

Symplectic embedding problem: $(X_1, \omega_1) \stackrel{?}{\hookrightarrow} (X_2, \omega_2)$.

Rigidity vs Flexibility

- Rigidity: (Gromov '85) $B^{2n}(1) \hookrightarrow Z^{2n}(r) \Leftrightarrow r \geq 1$, where

$$B^{2n}(1) = \left\{ \mathbf{z} \in \mathbb{C}^n \mid \sum_{j=1}^{n} \pi |z_j|^2 < 1 \right\}, \; Z^{2n}(r) = B^2(r) \times \mathbb{C}^{n-1}.$$

- Flexibility: (McDuff and Schlenk '12, Frenkel and Müller '15, ...) embeddings of 4-d ellipsoids into balls and polydiscs.

Lagrangian products

Definition (Lagrangian products)

Given open subsets $A, B \subset \mathbb{R}^n$,

$$A \times_L B := \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \mathbf{x} \in A\,,\ \mathbf{y} \in B\right\} \subset \left(\mathbb{R}^{2n}, \omega_{\mathrm{can}}\right).$$

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Motivation:

- Ostrover '14: phase spaces of billiards.
- Ramos '15: embeddings of 4-d Lagrangian bidisc using an integrable billiard.

Some subsets of $\mathbb{R}^n$

Definition

$A \subset \mathbb{R}^n$ open, bounded.

- Balanced:
  $(x_1, \dots, x_n) \in A \Rightarrow [-|x_1|, |x_1|] \times \dots \times [-|x_n|, |x_n|] \subset A$.
- Symmetric: balanced and invariant under permutation of coordinates.
- Convex: convex, Concave: $\mathbb{R}^n_{\geq 0} \setminus A$ convex.

Example

- $\bigcirc$: convex, symmetric, $\diamondsuit$: concave, balanced, $\heartsuit$: none.
- $\forall p \in [1, +\infty]$, B^n_p = unit ball in L^p-norm: symmetric.

Main result: a rigidity theorem

Theorem (Ramos and S., '17)

If $A, A' \subset \mathbb{R}^n$ open and bounded such that

- $A \in \{B_1^n, B_\infty^n\}$, A' *convex/concave balanced, or*
- A *convex symmetric,* $A' \in \{B_1^n, B_\infty^n\}$, *or*
- A *convex symmetric,* A' *concave symmetric, or*
- $A = B_p^n$, $A' = rB_q^n$, $p, q \in [1, +\infty]$, $r \in]0, +\infty[$,

then

$$B_\infty^n \times_L A \hookrightarrow B_\infty^n \times_L A' \Leftrightarrow A \subset A'.$$

Example

$$\square \times_L \bigcirc \hookrightarrow \square \times_L \diamondsuit \Leftrightarrow \bigcirc \subset \diamondsuit$$
$$\square \times_L \square \hookrightarrow \square \times_L \diamondsuit \Leftrightarrow \square \subset \diamondsuit.$$

Strategy of proof

Sketch of proof.

Fix $A, A' \subset \mathbb{R}^n$ as above.

1. Prove that $B^n_\infty \times_L A, B^n_\infty \times_L A'$ are symplectomorphic to toric domains.
2. Use Gromov width and cube capacity (Gutt and Hutchings '17).

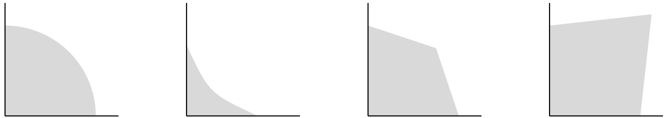
$\square$

Toric domains

Standard toric action $(\mathbb{R}/\mathbb{Z})^n \curvearrowright \left(\mathbb{R}^{2n}, \omega_{\mathrm{can}}\right) \cong (\mathbb{C}^n, \omega_{\mathrm{can}})$ with moment map $\mu\left(z_1, \ldots, z_n\right) = \pi\left(|z_1|^2, \ldots, |z_n|^2\right)$.

Definition

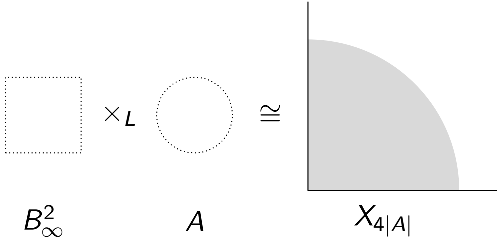
If $\Omega \subset \mathbb{R}^n_{\geq 0}$ open, the toric domain associated to Ω is $X_\Omega := \mu^{-1}(\Omega)$.



Toric Lagrangian products

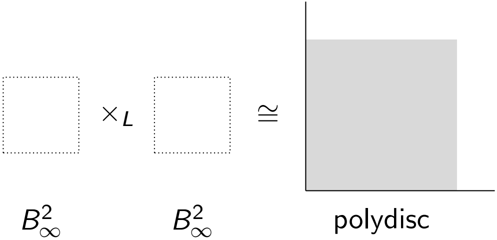
Theorem (Ramos and S., '17)

$\forall A \subset \mathbb{R}^n$ balanced, $B^n_\infty \times_L A \cong X_{4|A|}$, where $|A| := A \cap \mathbb{R}^n_{\geq 0}$.



Previously known cases

- $n = 1$: rectangle $\cong$ disc;
- $n = 2$: only two cases (Schlenk '03, Latschev, McDuff, Schlenk '13);



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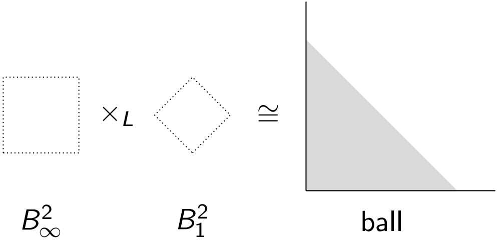
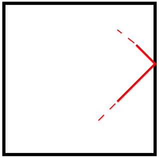


Previously known cases

- $n = 1$: rectangle $\cong$ disc;
- $n = 2$: only two cases (Schlenk '03, Latschev, McDuff, Schlenk '13);
- $n \geq 3$: $B^n_\infty \times_L B^n_\infty \cong$ polydisc (Schlenk '03).

Billiard in the cube

$$H : (\mathrm{T}^* \bar{B}^n_\infty, \omega_{\mathrm{can}}) \cong \bar{B}^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}$$
$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \|\mathbf{y}\|^2$$
$+$ reflection law.



Conserved quantities: $\frac{1}{2}y_1^2, \frac{1}{2}y_2^2, \ldots, \frac{1}{2}y_n^2 \rightsquigarrow$ integrable!

A naïve argument

Aim: $A \subset \mathbb{R}^n$ balanced $\Rightarrow B^n_\infty \times_L A \cong X_{4|A|}$.

Heuristic argument.

- 'Actions' $\mathbf{I}_0$ for $\Phi_0 : \bar{B}^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$,

$$\Phi_0(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2}y_1^2, \ldots, \frac{1}{2}y_n^2\right).$$

□

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$$\Phi_0(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2} y_1^2, \ldots, \frac{1}{2} y_n^2 \right).$$

- Karshon and Lerman '15 $\Rightarrow \exists\, \Psi : \bar{B}^n_\infty \times_L \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^{2n}$ with $\Psi\left(B^n_\infty \times_L A\right) = X_{4|A|}$.

□

Approximating the billiard (Benci and Giannoni, '89)

Family of integrable systems: for $\epsilon > 0$,

$$\Phi_\epsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$$
$$\Phi_\epsilon(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{2}\left(y_1^2 + \frac{\epsilon}{1 - x_1^2} \right), \ldots, \frac{1}{2}\left(y_n^2 + \frac{\epsilon}{1 - x_n^2} \right) \right).$$

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For each $\epsilon > 0$,

- actions $\mathbf{I}_\epsilon : \Phi_\epsilon(B^n_\infty \times_L \mathbb{R}^n) = \left[\frac{\epsilon}{2}, +\infty\right[^n \to \mathbb{R}^n_{\geq 0}$;
- symplecto $\Psi_\epsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^{2n}$ with $\mu \circ \Psi_\epsilon = \mathbf{I}_\epsilon \circ \Phi_\epsilon$.

A proof (inspired by Ramos '15)

Aim: $A \subset \mathbb{R}^n$ balanced $\Rightarrow B^n_\infty \times_L A \cong X_{4|A|}$.

Proof.

- For $\epsilon > 0$, $P_\epsilon := \Phi_\epsilon^{-1}\left(\mathbf{I}_0^{-1}(4|A|)\right)$. Then
  - $\bigcup_{\epsilon>0} \mathrm{cl}\,(P_\epsilon) = B^n_\infty \times_L A$;
  - $\bigcup_{\epsilon>0} \Psi_\epsilon\,(\mathrm{cl}\,(P_\epsilon)) = X_{4|A|}$;
  - $\epsilon_1 > \epsilon_2 \Rightarrow \mathrm{cl}\,(P_{\epsilon_1}) \subset \mathrm{cl}\,(P_{\epsilon_2})\,,\ \Psi_{\epsilon_1}(\mathrm{cl}\,(P_{\epsilon_1})) \subset \Psi_{\epsilon_2}(\mathrm{cl}\,(P_{\epsilon_2}))$.

□

A proof (inspired by Ramos '15)

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Proof.

- For $\epsilon > 0$, $P_\epsilon := \Phi_\epsilon^{-1}\left(\mathbf{I}_0^{-1}(4|A|)\right)$. Then
  - $\bigcup_{\epsilon>0} \mathrm{cl}\left(P_\epsilon\right) = B^n_\infty \times_L A$;
  - $\bigcup_{\epsilon>0} \Psi_\epsilon\left(\mathrm{cl}\left(P_\epsilon\right)\right) = X_{4|A|}$;
  - $\epsilon_1 > \epsilon_2 \Rightarrow \mathrm{cl}\left(P_{\epsilon_1}\right) \subset \mathrm{cl}\left(P_{\epsilon_2}\right)$, $\Psi_{\epsilon_1}\left(\mathrm{cl}\left(P_{\epsilon_1}\right)\right) \subset \Psi_{\epsilon_2}\left(\mathrm{cl}\left(P_{\epsilon_2}\right)\right)$.
- Use symplectic isotopy extension theorem.

□

Further work

- Dimension 2: polygons and ellipses.
- Higher dimensions: non-commutatively integrable billiards and multiplicity-free Hamiltonian actions, *e.g.* $B_2^3 \times_L B_2^3$.
- Non-integrable case?