# Noncommutative Painlevé equations and systems of Calogero type.

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# Plan of the talk

- Painlevé equations : isomonodromic deformations and confluence.
- Calogero–Painlevé correspondence : classical and quantum.
- Multi–particles systems and their isomonodromic formulation.
- The case of PII : Stokes data and applications.

Collaboration with M. Bertola, M. Cafasso .

# Painlevé equations

Painlevé property : The only movable singularities are poles

$$(PVI)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\frac{\gamma(t-1)}{(\lambda-1)^2}+\frac{\delta t(t-1)}{(\lambda-t)^2}\right).$$

$$(PV)\quad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)^2}{t^2}\left(\alpha+\frac{\beta}{\lambda^2}+\frac{\gamma t}{(\lambda-1)^2}+\frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right).$$

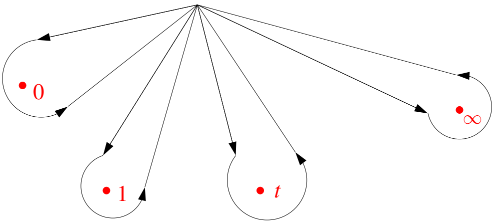
$$(PIV)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha)\lambda + \frac{\beta}{\lambda}.$$

$$(PIII)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\lambda^2}{4t^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\gamma\lambda+\frac{\delta t^2}{4\lambda^3}\right).$$

$$(PII)\quad \frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha.$$

$$(PI)\quad \frac{d^2\lambda}{dt^2} = 6\lambda^2 + t.$$

# Isomonodromic deformations




$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}(\theta_\infty, -\theta_\infty).$$

Isomonodromic deformations $\longleftrightarrow$ Schlesinger equations :

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}.$$

$$\Updownarrow$$

Painlevé VI for $\lambda =$ (simple) zero of $(\mathcal{A})_{12}$.

## Isomonodromic deformations-Reminder

Consider a Fuchsian system of rank $N$ sur $\mathbb{P}^1$ :

$$\partial_z \Phi = A(z)\Phi, \qquad \Phi(z) \in GL(N, \mathbb{C})$$

$$A(z) = \sum_{\nu=1}^{n} \frac{A_\nu}{z - a_\nu}$$

- $n$ regular singular points singuliers $a_1, \ldots, a_n$
- No singularity in $\infty \Longrightarrow A_\infty := -\sum_\nu A_\nu = 0$

Monodromy Representation :

$$\rho : \ \pi_1(\mathbb{P}^1 \backslash \{a\}) \to GL(N, \mathbb{C}) \qquad (1)$$

- On can suppose $\operatorname{tr} A_\nu = 0$, monodromy matrices $M_\nu \in SL(N, \mathbb{C})$
- Different choice of base in solution space $\Longrightarrow$ equivalent representations.

Riemann-Hilbert correspondence :

$$\mathcal{RH} : \begin{array}{c}\text{set of parameteres } \mathcal{P} \\ \text{of linear system}\end{array} \longrightarrow \begin{array}{c}\text{space } \mathcal{M} \\ \text{of the monodromy data}\end{array} \qquad (2)$$

## Isomonodromic Deformation

Schlesinger equations :

$$\partial_{a_\mu} A_\nu = \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \qquad \mu \neq \nu \tag{3}$$

- Hamiltonian non-autonomous system.
- $a_\nu$'s play the role of "time"
- **Tau-function** generates the hamiltonians isomonodromic flows :

$$H_\mu := \partial_{a_\nu} \ln \tau(a) = \frac{1}{2} \operatorname{res}_{z=a_\nu} \operatorname{tr} A^2(z) \tag{4}$$

- $\mathrm{Tr}(A_\nu)$ is conserved due to the Lax form
- it is useful to consider the space

$$\mathcal{M}_{\vec{\theta}} := \mathrm{Hom}(\pi_1(\mathbb{P}^1 \backslash \{a\}, SL(N, \mathbb{C}))/ \sim$$

  of local fixed monodromy

**Exemple** : $N = 2$

- Schlesinger $\Longrightarrow$ Garnier system $\mathcal{G}_{n-3}$
- $\dim \mathcal{M}_{\vec{\theta}} = 3(n-1) - 3 - n = 2(n-3)$
  (the complet set of quantites conserved for $\mathcal{G}_{n-3}$ !)
- $n = 4 \Longrightarrow$ Painlevé VI ; $a = \{0, t, 1, \infty\}$

## The Monodromy Variety of Painlevé VI

Let the $\zeta(t)$ be defined by

$$\zeta(t) = t(t-1)\frac{d}{dt}\ln\tau(t)$$

Sigma Painlevé VI :

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

The monodromy data :

- To each solution on associate (the conjugation class of a triple $(M_0, M_t, M_1)$
- $x_\nu = 2\cos 2\pi\theta_\nu = \operatorname{tr} M_\nu$ (with $\nu = 0, t, 1, \infty$) give 4 parameters of PVI
- two other coordinates $\Rightarrow$ integration constants
- introduce $x_{\mu\nu} = 2\cos 2\pi\sigma_{\mu\nu} = \operatorname{tr} M_\mu M_\nu$, then [Jimbo, '82]

$$x_{0t}x_{1t}x_{01} + x_{0t}^2 + x_{1t}^2 + x_{01}^2 + x_0^2 + x_t^2 + x_1^2 + x_\infty^2 + x_0x_tx_1x_\infty =$$
$$= (x_0x_t + x_1x_\infty)\,x_{0t} + (x_1x_t + x_0x_\infty)\,x_{1t} + (x_0x_1 + x_tx_\infty)\,x_{01} + 4$$

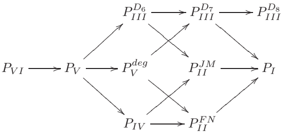
The triple $\boldsymbol{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$ verify the above identity and one can interprete as a pair of integration constants for PVI.

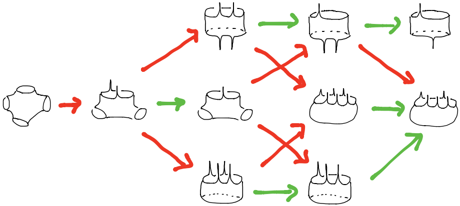
# Confluence and Lax systems

Each of the Painlevé equations can be written as a Lax system (of isomonodromic type)

$$\begin{cases} \dfrac{\partial}{\partial z}\Psi(t;z) & = & U(t;z)\Psi(t;z), \\ \dfrac{\partial}{\partial t}\Psi(t;z) & = & V(t;z)\Psi(t;z). \end{cases} \qquad \Longrightarrow \qquad \frac{\partial V}{\partial z} - \frac{\partial U}{\partial t} = [U, V].$$

Confluence of Painlevé equations :



# Calogero–Painlevé correspondence for PVI

Take the elliptic curve $y^2 = z(z-1)(z-t)$ and the associated Weierstrass $\wp$ function

$$\wp(u;1,\tau) := \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left( \frac{1}{(u+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right).$$

Define $e_n := \wp(\omega_n), \quad \omega_1 := 1/2,\ \omega_2 := -(1+\tau)/2,\ \omega_3 := \tau/2.$

Theorem (Fuchs, Painlevé, Lamé, Manin) :

Let $q$ be implicitely defined by

$$\lambda = \frac{\wp(q) - e_1}{e_2 - e_1} \quad \text{and} \quad \dot{q} := \frac{dq}{d\tau}.$$

Then the PVI equation is equivalent to the Hamiltonian system

$$\dot{q} = \frac{1}{2\pi i}\frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{1}{2\pi i}\frac{\partial H}{\partial q}, \quad H(p,q,\tau) := \frac{p^2}{2} - \sum_{n=0}^{3} g_n \wp(q+\omega_n)$$

with

$$g_0 = \alpha,\ g_1 = -\beta,\ g_2 = \gamma,\ g_3 = -\delta + \frac{1}{2}.$$

# Why “Calogero – Painlevé” ?

Remark :

Levin et Olshanetsky observed that Manin’s Hamiltonian

$$H(p,q,\tau) := \frac{p^2}{2} - \sum_{n=0}^{3} g_n \wp(q+\omega_n)$$

is the rank–one case of a system of $n$ particules $q_1, \ldots, q_n$ introduced by Inozemtsev

$$H_{VI} = \sum_{j=1}^{n} \left( \frac{p_j^2}{2} + \sum_{n=0}^{3} g_n^2 \wp(q_j+\omega_n) \right) + g_4^2 \sum_{j \neq k} \left( \wp(q_j - q_k) + \wp(q_j + q_k) \right)$$

and generalising the elliptic Calogero–Moser system.

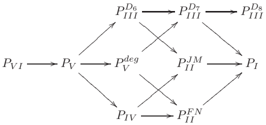
Manin’s system is non–autonomous, while Inozemtsev’s system is an (integrable) autonomous Hamiltonian system.

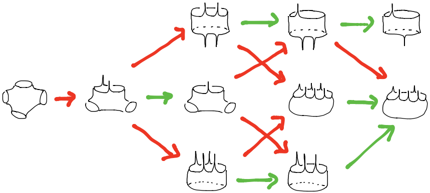
# Takasaki : “Painlevé–Calogero revisted” (2000)

Theorem (Takasaki) :
Each of the Painlevé equation can be written, using the confluence scheme, as an Hamiltonian system with Hamiltonian of the type

$$H(q,p;t) = \frac{p^2}{2} - V(q;t),$$

and there is a canonical transformation between these Hamiltonians and the Okamoto’s (polynomial) ones.
The correspondence extends to the case of many particles.



$$H_{VI}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}+\sum_{n=0}^{3} g_n^2\wp(q_j+\omega_n)\right)+g_4^2\sum_{j\neq k}\left(\wp(q_j-q_k)+\wp(q_j+q_k)\right).$$

$$H_{V}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{\sinh^2(q_j/2)}-\frac{\beta}{\cosh^2(q_j/2)}+\frac{\gamma t}{2}\cosh(q_j)+\frac{\delta t^2}{8}\cosh(2q_j)\right)+$$
$$+g_4^2\sum_{j\neq k}\left(\frac{1}{\sinh^2((q_j-q_k)/2)}+\frac{1}{\sinh^2((q_j+q_k)/2)}\right).$$

$$H_{IV}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(\frac{q_j}{2}\right)^6-2t\left(\frac{q_j}{2}\right)^4-2(t^2-\alpha)\left(\frac{q_j}{2}\right)^2+\beta\left(\frac{q_j}{2}\right)^{-2}\right)+$$
$$+g_4^2\sum_{j\neq k}\left(\frac{1}{(q_j-q_k)^2}+\frac{1}{(q_j+q_k)^2}\right).$$

$$H_{III}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{4}e^{q_j}+\frac{\beta t}{4}e^{-q_j}-\frac{\gamma}{8}e^{2q_j}+\frac{\delta t^2}{8}e^{-2q_j}\right)+g_4^2\sum_{j\neq k}\frac{1}{\sinh^2\left((q_j-q_k)/2\right)}.$$

$$H_{II}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(q_j^2+\frac{t}{2}\right)^2-\alpha q_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

$$H_{I}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-2q_j^3-tq_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

# Quantum Calogero–Painlevé correspondence

Theorem (Zabrodin–Zotov, Suleimanov) :
For each of the Painlevé equations it exists a Lax pair

$$\begin{cases} \frac{\partial}{\partial z}\Phi(t;z) & = & \tilde{U}(t;z)\Phi(t;z), \\ \frac{\partial}{\partial t}\Phi(t;z) & = & \tilde{V}(t;z)\Phi(t;z). \end{cases} \implies \frac{\partial \tilde{V}}{\partial z} - \frac{\partial \tilde{U}}{\partial t} = [\tilde{U}, \tilde{V}].$$

such that the first component $\phi(t;z) = \phi_1(t;z)$ of

$$\Phi(t;z) = \begin{pmatrix} \phi_1(t;z) \\ \phi_2(t;z) \end{pmatrix}$$

satisfies the equation

$$\partial_t \phi(t;z) = \Big[H(z,\partial_z) - H(q,p)\Big]\phi(t;z).$$

Remark :
For the case of PII and PIII these results apply to the study of $\beta$–models.

$$\frac{\beta}{2}\partial_t \phi(t;z) = \Big[H(z,\partial_z) - H(q,p)\Big]\phi(t;z).$$

Ramirez, Rider, Virag, Rumanov, Grava, Its, Kapaev, Mezzadri....

Our aim :

Describing an isomonodromic formulation of multi–particles Calogero–Painlevé systems.

- *A central issue will be to find an isomonodromic description of the multi–component Painlevé equations. If such an isomonodromic description does exist, it should be related to a new geometric structure* (Takasaki).
- Applying the classical tool of isomonodromic deformations to such systems.
- Quantization, application to beta models.

Our procedure :

Applying an Hamiltonian reduction à la Kazdan-Konstant-Sternberg on a matrix–valued version of Painlevé equations.

# The simplest example : PI

A matrix–valued Lax pair for the first Painlevé equation :

$$\begin{cases} \frac{\partial}{\partial z}\Psi(t;z) = \begin{pmatrix} \mathbf{p} & z-\mathbf{q} \\ z^2+z\mathbf{q}+\mathbf{q}^2+\frac{t}{2} & -\mathbf{p} \end{pmatrix}\Psi(t;z) \\ \frac{\partial}{\partial t}\Psi(t;z) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{z}{2}+\mathbf{q} & 0 \end{pmatrix}\Psi(t;z) \end{cases} \Longrightarrow \begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = \frac{3}{2}\mathbf{q}^2+\frac{t}{4}. \end{cases}$$

Lemma I :
The Lax equations are Hamiltonian on $\mathcal{M} := T^*\mathfrak{gl}_n$ with respect to the standard symplectic structure $\omega := d\mathbf{q}\wedge d\mathbf{p}$ and

$$H := \mathrm{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - t\frac{\mathbf{q}}{4}\right).$$

Moreover, the commutator $[\mathbf{p},\mathbf{q}]$ is conserved along the flow.

# Reduction à la Kazhdan-Konstant-Sternberg

$$\mathcal{M}_{g_4} := \left\{ (\mathbf{q}, \mathbf{p}) \in \mathcal{M} \text{ s.t. } [\mathbf{p}, \mathbf{q}] = i g_4 (\mathbf{1}_n - v^T v) \right\}, \quad v := (1, \ldots, 1).$$

Lemma II :

Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{M}_{g_4}$ with $\mathbf{q}$ diagonalizable. Then it exists $G$ such that

$$G^{-1} \mathbf{q} G = X = \mathrm{diag}(q_1, \ldots, q_n),$$

and, if $Y = G^{-1} \mathbf{p} G$, then

$$[Y, X] = i g_4 (\mathbf{1}_n - v^T v).$$

Corollary :

$$Y_{ij} = -\frac{i g_4}{q_i - q_j}, \ i \neq j = 1, \ldots, n.$$

The variables $p_i := Y_{ii}, i = 1, \ldots, n$ are the conjugated variables of $\{q_1, \ldots, q_n\}$ for the reduced system.

More precisely : $\mu : \mathcal{M} \to \mathfrak{gl}_\ell^*$, $\mu(\mathbf{q}, \mathbf{p}) := [\mathbf{p}, \mathbf{q}]$ is the moment map and $\{q_i, p_j\}$ are the symplectic coordinates on the quotient $\mu^{-1}(\mathcal{O})/PGL_n(\mathbb{C})$, with $\mathcal{O}$ orbit of $\mathrm{diag}(n-1, -1, \ldots, -1)$.

# Reduction à la Kazhdan-Konstant-Sternberg II

The gauge–transformed eigenfunction

$$\Phi(t;z) := (G^{-1}(t) \otimes \mathbf{1}_2)\Psi(t;z) = \mathbf{G}^{-1}(t)\Psi(t;z)$$

satisfies the Lax pair

$$\begin{cases} \dfrac{\partial}{\partial z}\Phi(t;z) & = & \tilde{U}(t;z)\Phi(t;z) \\ \dfrac{\partial}{\partial t}\Phi(t;z) & = & \tilde{V}(t;z)\Phi(t;z) \end{cases}$$

$$\tilde{U} := \mathbf{G}^{-1}(t)\begin{pmatrix} \mathbf{p} & z-\mathbf{q} \\ z^2+z\mathbf{q}+\mathbf{q}^2+\frac{t}{2} & -\mathbf{p} \end{pmatrix}\mathbf{G}(t) = \begin{pmatrix} Y & z-X \\ z^2+zX+X^2+\frac{t}{2} & -Y \end{pmatrix}$$

$$\tilde{V} := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{z}{2}+X & 0 \end{pmatrix} - \mathbf{G}^{-1}(t)\dot{\mathbf{G}}(t) = \begin{pmatrix} -F & \frac{1}{2} \\ \frac{z}{2}+X & -F \end{pmatrix}, \quad F(t) := G^{-1}(t)\dot{G}(t)$$

# Reduction à la Kazhdan-Konstant-Sternberg III

Compatibility conditions yields

$$\begin{cases} \dot{X} &= Y - [F, X], \\ \dot{Y} &= \frac{3}{2}X^2 + \frac{t}{4} - [F, Y], \end{cases}$$

$$\Longrightarrow F_{j,k} = -\frac{ig_4}{(x_j - x_k)^2},\ j \neq k, \quad \ddot{q}_j = \frac{3}{2}q_j^2 + \frac{tq_j}{4} - \sum_{k\neq j} \frac{g_4^2}{(x_j - x_k)^3}$$

On the other hand

$$H(\mathbf{q}, \mathbf{p}) = H(X, Y) = \frac{Y^2}{2} - \frac{X^3}{2} - \frac{tX}{4} = \sum_{j=1}^{n} \left( \frac{p_j^2}{2} - \frac{q_j^3}{2} - \frac{tq_j}{4} \right) + g_4^2 \sum_{j\neq k} \frac{1}{(q_j - q_k)^2}.$$

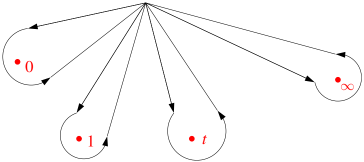
# Painlevé II

$$\begin{cases} \dfrac{\partial}{\partial z}\Psi(t;z) = \begin{pmatrix} i\dfrac{z^2}{2} + i\mathbf{q}^2 + i\dfrac{t}{2} & z\mathbf{q} - i\mathbf{p} - \dfrac{\theta}{z} \\ z\mathbf{q} + i\mathbf{p} - \dfrac{\theta}{z} & -i\dfrac{z^2}{2} - i\mathbf{q}^2 - i\dfrac{t}{2} \end{pmatrix}\Psi(t;z), \\ \dfrac{\partial}{\partial t}\Psi(t;z) = \begin{pmatrix} i\dfrac{z}{2} & \mathbf{q} \\ \mathbf{q} & -i\dfrac{z}{2} \end{pmatrix}\Psi(t;z), \end{cases}$$

$$\Longrightarrow \begin{cases} \dot{\mathbf{q}} = \mathbf{p}, \\ \dot{\mathbf{p}} = 2\mathbf{q}^3 + t\mathbf{q} + \theta \end{cases}$$

$$\begin{aligned} H(X,Y) &= \frac{Y^2}{2} - \frac{1}{2}\left(X^2 + \frac{t}{2}\right)^2 - \theta X \\ &= \sum_{i=1}^{n}\left(\frac{p_i^2}{2} - \frac{1}{2}\left(q_i^2 + \frac{t}{2}\right)^2 - \theta q_i\right) + \sum_{j<k}\frac{g^2}{(q_j - q_k)^2}. \end{aligned}$$

# Spectral type and Fuchsian systems




$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}(\theta_\infty, -\theta_\infty).$$

$A_k$ with eigenvalues $(\theta_k, -\theta_k)$, $\quad k = 0, 1, t$.

$$\Downarrow$$

Painlevé VI = Fuchsian system of spectral type $11, 11, 11, 11$.
This is, essentially, the only Fuchsian system with phase space of dimension two (“accessory parameters”) and one–dimensional deformation.

Given a Fuchsian system of size $k$ with $N$ singular points, its spectral type is given by $N$ partitions $Y_1, \ldots, Y_N$ of $k$ and the dimension of its phase space is given by (Katz)

$$2 + (N-2)k^2 - \sum (Y_{j,\ell})^2.$$

# Spectral type and Fuchsian systems II

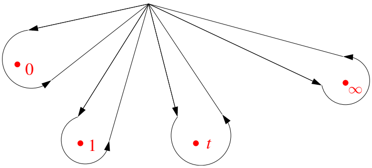
Proposition (Oshima) :

There are (essentially) just $4$ Fuchsian systems whose phase space has dimension $4$.
One is $11, 11, 11, 11, 11$, giving the Garnier system in two variables. The other three, which admits a one–dimensional deformation, are

$$21, 21, 111, 111, \quad 31, 22, 22, 1111, \quad 22, 22, 22, 211.$$

Kawakami : The Fuchsian system $n\,n, n\,n, n\,n, n\,n-1\,1$ gives a "matrix version" of the Painlevé VI equation, and its degenerations (confluence) yields a matrix version of all the other equations.

# Matrix Painlevé VI equation




$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_k \sim \begin{pmatrix} 0_n & 0_n \\ 0_n & \theta^k \mathrm{I}_n \end{pmatrix} \quad k = 0, 1, t$$

$$A_\infty = \mathrm{diag}(\theta_1^\infty, \ldots, \theta_1^\infty, \theta_2^\infty, \ldots, \theta_2^\infty, \theta_3^\infty).$$

Theorem (Kawakami) :
$\mathcal{A}(z) = \mathcal{A}(t, \mathbf{q}, \mathbf{p}; z)$ in such a way that

$$[\mathbf{p}, \mathbf{q}] = (\theta^0 + \theta^1 + \theta^t + \theta_1^\infty)\mathrm{I} + \mathrm{diag}(\theta_2^\infty, \ldots, \theta_2^\infty, \theta_3^\infty)$$

and the isomonodromic deformation of the system is governed by the Hamiltonian ($\theta := \theta^0 + \theta^1 + \theta^t$).

$$t(t-1)H_{VI} = \mathrm{Tr}\Big[\mathbf{q}(\mathbf{q}-1)(\mathbf{q}-t)\mathbf{p}^2 + \\ + \Big(\big(\theta^0 + 1 - [\mathbf{p}, \mathbf{q}]\big)\mathbf{q}(\mathbf{q}-1) + \theta^t(\mathbf{q}-1)(\mathbf{q}-t) + (\theta + 2\theta_1^\infty - 1)\mathbf{q}(\mathbf{q}-t)\Big)\mathbf{p} + \\ + (\theta + \theta_1^\infty)(\theta^0 + \theta^t + \theta_1^\infty)\mathbf{q}\Big]$$

# Hamiltonians by confluence

$$
\begin{aligned}
tH_V &= \mathrm{Tr}\Big[\mathbf{p}(\mathbf{p}+t)\mathbf{q}(\mathbf{q}-1)+\beta\mathbf{p}\mathbf{q}+\gamma\mathbf{p}-(\alpha+\gamma)t\mathbf{q}\Big],\\
tH_{IV} &= \mathrm{Tr}\Big[\mathbf{p}\mathbf{q}(\mathbf{p}-\mathbf{q}-t)+\beta\mathbf{p}+\alpha\mathbf{q}\Big],\\
tH_{III(D6)} &= \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2-(\mathbf{q}^2-\beta\mathbf{q}-t)\mathbf{p}-\alpha\mathbf{q}\Big],\\
tH_{III(D7)} &= \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\alpha\mathbf{p}\mathbf{q}+t\mathbf{p}+\mathbf{q}\Big],\\
tH_{III(D8)} &= \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\mathbf{p}\mathbf{q}-\mathbf{q}-t\mathbf{q}^{-1}\Big],\\
tH_{II} &= \mathrm{Tr}\Big[\mathbf{p}^2-(\mathbf{q}^2+t)\mathbf{p}-\alpha\mathbf{q}\Big],\\
tH_{I} &= \mathrm{Tr}\Big[\mathbf{p}^2-\mathbf{q}^3-t\mathbf{q}\Big].
\end{aligned}
$$

$$[\mathbf{p},\mathbf{q}] = \mathrm{const}$$

# General procedure :

- We start with a Lax pair of type

$$\begin{cases} \frac{\partial}{\partial z}\Psi(z;t) = A(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Psi(z;t) \\ \frac{\partial}{\partial t}\Psi(z;t) = B(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Psi(z;t), \end{cases} \Longrightarrow \dot{\mathbf{q}} = \mathcal{A}(\mathbf{q},\mathbf{p},t), \qquad \dot{\mathbf{p}} = \mathcal{B}(\mathbf{q},\mathbf{p},t)$$

  with $\mathcal{A}, \mathcal{B}$ polynomials (rationals) in $\mathbf{q}, \mathbf{p}$ such that the equations above are Hamiltonians and

  $\mathcal{A}$ is of degree at most $1$ in $\mathbf{p}$ and $\mathcal{B}$ is of degree at most $2$ in $\mathbf{p}$.

- $[\mathbf{p},\mathbf{q}]$ is a constant of motions, hence we apply KKS reduction and we get the Lax pair

$$\begin{cases} \frac{\partial}{\partial z}\Phi(z;t) = A(z;X,X^{-1},Y,t)\Phi(z;t) \\ \frac{\partial}{\partial t}\Phi(z;t) = \Big(B(z;X,X^{-1},Y,t) - F(X,X^{-1},Y)\Big)\Phi(z;t) \end{cases}$$

  with $X = \mathrm{Diag}(q_1,\dots,q_n)$, $Y = \mathrm{Diag}(p_1,\dots,p_n) + \left(\frac{ig_4}{q_j - q_k}\right)_{j\neq k}$,

$$\Longrightarrow \begin{cases} \dot{X} &= \mathcal{A}(X,Y,t) + [X,F], \\ \dot{Y} &= \mathcal{B}(X,Y,t) + [Y,F]. \end{cases}$$

# General procedure II :

Proposition :

$$(x_i - x_j)^2 F_{i,j} = \Big([\mathcal{A}(X,Y),X]\Big)_{i,j}, \quad i \neq j,$$

$$F_{jj} = -\sum_{k:k\neq j} F_{jk} + K, \quad K := \frac{1}{n}\sum_{\ell,m:\ell\neq m} F_{\ell,m}.$$

All entries of $F$ are rational functions of $(x_1,\ldots,x_n)$ only.

Proof :

$$[X,\dot{X}] = 0 \Longrightarrow \Big[X,[X,F]\Big] = \Big[\mathcal{A}(X,Y),X\Big]. \quad \text{(This gives the first equation).}$$

$$0 = \frac{d}{dt}[X,Y] = [\mathcal{A}(X,Y),X] + [Y,\mathcal{B}(X,Y)] + \Big(\Big[[X,F],Y\Big] + \Big[X,[Y,F]\Big]\Big).$$

On the other hand

$$[\mathcal{A}(\mathbf{q},\mathbf{p}),\mathbf{p}] + [\mathbf{p},\mathcal{B}(\mathbf{q},\mathbf{p})] = 0 \Longrightarrow [\mathcal{A}(X,Y),X] + [Y,\mathcal{B}(X,Y)] = 0.$$

Hence

$$0 = \Big[[X,F],Y\Big] + \Big[X,[Y,F]\Big] = -\Big[[Y,X],F\Big] = [ig_4(v^T v),F]$$

The off–diagonal entries of the equation above give the linear system of equations

$$f_i + \sum_{j\neq i} F_{i,j} - f_k - \sum_{j\neq k} F_{j,k} = 0, \quad i,k = 1,\ldots,n;\ i \neq k.$$

# General procedure III :

Final result : multi–component Painlevé Hamiltonians of Okamoto type

$\Downarrow$

Calogero–Painlevé systems.

(Using Takasaki's canonical transformations.)

# Monodromy : the case of PII

$$\frac{d}{dz}\Psi(\mathbf{t};z) = A(\mathbf{t};z)\Psi(\mathbf{t};z); \quad A(\mathbf{t};z) := \begin{pmatrix} i\frac{z^2}{2} + i\mathbf{q}^2 + i\frac{\mathbf{t}}{2} & z\mathbf{q} - i\mathbf{p} - \frac{\theta}{z} \\ z\mathbf{q} + i\mathbf{p} - \frac{\theta}{z} & -i\frac{z^2}{2} - i\mathbf{q}^2 - i\frac{\mathbf{t}}{2} \end{pmatrix}.$$

Slight generalisation :

$$t \longmapsto \mathbf{t} := \mathrm{diag}(t_1, \ldots, t_n), \quad \frac{d}{dt} \longmapsto \frac{d}{d\mathbf{t}} := \sum_{i=1}^{n} \frac{d}{dt_i}$$

$$\Downarrow$$

$$\ddot{\mathbf{q}} = 2\mathbf{q}^3 + \frac{1}{2}[\mathbf{t}, \mathbf{q}]_+ + \theta$$

(Retakh - V. R., Bertola–Cafasso)

Theorem : Given the equation

$$\frac{d}{dz}\Psi(\mathbf{t};z) = A(\mathbf{t};z)\Psi(\mathbf{t};z),$$

There exists a unique piecewise analytic solution $\Psi = \{\Psi_\nu,\ \nu = 0,\dots,7\}$ satisfying

$$\Psi(\mathbf{t};z) \sim \left(\mathbf{1} + \frac{\alpha_1 \otimes \sigma_3 - \mathbf{q} \otimes \sigma_2}{z} + \mathcal{O}(z^{-2})\right) \mathrm{e}^{(\ln z + i\pi\epsilon)[\mathbf{q},\mathbf{p}]\otimes \mathbf{1}} \mathrm{e}^{\frac{i}{2}\left(\frac{z^3}{3} + \mathbf{t}z\right)\widehat{\sigma}_3},$$

The corresponding (matrix) Stokes operator $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ satisfy the relations

$$(\mathbf{X} + \mathbf{Z} + \mathbf{XYZ})Q + Q^{-1}\mathbf{Y} = 2i\sin(\pi\theta)$$

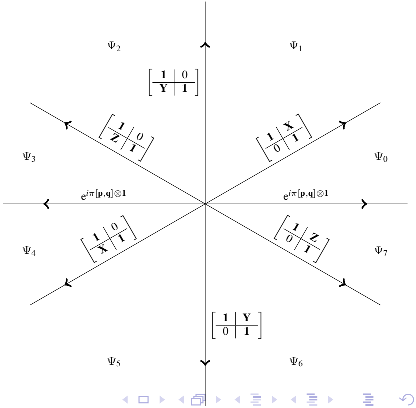
$$(\mathbf{XY} + \mathbf{1})Q - Q^{-1}(\mathbf{YX} + \mathbf{1}) = 0$$

$$\mathbf{Z}Q\mathbf{X} - \mathbf{X}Q^{-1}\mathbf{Z} + Q - Q^{-1} = 0$$

$$(\mathbf{YZ} + \mathbf{1})Q - Q^{-1}(\mathbf{ZY} + \mathbf{1}) = 0$$

$$\mathbf{Y}Q + Q^{-1}(\mathbf{X} + \mathbf{Z} + \mathbf{ZYX}) = 2i\sin(\pi\theta),$$

$$Q := \mathrm{e}^{i\pi[\mathbf{p},\mathbf{q}]}.$$



# “Classical” PII cubic I

Flaschka-Newell (’82) :

$$\begin{aligned} A(\lambda) &= \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & -4w \\ -4w & 0 \end{pmatrix}\lambda + \\ &+ \begin{pmatrix} -2w^2 - z & 2w_z \\ -2w_z & 2w^2 + z \end{pmatrix} - \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\frac{1}{\lambda} \end{aligned}$$

**Monodromy data** :

$$M_0 = C\begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{-2i\pi\alpha} \end{pmatrix}C^{-1},$$

$$S_{2j} = \begin{pmatrix} 1 & x_{2j} \\ 0 & 1 \end{pmatrix}, \qquad S_{2j-1} = \begin{pmatrix} 1 & 0 \\ x_{2j-1} & 1 \end{pmatrix}, \qquad j = 1, 2, 3.$$

$$x_{i+3} = x_i, \quad \text{and} \quad x_1x_2x_3 + x_1 + x_2 + x_3 = 2\sin(\pi\alpha)$$

# “Classical” and deformed cubics I

If $Q = e^{i\pi[\mathbf{p},\mathbf{q}]} = \pm 1$ then

$$[\mathbf{X},\mathbf{Y}] = [\mathbf{X},\mathbf{Z}] = [\mathbf{Y},\mathbf{Z}] = 0, \quad \mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{XYZ} = \mathrm{const},$$

as in the classical case.

Example (Bertola, Cafasso) :

$C = \left(c_{ij}\right)_{i,j=1}^{n}$ Hermitian,

$$\mathcal{A}i_{\mathbf{t}} : \left(L^2(\mathbb{R}_+)\otimes\mathbb{R}^n\right)^{\circlearrowleft}, \ (\mathcal{A}i_{\mathbf{t}}\vec{f})_i(x) := \int_{\mathbb{R}_+} c_{i,j}\mathrm{Ai}(x+y+t_i+t_j)f_j(y)dy.$$

$$-\frac{\partial^2}{\partial \mathbf{t}^2}\log\det(\mathrm{I}-\mathcal{A}i_{\mathbf{t}}^2) = \mathrm{Tr}(\mathbf{q}^2),$$

where $\mathbf{q}$ is the unique solution with asymptotics

$$\mathbf{q}_{ij}(\mathbf{t}) = c_{ij}\mathrm{Ai}(t_i+t_j) + \mathcal{O}\left(\sqrt{T}e^{-\frac{4}{3}(2T-2m)^{3/2}}\right)$$

$$T := \frac{1}{n}\sum t_j, \ m := \max_j (t_i - T), \ T\to\infty.$$

# “Classical” and deformed cubics II

Suppose $[\mathbf{p},\mathbf{q}] = i\hbar$ multiple of the identity. Then

- Upon identification $\mathbf{p} = i\hbar\frac{\partial}{\partial \mathbf{q}}$, the term $\alpha_1$ in

$$\Psi(\mathbf{t};z) \sim \left(\mathbf{1} + \frac{\alpha_1 \otimes \sigma_3 - \mathbf{q}\otimes\sigma_2}{z} + \mathcal{O}(z^{-2})\right) e^{(\ln z + i\pi\epsilon)[\mathbf{q},\mathbf{p}]\otimes \mathbf{1}} e^{\frac{i}{2}\left(\frac{z^3}{3}+\mathbf{t}z\right)\widehat{\sigma}_3},$$

  gives the quantum Hamiltonian of Painlevé II.
- The Stokes relations read

$$(\mathbf{X}+\mathbf{Z}+\mathbf{XYZ})Q + Q^{-1}\mathbf{Y} = 2i\sin(\pi\theta)$$

$$Q\mathbf{XY} - Q^{-1}\mathbf{YX} = Q^{-1} - Q$$

$$Q\mathbf{ZX} - Q^{-1}\mathbf{XZ} = Q^{-1} - Q$$

$$Q\mathbf{YZ} - Q^{-1}\mathbf{ZY} = Q^{-1} - Q.$$

  These relations are the same obtained by Mazzocco and V.R. (2012) as a result of the quantisation of the Poisson structure of the classical cubic of the monodromy surface of Painlevé II.

# Coupling $n$ scalar solution of Painlevé II

Example : How to construct solutions for $[\mathbf{p},\mathbf{q}] = ig_4(\mathbf{1}-v^T v),\ ig_4 = r \in 2\mathbb{Z}$ :

- Take $r=0$ and construct the block–diagonal eigenfunction

$$\Psi_0(z;\mathbf{X},\mathbf{Y},\mathbf{Z};\theta) = \left[\begin{array}{c|c} \mathrm{diag}[\psi_{11}^{(j)}(z;x^{(j)},y^{(j)},z^{(j)};\theta)]_{j=1}^n & \mathrm{diag}[\psi_{12}^{(j)}(z;x^{(j)},y^{(j)},z^{(j)};\theta)]_{j=1}^n \\ \hline \mathrm{diag}[\psi_{21}^{(j)}(z;x^{(j)},y^{(j)},z^{(j)};\theta)]_{j=1}^n & \mathrm{diag}[\psi_{22}^{(j)}(z;x^{(j)},y^{(j)},z^{(j)};\theta)]_{j=1}^n \end{array}\right],$$

  with asymptotics

$$\Psi_0 \sim \left(\mathbf{1}_{2n} + \mathcal{O}(z^{-1})\right) z^0 \mathrm{e}^{\frac{i}{2}\left(\frac{z^3}{3}+tz\right)\widehat{\sigma}_3},$$

- Define

$$\widehat{\Psi}_0(z) := (\mathcal{K}\otimes\mathbf{1})\Psi_0(z)(\mathcal{K}\otimes\mathbf{1})^{-1},$$

  where

$$\mathcal{K}^{-1}(\mathbf{1}-vv^t)\mathcal{K} = \mathrm{diag}(1-n,1,\dots,1).$$

- Combining a finite number of Schlesinger transformations, it exists (Jimbo–Miwa–Ueno) $\widehat{R}(z)$ such that

$$\widehat{\Psi}(z) := \widehat{R}(z)\widehat{\Psi}_0(z)$$

is still an eigenfunction with asymptotics

$$\widehat{\Psi}(z) = \left(\mathbf{1}_{2n} + \mathcal{O}(z^{-1})\right) z^{r\,\mathrm{diag}(1-n,1,\ldots,1)\otimes\mathbf{1}} \mathrm{e}^{\frac{i}{2}\left(\frac{z^3}{3}+tz\right)\widehat{\sigma}_3}.$$

- Finally,

$$\Psi(z) := (\mathcal{K}\otimes\mathbf{1})^{-1}\widehat{\Psi}(z;)(\mathcal{K}\otimes\mathbf{1})$$

is still an eigenfunction with the “good” exponent of formal monodromy $z^{r(\mathbf{1}-v^Tv)}$. Note that

$$\Psi(z) = R(z)\Psi_0(z), \quad R(z) := (\mathcal{K}\otimes\mathbf{1})^{-1}\widehat{R}(z)(\mathcal{K}\otimes\mathbf{1}).$$

Remark : These are “classical” solutions, with mutually commuting Stokes parameters.

# Two interesting questions for future

- "NC Ruijsenaars duality" :

  We can also use “dual coordinates” imposing

$$X := \mathrm{diag}(p_1, \ldots, p_n), \quad Y := \mathrm{diag}(q_1, \ldots, q_n) - \left( \frac{ig_4}{p_i - p_j} \right)_{i \neq j}.$$

  and, in these coordinates, the "dual" Hamiltonian reads

$$H^D := \sum_{i=1}^{n} \left( \frac{p_i^2}{2} - \frac{q_i^3}{2} - \frac{tq_i}{4} \right) - \sum_{j<k} \frac{g_4^2 (q_j + q_k)}{(p_j - p_k)^2}. \tag{5}$$

- What about other Calogero-Painlevé ?
- What about $q-$ Painlevé ?

Thanks !