Hamiltonian S^1-spaces with large minimal Chern number

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- otherwise $k = \infty$

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Symplectic S^1-actions

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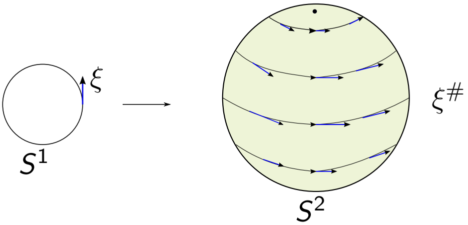
$\xi^{\#}$: vector field associated to the flow of symplectomorphisms



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- Otherwise *non-Hamiltonian*.

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Hamiltonain $S^1 \curvearrowright (M, \omega)$, $\psi \colon M \to \mathbb{R}$

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$$S^1 \ni \lambda * [z_0 : z_1 : \ldots : z_n] = [z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2 : \ldots : \lambda^{a_n} z_n]$$

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   Charton, *"Hamiltonian manifolds with high index"*, Master thesis. University of Cologne, 2017.

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- $k_0 = 0$ exactly if c_1 is torsion.

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So

$$\boxed{1 \leq k}$$

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$\Longrightarrow\ k-1 \leq \deg(H) \leq n.$

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- $\mathcal{T}(M)$ is the total Todd class of M, i.e. $\mathcal{T}(M) = \sum_j T_j$

T_j: j-th Todd polynomial, e.g.
$T_0 = 1, \quad T_1 = \frac{c_1}{2}, \quad T_2 = \frac{c_1^2 + c_2}{12}, \quad T_3 = \frac{c_1 c_2}{24}, \ldots$

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- If $k \geq 2$ then $H(-1) = H(-2) = \ldots = H(-k+1) = 0$ (S. '17)

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5. using this formula, compute the limits of $P_m(t)$ for $t \to \infty$ and $t \to 0$, and realize that these limits are both zero for all $m \in \{-1, \ldots, -k+1\}$.

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5. using this formula, compute the limits of $P_m(t)$ for $t \to \infty$ and $t \to 0$, and realize that these limits are both zero for all $m \in \{-1, \ldots, -k+1\}$.
6. Hence $P_m(t) \equiv 0$ for all $m \in \{-1, \ldots, -k+1\} \Longrightarrow$ $H(m) = 0$ for all $m \in \{-1, \ldots, -k+1\}$.

Relations between Betti numbers and k

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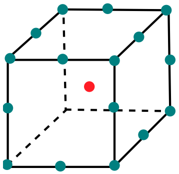

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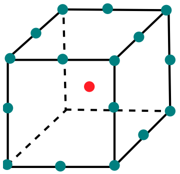

$e \longleftrightarrow S_e^2 := \mu^{-1}(e)$.

$$\textbf{Fact: } c_{n-1} = \mathrm{PD}\left[\bigcup_{e \in E} S_e^2\right].$$

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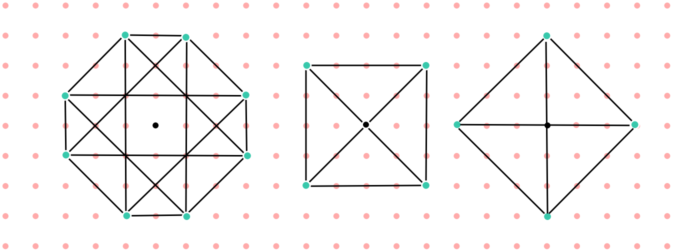
$\mu(\cup_{e\in E} S^2_e)$ is a graph in $\mathrm{Lie}(\mathbb{T})^*$:



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More precisely:

- If n is *even*

$$\sum_{e\in E} c_1[S^2_e] + \frac{n}{2}\,\chi(M) = 12 \sum_{k=1}^{\frac{n}{2}} \left[k^2 b_{n-2k}(M)\right]$$

- If n is *odd*

$$\sum_{e\in E} c_1[S^2_e] + \left(\frac{n-3}{2}\right)\chi(M) = 12 \sum_{k=1}^{\frac{n-1}{2}} \left[k(k+1) b_{n-1-2k}(M)\right]$$

Special cases:

Under the same hypotheses:

- If $n = 2$ then
$$\sum_{e \in E} c_1[S^2_e] + \chi(M) = \mathbf{12}\,.$$
- If $n = 3$ then
$$\sum_{e \in E} c_1[S^2_e] = \mathbf{24}\,.$$

Sketch of the Proof:

- Since $c_{n-1} = \mathrm{PD}\left[\cup_{e\in E} S^2_e\right]$,

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- $c_1 c_{n-1}[M]$ only depends on the Betti numbers ("Rigidity of Hirzebruch genus") (Godinho-S. '12).

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Both $\sum_{e\in E} c_1[S^2_e]$ and $|E|$ depend on the Betti numbers.

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(M, ω, S^1) *positive* Hamiltonian S^1-space with a toric one-skeleton and minimal Chern number k:

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k	$n = 2$ b_2	$n = 3$ b_2	$n = 4$ (b_2, b_4)	$n = 5$ (b_2, b_4)
1	$b_2 \leq 4$	$b_2 \leq 7$		
2	2	$b_2 \leq 3$		
3	1	1	$(1, 2), (2, 3), (3, 1), (4, 2), (6, 1)$	
4		1	$(1, 2)$	
5			$(1, 1)$	$(1, 1), (6, 1)$
6				$(1, 1)$

Table: List of allowed values of b_2 and b_4 for $2 \leq n \leq 5$.

Relations between Betti numbers and k

Unimodality of even Betti numbers:
The sequence of even Betti numbers of (M, ω, S^1) is *unimodal* if:

$$b_0 \leq b_2 \leq \ldots \leq b_{2\lfloor \frac{n}{2} \rfloor}$$

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E.g. Transversality of unstable and stable manifolds of moment map $\implies$ index increasing.

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- $\sum_{e \in E} c_1[S_e^2] - k|E|$ is a non-negative multiple of k
- $\sum_{e \in E} c_1[S_e^2]$ and $|E|$ can be expressed as linear combinations of (even) Betti numbers.

From $\chi(M) = n + 1$ to $M \simeq \mathbb{C}P^n$

Local form of the action around $p \in M^{S^1}$: ***Weights*** *at p*

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If $p \in M^{S^1}$ then $S^1 \curvearrowright T_pM \simeq \mathbb{C}^n$:

$$\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{w_{1p}} z_1, \ldots, \lambda^{w_{np}} z_n)$$

$w_{1p}, \ldots, w_{np} \in \mathbb{Z}$ are the **weights** of the S^1 action at p.

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The S^1-action on $TM|_{M^{S^1}}$ coincides with the standard S^1-action on $\mathbb{C}P^n$.

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$\exists \quad N_i \in \mathbb{Z}\setminus\{0\}$ s.t. $\frac{c_1^i}{N_i}$ is a generator of $H^{2i}(M;\mathbb{Z})$, $\quad i = 0, \dots, n$

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Hence: if the weights agree with those of $\mathbb{C}P^n$, the cohomology ring does as well.

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If $k = n + 1$, then M is homotopy equivalent to $\mathbb{C}P^n$.

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- (Corollary of Hurewicz) $\implies$ f is a homotopy equivalence.

Thank you!