L^2 geometry of symplectic vortices

Nuno M. Romão

nuno.romao@math.uni-augsburg.de

Geometric Aspects of Momentum Maps and Integrability

CSF Ascona, Switzerland

12.04.2018

Outline

Gauged sigma-models and the vortex equations

The L^2 metric on vortex moduli spaces

Vortices with toric targets

Asymptotic geometry of L^2 metrics

SUSY QM on vortex moduli spaces

On-going project: SUSY QM on vortex moduli spaces

Main collaborators:

- Marcel Bökstedt (Aarhus)
- Ákos Nagy (Duke NC)
- Martin Speight (Leeds)
- Christian Wegner (Bonn)

Some references:

arXiv:1010.1488 (MB+NR)
arXiv:1410.2429 (MB+NR)
arXiv:1605.07921 (MB+NR)
arXiv:1711.04624 (MB)
arXiv:180m.0nnnn (NR+MS)
(in preparation) (MB+NR), (NR+CW), (MB+NR+CW), (ÁN+NR)

On-going project: SUSY QM on vortex moduli spaces

Main collaborators:

- Marcel Bökstedt (Aarhus)
- Ákos Nagy (Duke NC)
- Martin Speight (Leeds)
- Christian Wegner (Bonn)

Some references:

`arXiv:1010.1488` (MB+NR)
`arXiv:1410.2429` (MB+NR)
`arXiv:1605.07921` (MB+NR)
`arXiv:1711.04624` (MB)
`arXiv:180m.0nnnn` (NR+MS)
(in preparation) (MB+NR), (NR+CW), (MB+NR+CW), (ÁN+NR)

Gauged sigma-models and the vortex equations

Gauged sigma-models

Ingredients:

- $(\Sigma, j_\Sigma, \omega_\Sigma)$ Kähler structure on an oriented surface (*base*)
- (X, j_X, ω_X) another Kähler manifold (*target*)

- G compact Lie group with invariant metric
  $\mathfrak{g} := \mathrm{Lie}(G)$
- $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ 'musical' isomorphism

- G-action on X: holomorphic, Hamiltonian
- $\mu^\sharp : X \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\sharp} \mathfrak{g}$ moment map

Gauged sigma-model: energy

Fields: $(A, \phi) \quad \in \quad \mathcal{A}(P) \times \Gamma(\Sigma, P^X)$

- A connection in principal G-bundle $P \to \Sigma$
- ϕ section of associated bundle $P^X := P \times_G X \to \Sigma$

Topological charge:

$[\phi]_2^G := ((\tilde{f} \times \phi)/G)_*[\Sigma] \ \in \ H_2^G(X;\mathbb{Z}) \quad$ for $P = f^*\mathrm{E}G$

Yang–Mills–Higgs functional and Bogomol'nyĭ's trick:

$$E(A,\phi) \;:=\; \frac{1}{2}\int_\Sigma \left(|F_A|^2 + |\mathrm{d}^A\phi|^2 + |\mu\circ\phi|^2\right)$$

$$= \; \langle [\omega_X - \mu]_G^2, [\phi]_2^G \rangle + \frac{1}{2}\int_\Sigma \left(\left| *F_A + \mu^\sharp \circ \phi \right|^2 + 2|\bar{\partial}^A\phi|^2 \right)$$

Vortex moduli spaces

The (symplectic) vortex equations:

$$(\mathrm{V1}) \quad \bar{\partial}^A \phi = 0$$

$$(\mathrm{V2}) \quad *F_A + \mu^\sharp \circ \phi = 0$$

NB: Same can be done for "antivortices" s.t. $\partial^A \phi = 0$ etc.
But they don't live with vortices in BPS configurations.

We'll see in a moment how to implement coexistence of vortices and antivortices in another sense.

Fix $\mathbf{h} \in H_2^G(X;\mathbb{Z})$. Moduli spaces defined:

$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) := \left\{ (A,\phi) \;\middle|\; \begin{array}{c} (V1),(V2) \\ \text{and } [\phi]_2^G = \mathbf{h} \end{array} \right\} / \mathcal{G}(P)$$

The L^2 metric on vortex moduli spaces

The L^2-metric

Can recast this quotient in terms of Kähler reduction:

- (V1) is invariant under complexification $\mathcal{G}(P)^{\mathbb{C}}$.
- RHS of (V2) interpreted as $\mathcal{G}(P)$-moment map.

Thus (the smooth part of) $\mathcal{M}^X_{\mathbf{h}}(\Sigma)$ receives a *Kähler structure*.
Tangent spaces:

$$\begin{aligned} \mathrm{T}_A\mathcal{A}(P) &= \Omega^1(\Sigma; P \times_{Ad} \mathfrak{g}) \\ \mathrm{T}_\phi\Gamma(\Sigma, P^X) &= \Gamma(\Sigma, \phi^*\mathrm{T}X/G) \end{aligned}$$

Complex structure:

$$(\dot{A}, \dot{\phi}) \mapsto (*\dot{A}, (\phi^* j_\Sigma)\dot{\phi})$$

L^2-metric:

$$(\dot{A}_1, \dot{\phi}_1) \cdot (\dot{A}_2, \dot{\phi}_2) := \int_\Sigma \left(\frac{1}{2}\langle \dot{A}_1 \overset{\wedge}{,} *\dot{A}_2 \rangle + (\phi^* g_X)(\dot{\phi}_1, \dot{\phi}_2)\omega_\Sigma \right)$$

Prototptypical example: vortices in line bundles

Take $X = \mathbb{C}$ with usual action of $G = \mathrm{U}(1)$,

$$\mu(x) = -\frac{1}{2}(|x|^2 - \tau)$$

Suppose Σ is closed and $\deg(P) = k$.

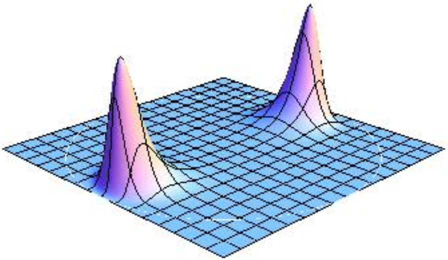
Then (V2) has solutions only if $\tau \mathrm{Vol}(\Sigma) \geq 4\pi k$.

Theorem (..., Bradlow,...): Assume $\tau \mathrm{Vol}(\Sigma) > 4\pi k$; then

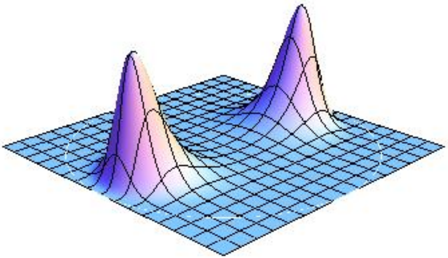
$$\mathcal{M}_k^{\mathbb{C}}(\Sigma) \cong \mathrm{Sym}^k(\Sigma).$$

This is a complex manifold with obvious complex structure J^{j_Σ}.
But describing g_{L^2} (or ω_{L^2}) is very hard.

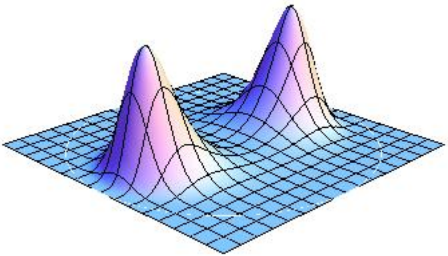
Moduli spaces: geodesic motion



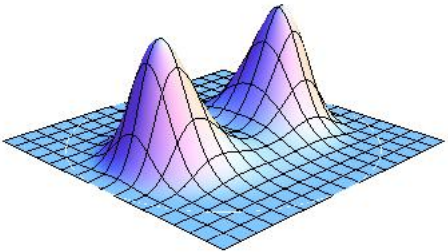
Moduli spaces: geodesic motion



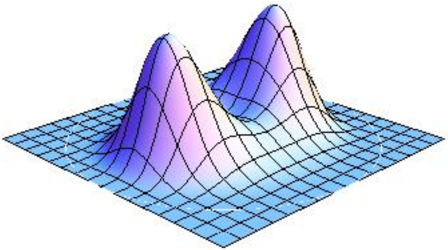
Moduli spaces: geodesic motion



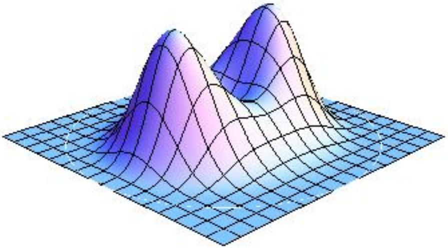
Moduli spaces: geodesic motion



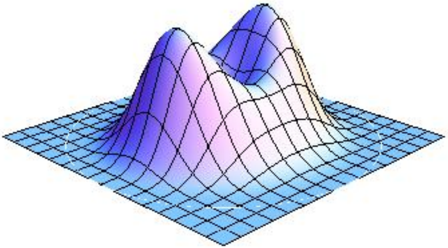
Moduli spaces: geodesic motion



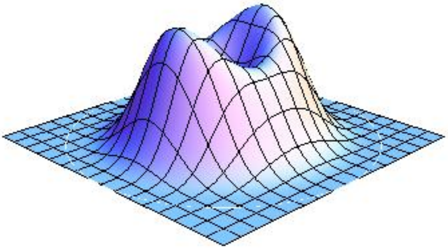
Moduli spaces: geodesic motion



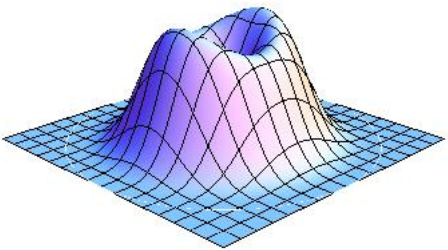
Moduli spaces: geodesic motion



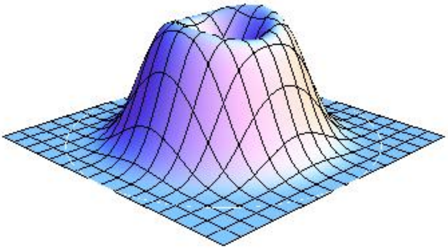
Moduli spaces: geodesic motion



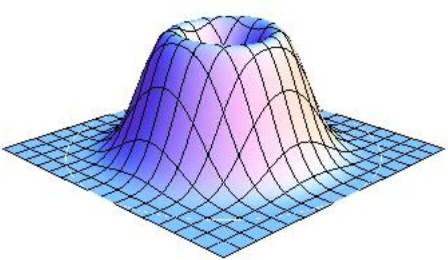
Moduli spaces: geodesic motion



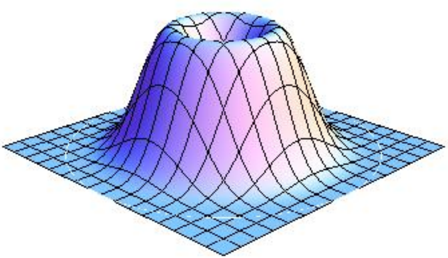
Moduli spaces: geodesic motion



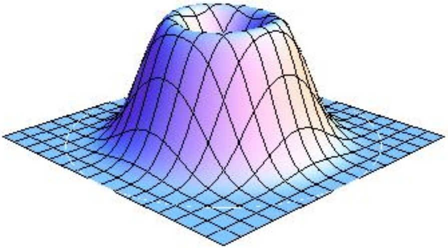
Moduli spaces: geodesic motion



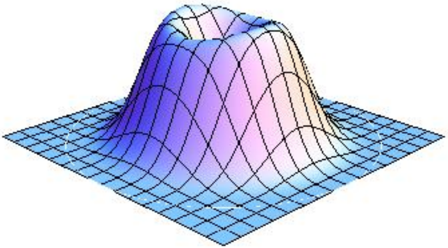
Moduli spaces: geodesic motion



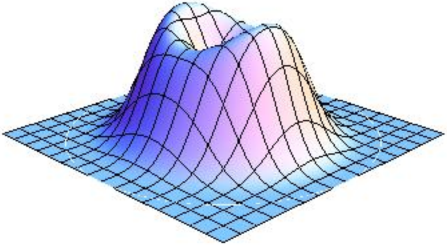
Moduli spaces: geodesic motion



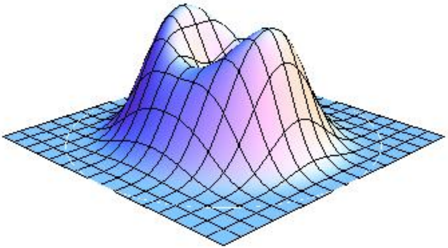
Moduli spaces: geodesic motion



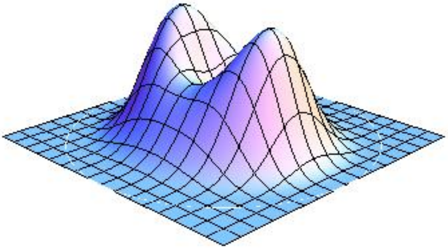
Moduli spaces: geodesic motion



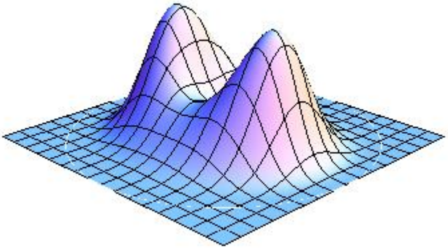
Moduli spaces: geodesic motion



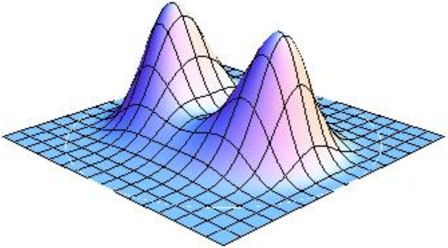
Moduli spaces: geodesic motion



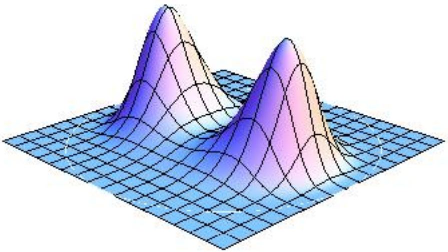
Moduli spaces: geodesic motion



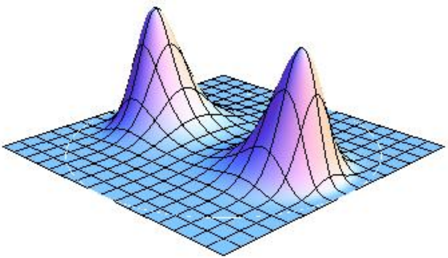
Moduli spaces: geodesic motion



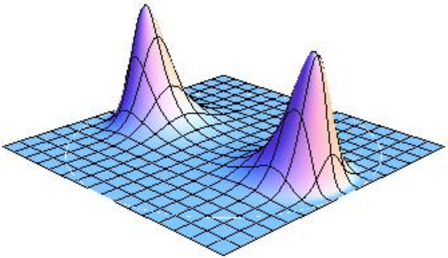
Moduli spaces: geodesic motion



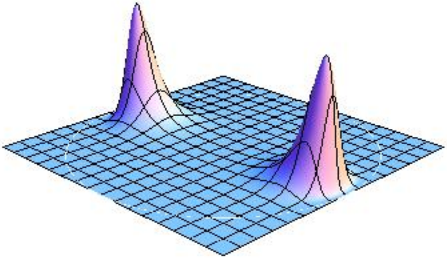
Moduli spaces: geodesic motion



Moduli spaces: geodesic motion



Moduli spaces: geodesic motion



Vortices with toric targets

Toric targets

An interesting setting:

- X Kähler toric manifold,
- $G = T \subset T^{\mathbb{C}} \subset X$ its (real) torus

Then for X, Σ compact we have a good description of $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$:

THEOREM (M Bökstedt + NR):
Suppose X is constructed as Fan_Δ for a Delzant polytope Δ, $\mathbf{h} \in {}^T\mathrm{BPS}_\Sigma^X$ with $a_{\mathbb{R}}^* \circ \mathbf{h}([\omega_\Sigma]^\vee) \in \mathrm{int}\,\Delta$ and

$$k_\rho = \langle c_1^T(D_\rho), \mathbf{h} \rangle \qquad \text{for } \rho \in \mathrm{Fan}_\Delta(1).$$

Then $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ is the smooth manifold

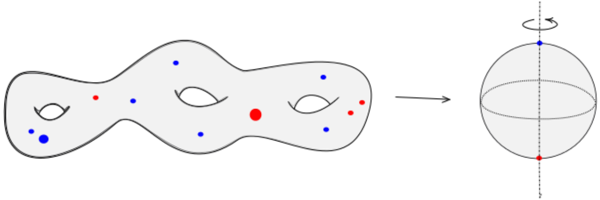
$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) = \mathrm{Div}_+^{\mathbf{k}}(\Sigma; (\partial\Delta)^\vee) \subset \prod_{\rho \in \mathrm{Fan}_\Delta(1)} \mathrm{Sym}^{k_\rho}(\Sigma)$$

$$=: \left\{ \mathbf{d} : [\lambda_0, \cdots, \lambda_\ell] \neq (\partial\Delta)^\vee \Rightarrow \bigcap_{i=0}^{\ell} \mathrm{supp}(d_{\lambda_\rho}) = \emptyset \right\}$$

Gauged $\mathbb{P}^1$-model: BPS vortices and antivortices

- For today's talk we focus on $X = \mathbb{P}^1 \cong S^2$, $T = \mathrm{U}(1) \cong S^1$ (Schroers)
- In this situation, $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ was already well understood as a complex manifold: (Mundet, Sibner-Sibner-Yang, Baptista)

$$\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) = \mathrm{Sym}^{k_+}(\Sigma) \times \mathrm{Sym}^{k_-}(\Sigma) \setminus \Delta_{(k_+,k_-)}$$



These spaces have a boundary even if Σ is compact.

Gauged $\mathbb{P}^1$-model: BPS vortices and antivortices

- Target: Riemann sphere $\mathbb{P}^1$ of unit radius.
- Moment map: (minus) height function, possibly translated:

$$\mu(x) = -\frac{1-|x|^2}{1+|x|^2} + \tau$$

- For Σ compact, have "Bradlow bounds"

$$-(1+\tau)\mathrm{Vol}(\Sigma) \leq 2\pi(k_+ - k_-) \leq (1-\tau)\mathrm{Vol}(\Sigma)$$

  obtained from integrating (V2). Here, $(k_+ - k_-) = \deg P$.
- Energy bound: $E(A,\phi) \geq 2\pi(1-\tau)k_+ + 2\pi(1+\tau)k_-$

  We will later focus on example with $k_+ = k_- = 1$, and more specifically at L^2-geometry close to 'pair anihilation'.

Asymptotic geometry of L^2 metrics

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

- Consider a gauged sigma-model with Σ compact,

$$X = \mathbb{C}^2, \qquad T^2 = \mathrm{U}(1)^2 = \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$$

- Vortex equations (for certain weights $Q^j_\pm$ and τ_j)

$$\bar{\partial}^{(A_1,A_2)}\varphi_\pm := \left(\bar{\partial} - \mathrm{i}\sum_{j=1}^{2} Q^j_\pm A^{(0,1)}_j\right)\varphi_\pm = 0$$

$$*F_{A_j} = -\mu^\sharp_j \circ \phi = \frac{e_j^2}{2}\left(Q^j_+|\varphi_+|^2 + Q^j_-|\varphi_-|^2 - \tau_j\right) \quad j = 1,2$$

- Gauged $\mathbb{P}^1$-model is formal limit as $e_1^2 = e^2 \to \infty$, $e_2^2 = 1$,
  i.e. $\quad g_{T^2} = e^{-2}\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2$.
  Take e.g. $Q_+ = (1,1), Q_- = (1,0), (\tau_1,\tau_2) = (4, 2-2\tau)$.

- Matching of charges: $k_+ = k_1 + k_2, \quad k_- = k_1$
- Bradlow bounds: $-(1+\tau) + \frac{2\pi k_-}{e^2 \mathrm{Vol}(\Sigma)} \leq \frac{2\pi(k_+ - k_-)}{\mathrm{Vol}(\Sigma)} \leq 1 - \tau$
- PROPOSITION: (NR + M Speight):
  For the strict inequalities and $k_+ \geq k_- > \max\{2g-2, 0\}$, the moduli space of the GLSM (for any e) is

$$\mathcal{M}^{\mathbb{C}^2}_{(k_+,k_-)}(\Sigma) = \mathrm{Sym}^{k_+}(\Sigma) \times \mathrm{Sym}^{k_-}(\Sigma).$$

- Note the inclusion $\iota : \mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) \hookrightarrow \mathcal{M}^{\mathbb{C}^2}_{(k_+,k_-)}(\Sigma)$.
- CONJECTURE: (NR + M Speight)
  The family of metrics $\iota^* g^e_{L^2}$ on $\mathrm{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ converges uniformly to g_{L^2} as $e \to \infty$.

Geometry of $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ from GLSMs

A little work leads to the result

$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{C}^2,e}_{(k_+,k_-)}(\Sigma)\right) = \sum_{\ell=0}^{g} \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^{\min\{g,k_\sigma\}} \frac{(2\pi)^{2\ell} C_\sigma^{k_\sigma - j_\sigma} D_\sigma^{j_\sigma - \ell}}{(j_\sigma - \ell)!(g-j_\sigma)!(k_\sigma - j_\sigma)!}.$$

for certain $C_\pm$, $D_\pm$ (depending on e), as well as to the

CONJECTURE (NR + M Speight) :

$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)\right) = \lim_{e\to\infty} \mathrm{Vol}\left(\mathcal{M}^{\mathbb{C}^2,e}_{(k_+,k_-)}(\Sigma)\right)$$

$$= \sum_{\ell=0}^{g} \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^{\min\{g,k_\sigma\}} \frac{(2\pi)^{k_\sigma + j_\sigma} [(1-\sigma\tau)V - 2\pi(k_\sigma - k_{-\sigma})]^{k_\sigma - j_\sigma}}{(j_\sigma - \ell)!(g-j_\sigma)!(k_\sigma - j_\sigma)!}.$$

From these formulae, one can compute Gibbs' partition function and study the thermodynamics of vortex-antivortex gas mixtures.

The case $\Sigma = \mathbb{R}^2$

- Topology is different for $\Sigma = \mathbb{R}^2$: P trivial, $\partial\Sigma = S^1_\infty$

  ϕ maps S^1_∞ to equator of $\mathbb{P}^1$

- THEOREM (Y Yang): ($\tau = 0$):

$$\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2) = \mathrm{Sym}^{k_+}\mathbb{C} \times \mathrm{Sym}^{k_-}\mathbb{C} \setminus \Delta_{(k_+,k_-)}$$

- (V1)+(V2) $\quad\Rightarrow\quad$ Taubes' equation for $h := \log|\phi|^2$:

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\sum_{r=1}^{k_+}\delta_{z_r^+} - \sum_{r=1}^{k_-}\delta_{z_r^-}\right)$$

AIM: Extract L^2-metric from this elliptic PDE.

The case $\Sigma = \mathbb{R}^2$

- $h := \log |\phi|^2$
- Taubes' equation: $(k_+, k_-) = (1, 1)$

$$\nabla^2 h - 2 \tanh \frac{h}{2} = 4\pi \left(\delta_{z_+} - \delta_{z_-} \right)$$

The case $\Sigma = \mathbb{R}^2$

- $h := \log |\phi|^2$
- Taubes' equation: $z_+ = -z_- = \varepsilon$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta_\varepsilon - \delta_{-\varepsilon}\right)$$

The case $\Sigma = \mathbb{R}^2$

- $h := \log |\phi|^2$
- Taubes' equation:

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta_\varepsilon - \delta_{-\varepsilon}\right)$$

- Regularize: $h(z) = \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \tilde{h}(z)$

The case $\Sigma = \mathbb{R}^2$

- $h := \log|\phi|^2$
- Taubes' equation:
- Regularize: $h(z) = \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \tilde{h}(z)$

$$\nabla^2 \tilde{h} - 2\,\frac{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} + |z+\varepsilon|^2} = 0$$

The case $\Sigma = \mathbb{R}^2$

- $h := \log|\phi|^2$
- Taubes' equation:
- Regularize: $h(z) = \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \tilde{h}(z)$

$$\nabla_z^2 \tilde{h} - 2\,\frac{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 \mathrm{e}^{\tilde{h}} + |z+\varepsilon|^2} = 0$$

- Rescale: $z =: \varepsilon w, \qquad \hat{h}(w) = \tilde{h}(\varepsilon w)$

The case $\Sigma = \mathbb{R}^2$

- $h := \log|\phi|^2$
- Taubes' equation:
- Regularize: $h(z) = \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \hat{h}(z)$
- Rescale: $z =: \varepsilon w$

$$\nabla_w^2 \hat{h} - 2\varepsilon^2 \frac{|w-1|^2 e^{\hat{h}} - |w+1|^2}{|w-1|^2 e^{\hat{h}} + |w+1|^2} = 0$$

- Solve with boundary condition $\hat{h}(w) \stackrel{|w|\to\infty}{\longrightarrow} 0.$

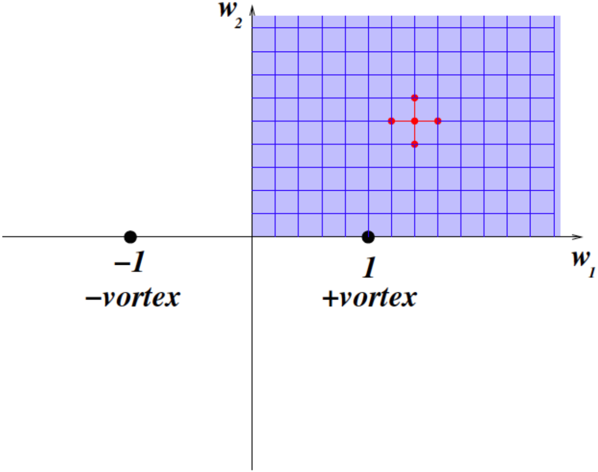
The case $\Sigma = \mathbb{R}^2$

- $h := \log |\phi|^2$
- Taubes' equation:
- Regularize: $h(z) = \log \left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2 + \hat{h}(z)$
- Rescale: $z =: \varepsilon w$

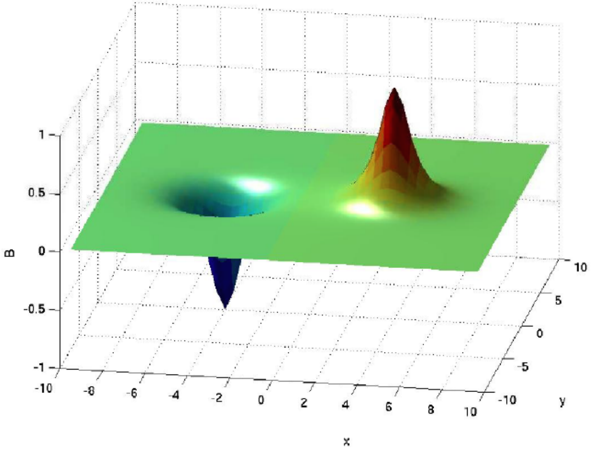
$$\nabla_w^2 \hat{h} - 2\varepsilon^2 \frac{|w-1|^2 e^{\hat{h}} - |w+1|^2}{|w-1|^2 e^{\hat{h}} + |w+1|^2} = 0$$

- Solve with boundary condition $\hat{h}(w) \xrightarrow{|w|\to\infty} 0.$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$



$\varepsilon = 4$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

- Assume all vortex positions remain distinct
- $\dot{\phi}$ solution of linearised Taubes' equation at ϕ
  Impose Coulomb gauge
  $\eta := \frac{\dot{\phi}}{\phi}$ provides 1-current in $\Sigma \times \mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$
  It evaluates at $v = (\mathbf{z}^\pm, \dot{\mathbf{z}}^\pm) \in \mathrm{T}\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$ to yield
  0-current η_v singular over $\{(z,(\phi)) : z \in \mathrm{supp}(\phi)\}$

$$\left(\nabla^2 - \mathrm{sech}^2\frac{h}{2}\right)\eta_v = 4\pi\left(\sum_{r=1}^{k_+}\dot{z}_r^+\partial_{z_r^+}\delta_{z_r^+} - \sum_{s=1}^{k_-}\dot{z}_s^-\partial_{z_s^-}\delta_{z_s^-}\right)$$

$$\Rightarrow \quad \eta_v = \sum_{r=1}^{k_+}\dot{z}_r^+\frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-}\dot{z}_s^-\frac{\partial h}{\partial z_s^-} \qquad \text{is (unique) solution}$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

- Assume all vortex positions remain distinct
- $\dot{\phi}$ solution of linearised Taubes' equation at ϕ
  Impose Coulomb gauge
  $\eta := \frac{\dot{\phi}}{\phi}$ provides 1-current in $\Sigma \times \mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$
  It evaluates at $v = (\mathbf{z}^\pm, \dot{\mathbf{z}}^\pm) \in \mathrm{T}\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$ to yield
  0-current η_v singular over $\{(z,(\phi)) : z \in \mathrm{supp}(\phi)\}$

$$\left(\nabla^2 - \mathrm{sech}^2\frac{h}{2}\right)\eta_v = 4\pi\left(\sum_{r=1}^{k_+}\dot{z}_r^+\partial_{z_r^+}\delta_{z_r^+} - \sum_{s=1}^{k_-}\dot{z}_s^-\partial_{z_s^-}\delta_{z_s^-}\right)$$

$$\Rightarrow \quad \eta_v = \sum_{r=1}^{k_+}\dot{z}_r^+\frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-}\dot{z}_s^-\frac{\partial h}{\partial z_s^-} \qquad \text{is (unique) solution}$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} = \frac{1}{2} \int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2 \right)$$

- Assume all vortex positions remain distinct
- $\dot{\phi}$ solution of linearised Taubes' equation at ϕ
  Impose Coulomb gauge
  $\eta := \frac{\dot{\phi}}{\phi}$ provides 1-current in $\Sigma \times \mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$
  It evaluates at $v = (\mathbf{z}^\pm, \dot{\mathbf{z}}^\pm) \in \mathrm{T}\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$ to yield
  0-current η_v singular over $\{(z, (\phi)) : z \in \mathrm{supp}(\phi)\}$

$$\left(\nabla^2 - \mathrm{sech}^2 \frac{h}{2} \right) \eta_v = 4\pi \left(\sum_{r=1}^{k_+} \dot{z}_r^+ \partial_{z_r^+} \delta_{z_r^+} - \sum_{s=1}^{k_-} \dot{z}_s^- \partial_{z_s^-} \delta_{z_s^-} \right)$$

$$\Rightarrow \quad \eta_v = \sum_{r=1}^{k_+} \dot{z}_r^+ \frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-} \dot{z}_s^- \frac{\partial h}{\partial z_s^-} \quad \text{is (unique) solution}$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_\Sigma \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

- Assume all vortex positions remain distinct
- $\dot{\phi}$ solution of linearised Taubes' equation at ϕ
  Impose Coulomb gauge
  $\eta := \frac{\dot{\phi}}{\phi}$ provides 1-current in $\Sigma \times \mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$
  It evaluates at $v = (\mathbf{z}^\pm, \dot{\mathbf{z}}^\pm) \in \mathrm{T}\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\mathbb{R}^2)$ to yield
  0-current η_v singular over $\{(z,(\phi)) : z \in \mathrm{supp}(\phi)\}$

$$\left(\nabla^2 - \mathrm{sech}^2\frac{h}{2}\right)\eta_v = 4\pi\left(\sum_{r=1}^{k_+}\dot{z}_r^+\partial_{z_r^+}\delta_{z_r^+} - \sum_{s=1}^{k_-}\dot{z}_s^-\partial_{z_s^-}\delta_{z_s^-}\right)$$

$$\Rightarrow \quad \eta_v = \sum_{r=1}^{k_+}\dot{z}_r^+\frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-}\dot{z}_s^-\frac{\partial h}{\partial z_s^-} \qquad \text{is (unique) solution}$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} \quad = \quad \frac{1}{2}\int_{\mathbb{R}^2} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \quad \frac{1}{2}\int_{\mathbb{R}^2} \left(4\,\partial_z \bar{\eta}_v \partial_{\bar{z}} \eta_v + \mathrm{sech}^2 \frac{h}{2} \bar{\eta}_v \eta_v\right)$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \frac{1}{2}\lim_{\epsilon\to 0}\int_{\mathbb{C}\setminus\cup_{r,s}B_\epsilon(z^{\pm}_{r,s})}\left(4\,\partial_z\bar{\eta}_v\partial_{\bar{z}}\eta_v + \mathrm{sech}^2\frac{h}{2}\bar{\eta}_v\eta_v\right)$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} \quad = \quad \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \quad \frac{1}{2}\lim_{\epsilon\to 0}\int_{\mathbb{C}\setminus\cup_{r,s}B_\epsilon(z^{\pm}_{r,s})}\left(4\,\partial_z(\bar{\eta}_v\partial_{\bar{z}}\eta_v) - (\nabla^2 - \mathrm{sech}^2\frac{h}{2})\bar{\eta}_v\eta_v\right)$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2+|\dot{\phi}|^2\right)$$

$$= -\mathrm{i}\lim_{\epsilon\to 0}\sum_{r,s=1}^{k_+,k_-}\oint_{\partial B_\epsilon(z^{\pm}_{r,s})}\bar{\eta}_v\,\bar{\partial}\eta_v$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= -\mathrm{i} \lim_{\epsilon\to 0} \sum_{r,s=1}^{k_+,k_-} \oint_{\partial B_\epsilon(z^\pm_{r,s})} \bar{\eta}_v\, \bar{\partial}\eta_v$$

$$\eta_v = \frac{\mp \dot{z}^\pm_{r,s}}{z - z^\pm_{r,s}} + O(1) \quad \text{as } z \to z^\pm_{r,s}$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}}\left(|\dot{A}|^2+|\dot{\phi}|^2\right)$$

$$= \pi\left(\sum_{r,s=1}^{k_+,k_-}|\dot{z}^{\pm}_{r,s}|^2+\sum_{p,q}^{+\text{ or }-}\frac{\partial b_q}{\partial z_p}\dot{z}_p\dot{\bar{z}}_q\right)$$

Asymptotics near vortex cores, $z = z^{\pm}_{r,s}$:

$$+h = \log|z-z_r^+|^2+a_r^++\frac{\bar{b}_r^+}{2}(z-z_r^+)+\frac{b_r^+}{2}(\bar{z}-\bar{z}_r^+)+O(|z-z_r^+|^2) \quad \text{as } z\to z_r^+$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|^2_{L^2} = \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \pi \left(\sum_{r,s=1}^{k_+,k_-} |\dot{z}^{\pm}_{r,s}|^2 + \sum_{p,q}^{+ \text{ or } -} \frac{\partial b_q}{\partial z_p} \dot{z}_p \dot{\bar{z}}_q \right)$$

Asymptotics near vortex cores, $z = z^{\pm}_{r,s}$:

$$+h = \log|z - z_r^+|^2 + a_r^+ + \frac{\bar{b}_r^+}{2}(z - z_r^+) + \frac{b_r^+}{2}(\bar{z} - \bar{z}_r^+) + O(|z - z_r^+|^2) \quad \text{as } z \to z_r^+$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A},\dot{\phi})\|^2_{L^2} \quad = \quad \frac{1}{2}\int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2\right)$$

$$= \quad \pi\left(\sum_{r,s=1}^{k_+,k_-} |\dot{z}^{\pm}_{r,s}|^2 + \sum_{p,q}^{+\text{ or }-} \frac{\partial b_q}{\partial z_p}\dot{z}_p\dot{\bar{z}}_q\right)$$

Asymptotics near vortex cores, $z = z^{\pm}_{r,s}$:

$$-h = \log|z - z_s^-|^2 + a_s^- + \frac{\bar{b}_s^-}{2}(z - z_s^-) + \frac{b_s^-}{2}(\bar{z} - \bar{z}_s^-) + O(|z - z_s^-|^2) \quad \text{as } z \to z_s^-$$

The case $\Sigma = \mathbb{R}^2$

$\mathcal{M}^{\mathbb{P}^1}_{(1,1)}(\mathbb{C}) \cong \mathbb{C}_{\mathrm{cm}} \times \mathbb{C}^*$

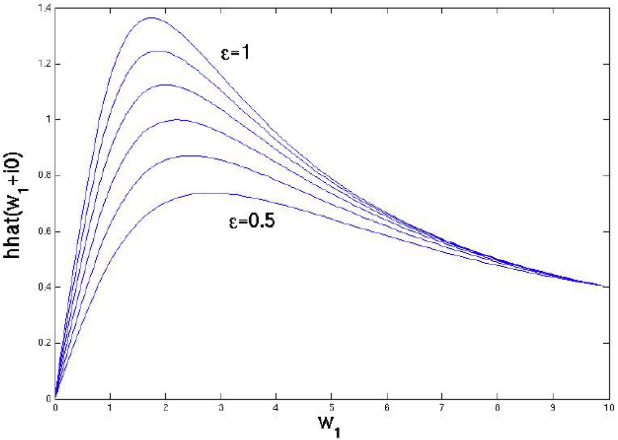
Aim: understand metric for centred $(+,-)$-pairs (i.e. on $\mathbb{C}^*$)
$z_+ = -z_- = \varepsilon \mathrm{e}^{\mathrm{i}\vartheta}$

$$g^{(0)}_{L^2} = F(\varepsilon)(\mathrm{d}\varepsilon^2 + \varepsilon^2 \mathrm{d}\vartheta^2)$$

where:

- $F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\varepsilon b(\varepsilon))\right)$
- $b(\varepsilon) := b_1^+(\varepsilon, -\varepsilon)$
- $\varepsilon b(\varepsilon) = \left.\frac{\partial \hat{h}}{\partial (\mathrm{Re}\, w)}\right|_{w=1} - 1$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Self-similarity conjecture: consider $\varepsilon << 1$.

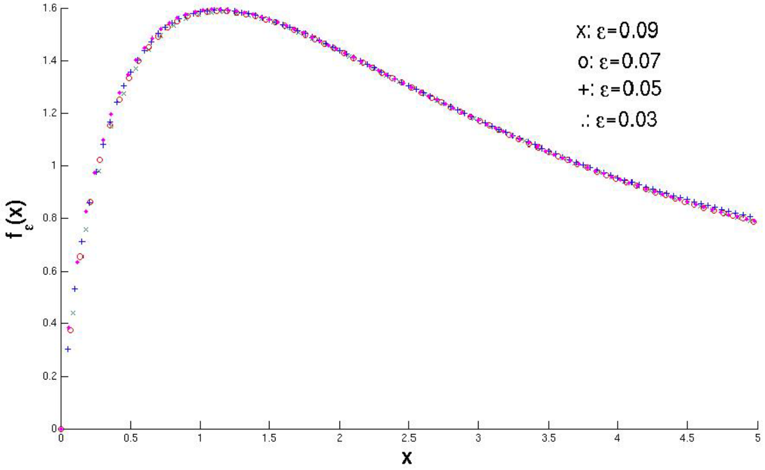
- Results suggest

$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \qquad \text{for } f_* \text{ fixed !?}$$

- To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon}\hat{h}\left(\frac{z}{\varepsilon}\right)$$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

- Results suggest

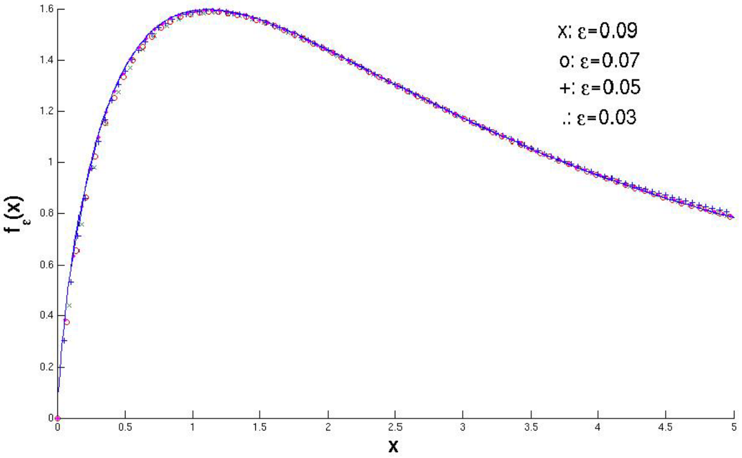
$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \qquad \text{for } f_* \text{ fixed !?}$$

- To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon}\hat{h}\left(\frac{z}{\varepsilon}\right) \quad \approx \quad f_*(z)$$

- Now plug this into Taubes' equation to obtain PDE for f_*
- Take formal limit as $\varepsilon \to 0$;
  obtain screened Poisson equation with a simple source.
- Solve equation to obtain asymptotics of metric $g_{L^2}^{(0)}$.

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Asymptotics of conformal factor:

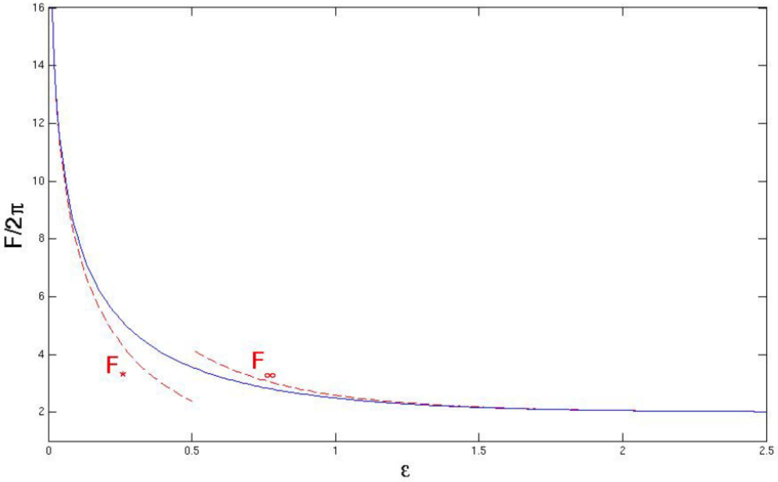
- As $\varepsilon \to 0^+$:

$$F_*(\varepsilon) = 2\pi(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)) \sim -8\pi \log \varepsilon$$

- As $\varepsilon \to \infty$ (different argument, cf. Manton & Speight):

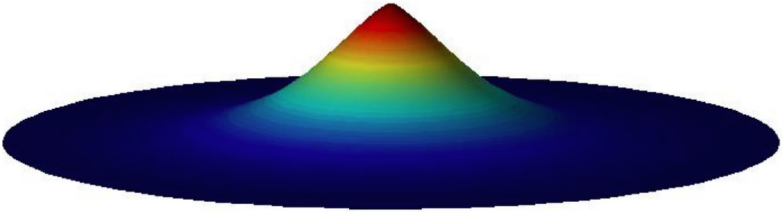
$$F_\infty(\varepsilon) = 2\pi \left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon) \right), \qquad q \approx -7.1388$$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Our formula for F_* implies incompleteness with unbounded positive Gauß curvature as $\varepsilon \to 0$.



The case $\Sigma = S^2_R$

Regularised Taubes equation for $(k_+, k_-) = (1, 1)$:

$$\nabla^2_w \hat{h} - \frac{8R^2\varepsilon^2}{(1+\varepsilon^2|w|^2)^2} \frac{|w-1|^2 e^{\hat{h}} - |w+1|^2}{|w-1|^2 e^{\hat{h}} + |w+1|^2} = 0.$$

THEOREM (NR + M Speight): The conjectural volume formula holds for $\tau = 0$, i.e.

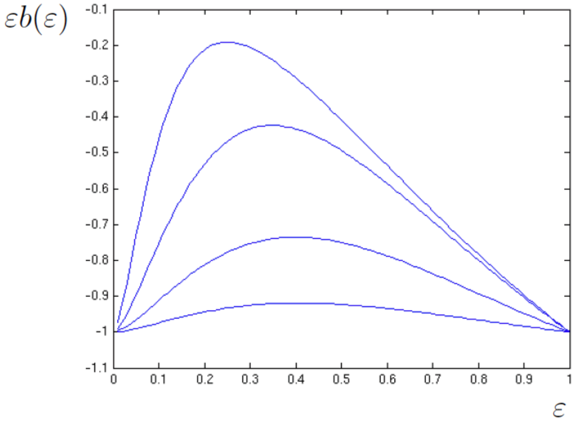
$$\mathrm{Vol}\left(\mathcal{M}^{\mathbb{P}^1}_{(1,1)}(S^2_R)\right) = (2\pi)^2(4\pi R^2)^2.$$

Ingredients in the proof:

- Strachan–Samols localisation (again)
- Use of $\mathrm{SO}(3) \times \mathbb{Z}_2$ isometry
- Elliptic estimates (and various tricks) to establish

$$\|\partial_{w_1}\hat{h}\|_{C^0(\bar{B}_{1/2}(1))} \leq \|\partial_{w_1}\hat{h}\|_{H^2(\bar{B}_{1/2}(1))} < C(R)\sqrt{\varepsilon}$$

The case $\Sigma = S^2_R$ (numerics)



$R = 8, 4, 2, 1$ (top to bottom)

NB: $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\varepsilon b(\varepsilon))$ bounded as $\varepsilon \to 0 \;\Rightarrow\; \mathcal{M}^{\mathbb{P}^1}_{(1,1)}(S^2_R)$ incomplete

The case $\Sigma = S^2_R$ (cont.)

Further analysis has established:

THEOREM (Á Nagy + NR): $\mathcal{M}^{\mathbb{P}^1}_{(1,1)}(S^2_R)$ is incomplete.

Proof requires sharpening estimates to

$$\|\partial_{w_1} \hat{h}\|_{C^0(\bar{B}_{1/2}(1))} < C(R)\,\varepsilon$$

Somewhat surprisingly:

Conformal factors for $\Sigma = \mathbb{R}^2$ and S^2_R exhibit different asymptotics!

Supersymmetric Quantum Mechanics on vortex moduli spaces