Some results on semitoric integrable systems

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portions joint with D.M. Kane, S. Hohloch, and Á. Pelayo

Geometric aspects of momentum maps and integrability
CSF Ascona

April 10, 2018

Semitoric integrable systems: definition

Definition

A *semitoric integrable system* is a triple $(M, \omega, (J, H))$ where (M, ω) is a 4-dimensional symplectic manifold and

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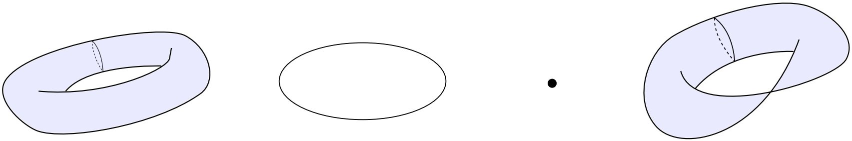
- *Simple* = at most one focus-focus point in each level set of J.

Semitoric integrable systems: fibers



regular | elliptic-regular | elliptic-elliptic | focus-focus

Semitoric integrable systems: classification

Simple semitoric systems have been classified in terms of five invariants:

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2. *the polygon invariant*;
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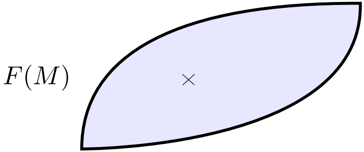

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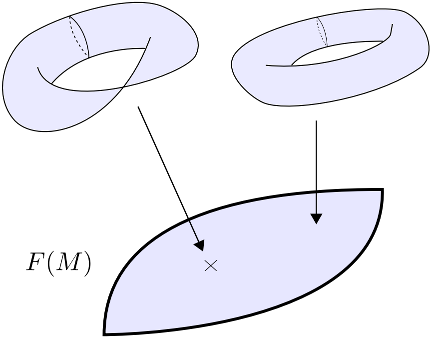

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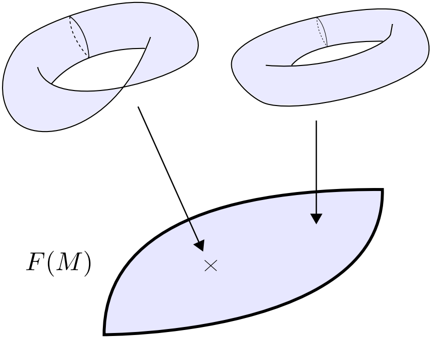

- Focus-focus points produce *monodromy* in the torus fibration

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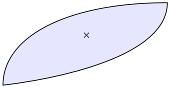

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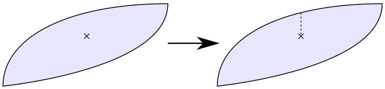

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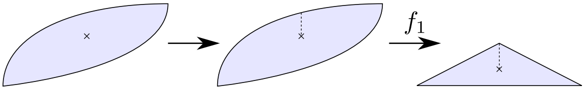

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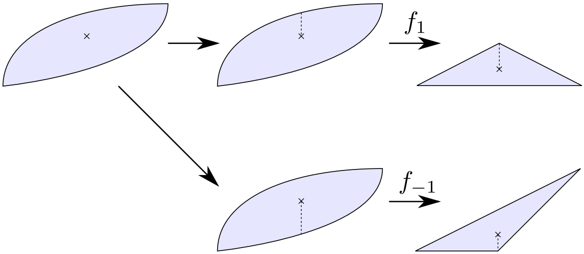

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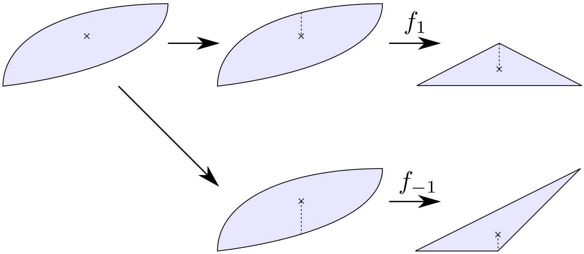

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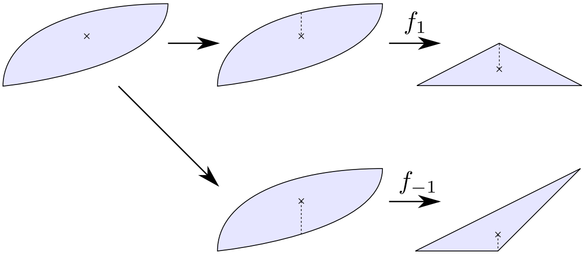

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- *Height invariant*: position of images of focus-focus points.

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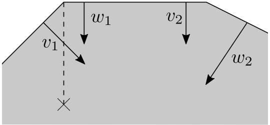
- $f_\varepsilon \circ F$ is a toric momentum map away from the cuts.
- If $v, w \in \mathbb{Z}^2$ are inwards pointing normal vectors of a corner:
  - Delzant: v and w span $\mathbb{Z}^2$;
  - Fake: $Tv = w$;
  - Hidden: Tv and w span $\mathbb{Z}^2$.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



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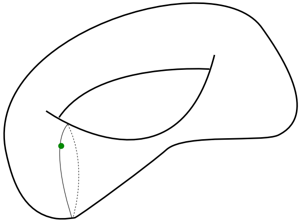

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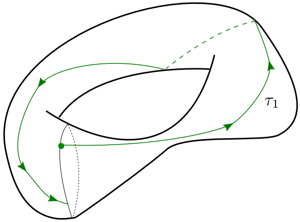

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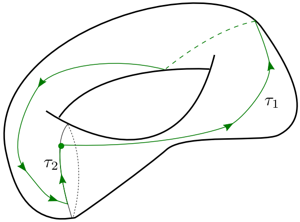

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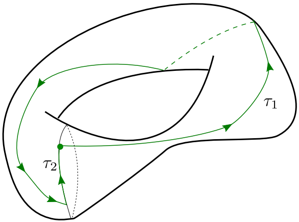

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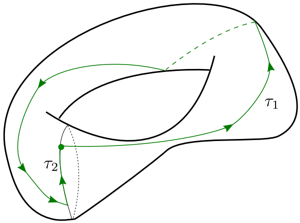

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- **Notice:** this construction only sees where the trajectory "lands" - it can't detect a Dehn twist.

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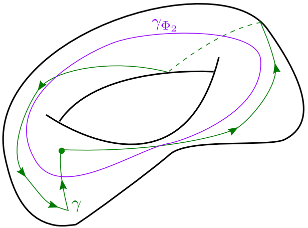

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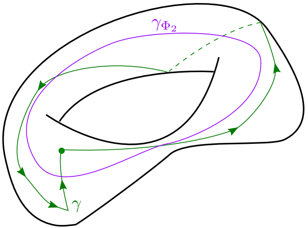
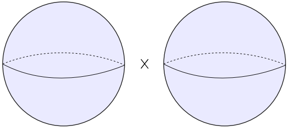


- This $k \in \mathbb{Z}$ is the *twisting index* [joint project with S. Hohloch].

(Also, $\Phi = T^k \nu$)

Coupled angular momenta

[Sadovskij and Zhilinskij, 1999]



- $M = S^2 \times S^2, \quad \omega = R_1\omega_1 \oplus R_2\omega_2$
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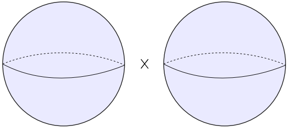

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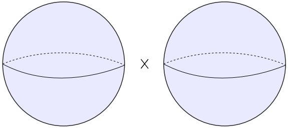

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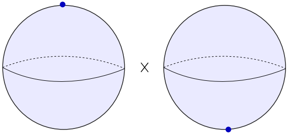

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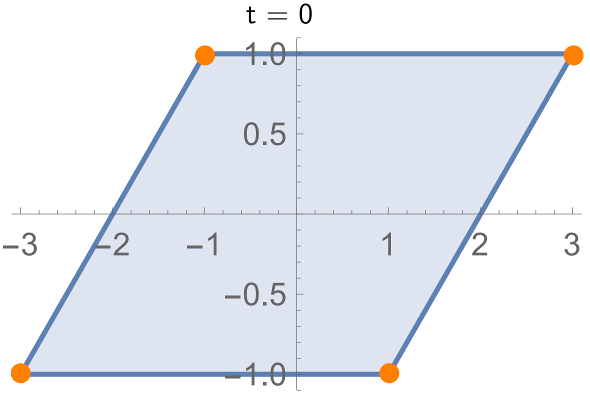

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(figure made in Mathematica)

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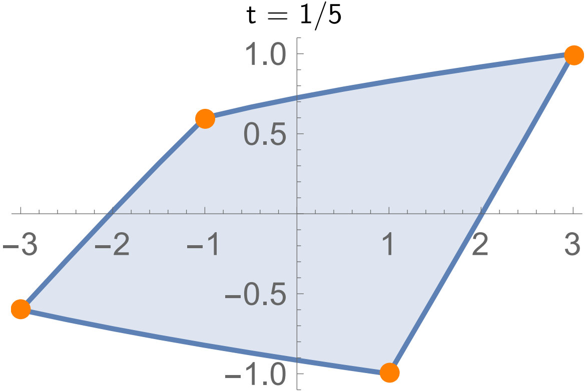

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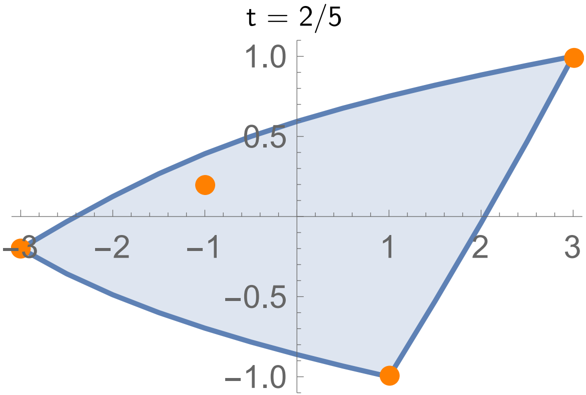

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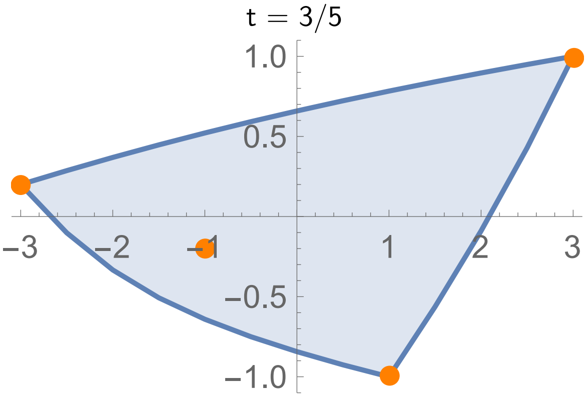

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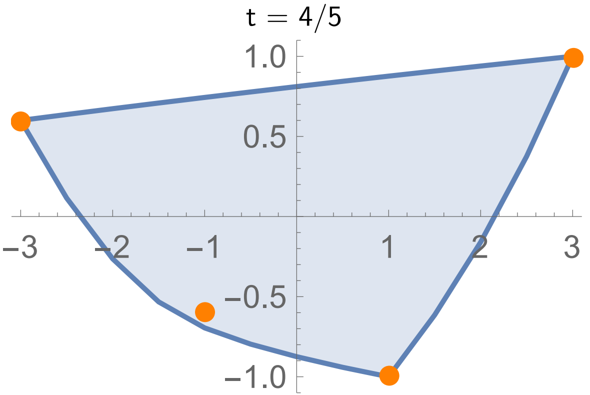

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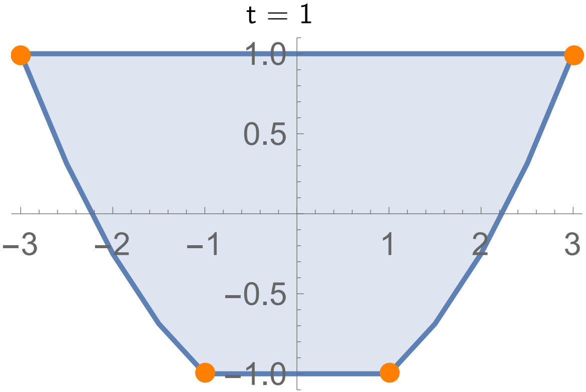

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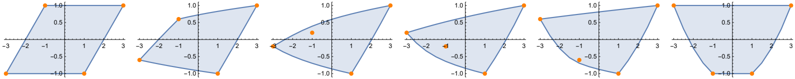

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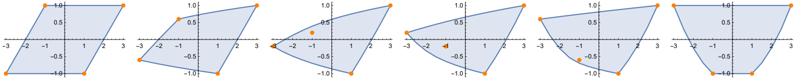

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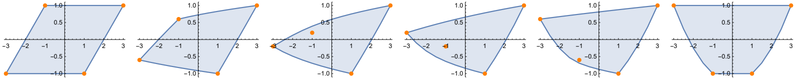

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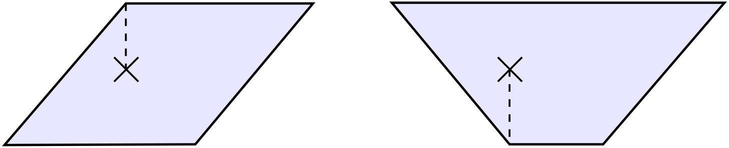

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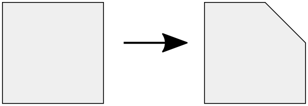
Goal

Find all compact semitoric systems which do not admit a blowdown.

- Then all systems can be obtained from these by performing a sequence of blowups.

Minimal models: blowups and corner chops

- Blowups correspond to a corner chop of the semitoric polygon.



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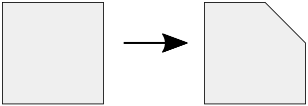
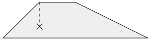


- Sometimes can be hard to see if blowdown is possible.



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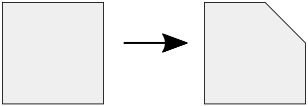
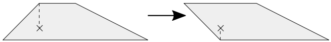


- Sometimes can be hard to see if blowdown is possible.



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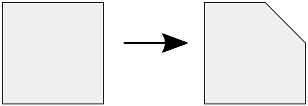

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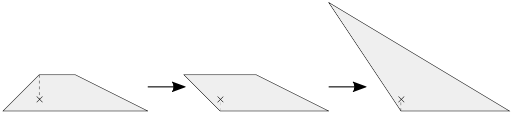

Minimal models: the semitoric helix

Definition

A *semitoric helix* is $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$, $v_i \in \mathbb{Z}^2$, such that:

1. v_i, v_{i+1} span $\mathbb{Z}^2$ (and positively oriented) for all $i \in \mathbb{Z}$;
2. $v_0, \ldots, v_{d-1}$ are arranged in counter-clockwise order;
3. $v_{i+d} = T^c v_i$ for all $i \in \mathbb{Z}$.

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$d \in \mathbb{Z}_{>0}$, number of elliptic-elliptic points
$c \in \mathbb{Z}_{\geq 0}$, number of focus-focus points

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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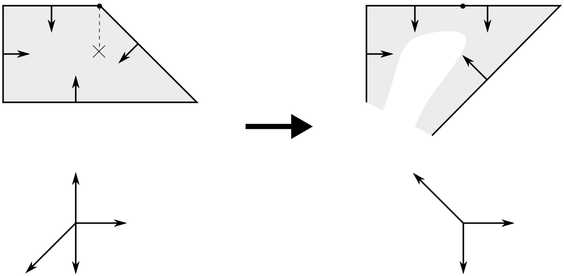

Minimal models: equations in $\mathrm{SL}_2(\mathbb{Z})$

- Let $A_i \in \mathrm{SL}_2(\mathbb{Z})$ be the matrix sending (v_i, v_{i+1}) to (v_{i+1}, v_{i+2}).
- If there are no focus-focus points (toric case):

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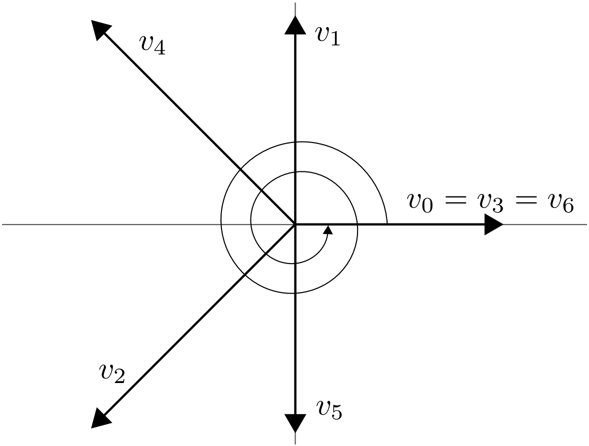
Minimal models: winding twice



Minimal models: equations in $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$

- Solution: lift equations to "universal cover of $\mathrm{SL}_2(\mathbb{Z})$"

$$\begin{array}{ccc} \widetilde{\mathrm{SL}_2(\mathbb{Z})} & \xhookrightarrow{\rho} & \widetilde{\mathrm{SL}_2(\mathbb{R})} \\ \downarrow & & \downarrow \\ \mathrm{SL}_2(\mathbb{Z}) & \xhookrightarrow{i} & \mathrm{SL}_2(\mathbb{R}) \end{array}$$

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Proposition (Kane-P.-Pelayo, 2016)

There is a one-to-one correspondence between solutions to

$$A_0 A_1 \dots A_{d-1} =_{\widetilde{\mathrm{SL}_2(\mathbb{Z})}} \widetilde{S}^4 X^{-1} T^c X$$

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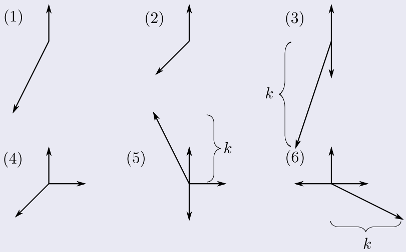
($S \in \mathrm{SL}_2(\mathbb{Z})$ corresponds to rotation by $\pi/2$, lifts to $\widetilde{S}$)

Minimal models: minimal helices

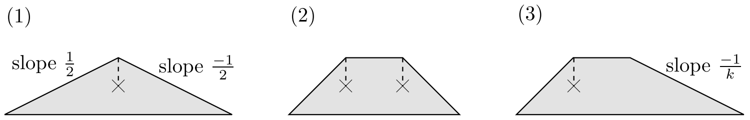
Theorem (Kane-P.-Pelayo, 2016)

The minimal helices come in 7 families:



with (1) $c = 1$; (2) $c = 2$; (3) $k \neq 2$, $c = 1$;
(4) $c \neq 2$; (5) $k \neq \pm 1, 0$, $c \neq 1$; (6) $k \neq -1, 1 - c$, $c > 0$
and type (7).

Minimal models: minimal polygons (1), (2), (3)



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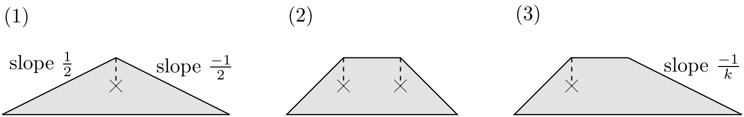

- The coupled angular momenta system is of type (3) with $k = -1$.

A system of type (2)

Goal

Find an explicit system which is minimal of type (2).

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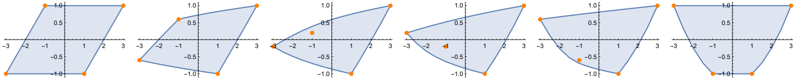

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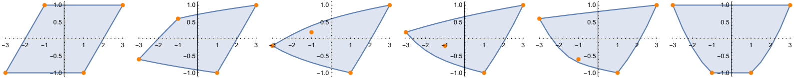

- The point NS passes through the interior and becomes focus-focus, can we do this with SN as well?

A system of type (2): 4 parameter family

$$\begin{cases} J & = R_1 z_1 + R_2 z_2 \\ H_{t_1,t_2,t_3,t_4} & = t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2 \end{cases}$$

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Theorem (Hohloch-P.)

Let $R_1 = 1$ and $R_2 = 2$. Then $(J, H_{\frac{1}{4},\frac{1}{4},\frac{1}{2},0})$ is a semitoric integrable system with exactly two focus-focus points (and so is every system in an open neighborhood of these parameters).

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- Can reparameterize as H_{s_1,s_2} where

$$(t_1, t_2, t_3, t_4) = \big(s_1(1-s_2),\ s_2(1-s_1),\ (1-s_1)(1-s_2)+s_1 s_2,\ (1-s_1)(1-s_2)-s_1 s_2\big)$$

to get a two parameter family.

A system of type (2): rewriting the system

If

$$\begin{cases} H_{0,0} & = x_1 x_2 + y_1 y_2 + z_1 z_2 \\ H_{1,0} & = z_1 \\ H_{0,1} & = z_2 \\ H_{1,1} & = x_1 x_2 + y_1 y_2 - z_1 z_2 \end{cases}$$

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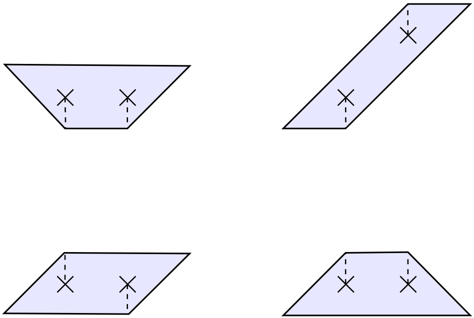
$$(J, H_{\frac{1}{4},\frac{1}{4},\frac{1}{2},0}) = (J, H_{\frac{1}{2},\frac{1}{2}})$$

A system of type (2): the semitoric polygons

The semitoric polygons for $(J, H_{\frac{1}{2},\frac{1}{2}})$:



A system of type (2): the momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0, 1]$

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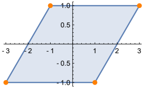

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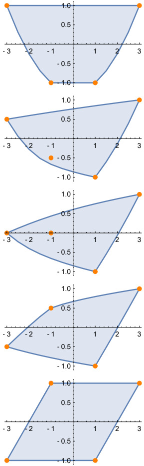

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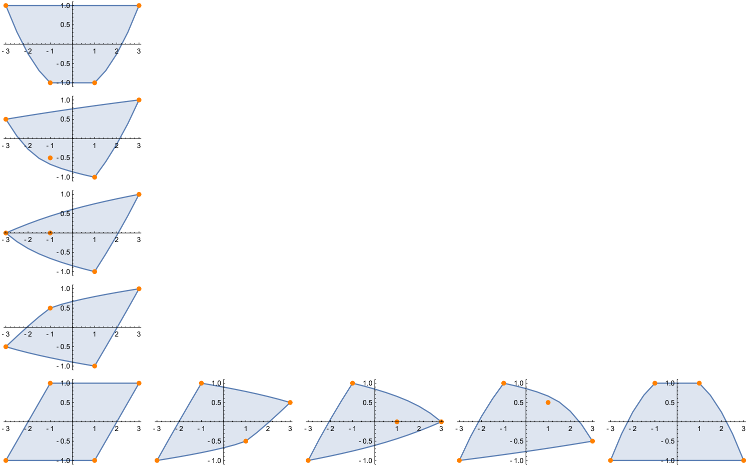

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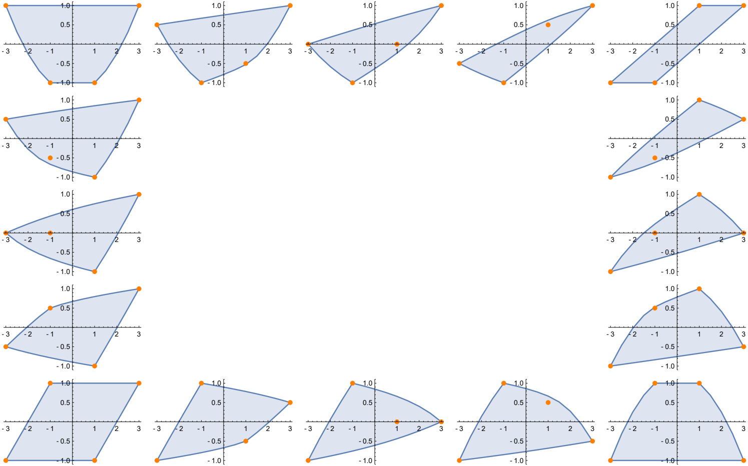

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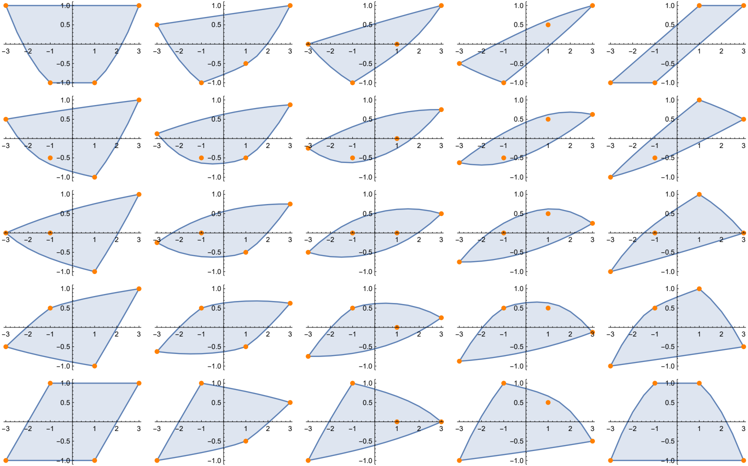

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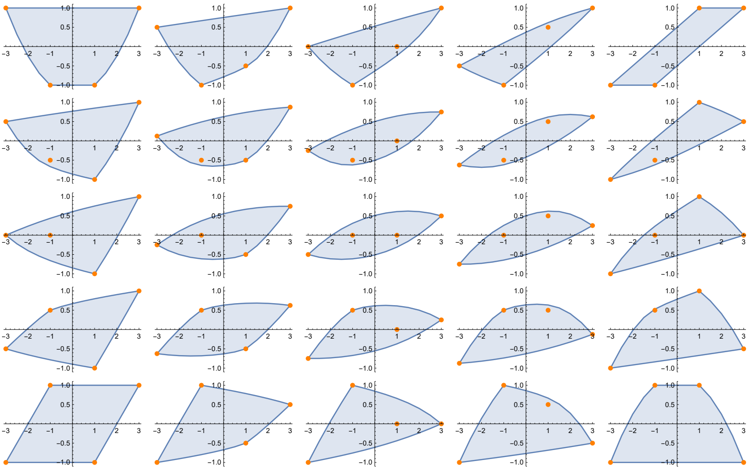
Thanks!



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