Geodesics of Finsler metrics of constant curvature

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Joint result with
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Definition of Finsler metric.

Finsler metric is a function $F : TM \to R$ such that for every $p \in M$ the restriction $F_{|T_pM}$ is a Minkowski norm, that is $\forall \xi, \nu \in T_pM, \ \forall \lambda > 0$

(a) $F(\lambda \cdot \xi) = \lambda \cdot F(\xi)$,

(b) $F(\xi + \nu) \leq F(\xi) + F(\nu)$,

(c) $F(\xi) = 0 \iff \xi = 0$.

- F is reversible if $F(v) = F(-v)$.

We in addition assume that the Finsler metric is strictly convex that is the matrix $\left(\frac{\partial^2 F^2}{\partial \xi_i \partial \xi_j}\right)$, is positive definite.

Everything in my talk is smooth.

Main message and plan of the talk

- We solve or at least advanced in a solution of a long-standing problem in Finsler geometry
  - I give all necessary definition
  - I explain motivation and history
- The solution would not be possible without the training I have in the integrable system group of Fomenko in the last millenium
  - Understanding symplectic topology of integrable systems was crucial
- The methods you possess are useful and important – trust to look around and apply them in other branches.
  - We (Topalov, Bolsinov, Bryant ...) applied them in the theory geodesically equivalent metrics – solved Lichnerowicz conjecture and two problems stated by Sophus Lie
  - We (Bolsinov, Rosemann, Eastwood ...) applied them in the theory of c-projectively equivalent metrics – solved Yano-Obata conjecture
  - In the present talk we (Bryant, Foulon, Ivanov, Ziller) apply them in Finsler geometry

History of Finsler metrics

- Appeared in Riemann's habilitation addressed who did not consider them interesting; were discussed in the Hermann Weyl's comment on Riemann's hablitation address who found them interesting and suggested to study what now is called Berwald and generalized Berwald metrics and I do not discuss in my talk.
- Were intensively studied by classics of calculus of variations – Cartheodory, Landsberg (beginning of the 20th century). They were:
  - interested in continuous optimal (variational) problems
  - impressed by description of qualitative behaviour of geodesics of Riemannian metrics with the help of Jacobi vector fields and sectional curvature
  - wanted to generalize them to the Finslerian setting.

Popular game in Finsler geometry: generalize methods and results from Riemannian geometry

Example: (Parameterized) Geodesics are solitions of the extremal problem $\mathcal{L}(c) \mapsto \min$ with

$$\mathcal{L}(c) := \int_{-1}^{1} F(c(t), \dot{c}(t))^2 dt.$$

Alternatively, one can view geodesic as soluions of the Hamiltonian system on T^*M with Hamiltonian F^* which is the $\frac{1}{2}F^2$-Legendre transform of F^2.

Example: flag curvature. It is a generalization of the sectional curvature. I will recall the definition later, in the proof of the main result; at the present point it is sufficient to know that in dimension two it is a function on a three-dimensional unit tangent bundle. In my talk I will mostly speak about the case when the flag curvature is constant.

Topic of my talk: metrics of constant flag curvature on closed surfaces: History

- Because of importance of Riemannian metrics of constant sectional curvature, Finsler metrics of constant flag curvature were intensively studied. In a slightly different form, the problem was asked already by Landsberg in 1908.
- Locally and microlocally, there is not a problem to prove the existence of metrics of constant flag curvature: in fact, the equation $K = 1$ is one PDE on the function F of 3 variables, and it has tons of microlocal solutions.
- In the case of negative or zero curvature everything is understood.

  Theorem (Akbar-Zadeh 1988). Every closed 2-dimensional surfaces with constant negative flag curvature is Riemannian. Every closed manifold with zero flag curvature is locally Minkowski.

  (The result for negative constant curvature was extended to all dimensions by Foulon 1997).

Positive constant curvature is different and similar: there are many irreversible examples and no reversible

- There are $\underbrace{\text{many examples}}_{\text{discuss at the end}}$ of Finsler metrics of positive constant flag curvature on the sphere.
  The most famous example is the so-called Katok metrics, I will recall the definition later.

- **Theorem (Bryant 2002–2006).** A <u>reversible</u> Finlser metric of constant positive flag curvature on the 2-sphere is Riemannian

Main Theorem

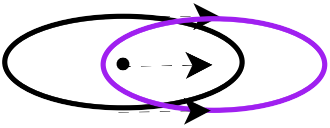
Main Theorem (Bryant-Foulon-Ivanov-Matveev-Ziller 2017). Geodesic flow of any metric of constant flag curvature $+1$ on the 2-sphere is symplectically conjugated to that of an appropriate $\underbrace{\text{Katok metric}}_{\text{to be explained}}$.

Here *symplectic conjugacy* means the existence a symplectic diffeomorphism of the split tangent bundles that takes F_1 to F_2.

Zermelo transformation and Katok metrics

Let F be a Finsler metric and v a vector field such that $F(-v) < 1$.

Zermelo transformation of a Finsler metric F by v is a Finsler metric $\tilde{F}$ whose unit ball $\{\xi \in T_x M \mid \tilde{F}(x,\xi) \leq 1\}$ at every x is the v-parallel transport of the the unit ball of F.



Def. Katok metrics are the Zermelo transformations of Riemannian metrics of constant positive curvature by Killing vector fields.

(Essentially, $g = d\theta^2 + \cos^2\theta d\phi^2$ and $v = \alpha\frac{\partial}{\partial\phi}$ with $|\alpha| < 1$)

Properties of Zermelo transformation w.r.t. a Killing vector field

Fact. Suppose v is a Killing vector field for F, and Ψ_t the flow of v. Suppose $F(-v) < 1$ and denote by $\tilde{F}$ the v-Zermelo transform of F. Then:

- For every arc-length geodesic $\gamma(t)$ of F the curve $\tilde{\gamma}(t) = \Psi_t(\gamma(t))$ is an arc-lenght $\tilde{F}$-geodesic (Known already by Katok 1979).
- The flag curvature $\tilde{K}$ of $\tilde{F}$ is given by $\tilde{K}(x, \xi + v) = K(x, \xi)$. In particular, if F has constant flag curvature, then so does $\tilde{F}$. (Was observed by P. Foulon in 2003 but not properly published; follows from works of Javaloyes – Vitorio 2014 and Huang – Mo 2015; we (Foulon-Matveev) wrote a simple selfcontained proof 2018.)

Geodesics of the Katok metric

- If α is rational, $\alpha = \frac{p}{q}$, then all geodesics are periodic, and all but at most two have the same length. For these two special geodesics, the sum of reciprocals of the length's is $\frac{1}{\pi}$.
- If α is irrational, precisely two geodesics are periodic, and again the sum of reciprocals of their length's is $\frac{1}{\pi}$.

Corollary. All these properties remain true for each metric of constant flag curvature $+1$.

Corollary. The length of the shortest geodesic determines the geodesic flow of a metric of constant flag curvature $+1$ on S^2.

Main technical statements which lead to the proof

Theorem A. Geodesic flow of a metric of positive constant flag curvature is Liouville integrable (We denote the integral by I). Thus, most geodesics (viewed as curves on the unit tangent bundle UM) are windings on two-dimensional Liouville tori.

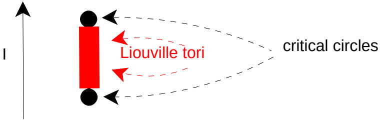
- We will see from the proof in many cases the corresponding systems are even superintegrable – all geodesics are closed.

Theorem B. The rotational number is the same for all Liouville tori.

- If it is rational, all geodesics are closed (and the geodesic flow is superintegrable).

Theorem C. (If one chooses I appropriately in the superintegrable case), the Liouville foliation has only two singular fiber which are circles and are ellipic singularities in the Morse-Bott classification.

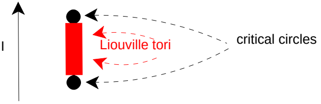
By Thm C, we have that $US^2 \setminus \{\text{two circles}\}$ is foliated by the tori.



Why Main Theorem follows from Theorems A, B, C?

Orbital equivalence ($\approx$ conjugacy but we allow reparameterisation) of integrable Hamiltonian systems (with Bott-Morse singularities) were actively studied in 1990th in Moscow by the group around Fomenko. In particular, they constructed a complete family of invariants (Bolsinov-Fomenko 1994–..., Kruglikov PhD Thesis 1995, Kruglikov 1997, Topalov PhD Thesis 1997)).

Fact. (Bolsinov-Fomenko, Kruglikov) In the class of systems under the consideration, because the topology of the Liouville-foliation is the same, the rotation number (viewed as a function on the red thick interval) is the only invariant



In our case the rotational function is a constant by Thm B, so it is sufficient to show that possible rotational numbers are the same as that for the Katok metric, this follows from some variational arguments which I do not have time discuss in this talk.

The symplectic conjugacy follows then, in the case when the rotational number is irrational, from the existence of action-angle coordinates, and in the case when the rotational number is rational, from the Weinstein's trick 1974 and from Wadsley 1975.

Plan for the rest of the talk

- Some basics of the Finsler geometry:
  - "Fundamental tensor" and why it is not as artificial as it looks
  - Flag curvature
  - Its relation with Jacobi vector fields
- Proof of (technical) Theorems A, B, C.
- Zoll metrics and (Bryant's) constructions of Finsler metrics of constant curvature and some new examples.

Fundamental tensor

Def. Fundamental tensor is the second differential by the ξ-coordinates of the function $\frac{1}{2}F^2$:

$$g_{(x,\xi)} = \frac{1}{2} d_\xi^2 F^2.$$

As a mathematical object, it is a bilinear symmetric form on TM which depends on the point of M and on a vector ξ tangent to this point. Its matrix is $n \times n$ matrix

$$g(x,\xi)_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \xi_i \partial \xi_j}$$

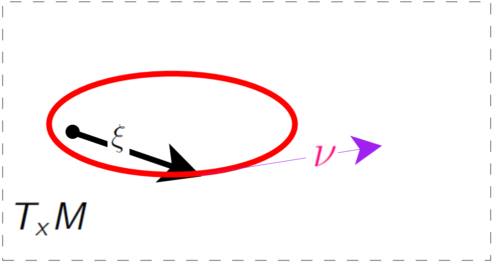
and it is positive definite by our definition of Finsler metric.

Fundamental tensor is natural and useful: I give two indications why:

Observation 1. $g_p(\xi, \xi) = F^2(p, \xi)$, in particular geodesics are extremals of

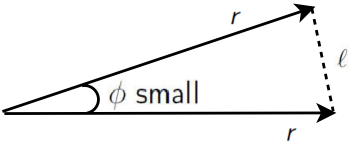
$$c(t) \mapsto \int g_{c(t),\dot c(t)}(\dot c(t), \dot c(t))dt.$$

Observation 2. If at $\xi \in T_xM$ with $F(x, \xi) = 1$ the vector ν is tangent to the unit sphere $\{\eta \in T_xM \mid F = 1\}$, then $g_\xi(\xi, \nu) = 0$.



Flag curvature in dimension 2

Consider the following triangle:



The black lines are geodesic segments of length r, one should think that r is small and ϕ is very small

In the Riemannian case,

$$\ell(r,\phi) = (r - \tfrac{1}{6}Kr^3)\phi + \text{terms of higher order},$$

where K is the sectional curvature, and this is a definition of the sectional curvature.

We will use the same picture and the same formula for the definition of the flag curvature in Finsler geometry (assuming dimension 2): we simply need to give sense to the notion "angle" ϕ and "distance" $\ell(r,\phi)$:

- Angle ϕ is calculated in the sense of $g_{\gamma(0),\dot\gamma(0)}$.
- Distance ℓ is calculated in the sense of $g_{\gamma(r),\dot\gamma(r)}$.

The flag curvature is a function on TM^2; it is homogeneous so one can view it as a function on UM^2; there is a complicated expression for it in g and its first and second derivatives.

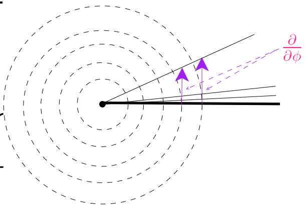
Jacobi vector fields

Def. The definition is the same as in the Riemannian geometry: Let $\gamma_s(t)$ be a family of geodesics. Jacobi vector field is the vector field along γ_0 given by

$$J(t) = \frac{\partial}{\partial s}\gamma_s(t)_{|s=0}.$$

Jacobi field is normal, if it is orthogonal in $g_{\gamma(t),\dot\gamma(t)}$ to $\dot\gamma(t)$ at all points.

Example. Consider the exponential mapping $\exp : \underbrace{T_pM}_{R^2} \to M$; then $d\exp\begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}$ is a normal Jacobi vector field (the vector $\begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}$ is simply $\frac{\partial}{\partial\phi}$ in the *polar coordinates* corresponding to the norm F. That is, the punctured lines on the picture are $\{F = \mathrm{const}\}$, and of course the coordinates are not linear.



The coordinates on T_pM^2 are nonlinear and are such that $\{F(0,\xi) = r\}$ is the standard r-circle

Analog of the Gauss Lemma: In the polar coordinates we have:

$$g_{(r,\phi),(1,0)} = \begin{pmatrix}1 & 0\\ 0 & G(r,\phi)^2\end{pmatrix}.$$

Comparison of normal Jacobi vector fields of Finsler and Riemannian geodesics.

Consider two metrics on the same $U \subseteq \mathbb{R}^2$: the Finsler metric F and the Riemannian metric $\tilde{g}$.

Assume the same curve $\gamma(t)$ is a geodesic of both metrics, and assume that $\tilde{g}_{\gamma(t)} = g_{\gamma(t),\dot{\gamma}(t)}$. Consider normal Jacobi vector fields $\tilde{J}(t)$ and $J(t)$ in $\tilde{g}$ and F along this geodesic such that $J(0) = \tilde{J}(0) = 0$.

Assume $K(\gamma(t), \dot{\gamma}(t)) \equiv \tilde{K}(\gamma(t))$, where $\tilde{K}$ is the (sectional) curvature of the Riemannian metric. Then,

$$J(t) = \mathrm{const} \cdot \tilde{J}(t).$$

Explanation. The differential equation for the length of J is govered by the first nontrivial coefficient in the Taylor polynomial which is K.

The isometry ψ and proof of Theorem A

Assume flag curvature is $+1$, take a point p and consider the exponential map $\exp : T_pS^2 \to S^2$.

Follows from the definition of the curvature/relation to Jacobi vector fields I explained before. The mapping restricted to the r-sphere $\{\xi \mid F(p,\xi) = r\}$ is a homothety w.r.t. the distance in $g_{p,\xi}$ on the sphere and the distance in $g_{(\gamma_{p,\xi}(r)),\dot\gamma_{p,\xi}(r))}$ on the $r-$sphere $\{x \in S^2 \mid d(p,x) = r\}$.

The coefficient of the homothety is $|\sin(r)|$ and vanishes for $r = k\pi$.

Corollary. All arc-lenght geodesics starting from p come to the same point in time π. We call this point $\psi(p)$.

Theorem. ψ is an isometry.

Proof that ψ is an isometry.

- Take two arbitrary points $x, y \in M^n$ consider an acr-length geodesic γ passing through these points, we assume $\gamma(0) = x$ and $\gamma(r) = y$.
- Then, by the construction of ψ we have $\psi(x) = \gamma(\pi)$ and $\psi(y) = \gamma(\pi + r)$ which implies that $d_F(x, y) = r = d_F(\psi(x), \psi(y))$. $\square$

Important properties of ψ.

- The mapping ψ^k sends the point x to the point $\gamma_{x,v}(k\pi)$, where v is an arbitrary vector of Finsler length 1.
- The differential $d_x\psi^k$ sends the vector v to the vector $\dot{\gamma}_{x,v}(k\pi)$.

Remark. If the metric is reversible, $\psi^2 = \mathrm{Id}$. For general metrics, $\psi^2 \neq \mathrm{Id}$. For example, for the Katok metrics $\psi(\theta,\phi) = (-\theta, \phi + \pi(1+\alpha))$, and if α is irrational, ψ^k is never an identity for $k \neq 0$.

The mapping ψ in the spherical coordinates (θ, ϕ)

Theorem. $\exists$ a spherical coordinate system s.t. $\psi(\theta, \phi) = (-\theta, \phi + \lambda\pi)$.

Proof. This is actually a general observation attributed to Poincare: *An isometry group of any Riemannian 2-sphere is a subgroup of the isometry group of a certain metric of constant curvature on the same sphere.*

The reduction Finsler $\longrightarrow$ Riemannian is given as follows: to each Finsler metric, one can associate a Riemannian metric such that isometries of the Finsler metrics g_F are isometries of the Riemannian metric, see e.g. [M$\sim$, Rademacher, Troyanov, Zeghib 2009] or [M$\sim$, Troyanov 2012].

The group generated by ψ and rough scheme of the further proof

Consider the group generated by ψ, i.e., the closure of $\{\psi, \psi^2 := \psi \circ \psi, \psi^3 = \psi \circ \psi \circ \psi, ...\}$ in the group of isometries of F.

Case 1. The group is finite, i.e., $\psi^k = Id$.
In this case, all geodesics are closed of length at most $k\pi$. Indeed, for every arc-length geodesic γ we have $\gamma(k\pi) = \gamma(0)$ and $\dot\gamma(k\pi) = d\psi^k(\dot\gamma(0)) = \dot\gamma(0)$.
Then, the system is even superintegrable, and the roational number is constant and is the same as for the Katok metric.

Case 2. The group is infinite. As we proved above, $\psi(\theta, \phi) = (-\theta, \phi + \lambda\pi)$. Then, λ is irrational and all rotations of the form $(\theta, \phi) \mapsto (\theta, \phi + \alpha)$ are isometries. Thus, there exists a Killing vector field and the integrability follows.
P.S. The rotational number is again the same as for the Katok metric.

More comments on Case 1: $\psi^k = \mathrm{Id}$ so all geodesics are closed of typical length of $2\pi k$.

In this case, the geodesic flow generate the locally free action of the circle S^1 and thus produces Seifert foliation on $US^2 = RP^3$.

Seifert foliation on RP^3 are completely understood (e.g. Orlik 1972) and the "rotational number" (called Euler number in the Seifert theory) describes it completely.

Proof of Theorem B under assumptions of Case 2 (geodesics are not periodic; there exists a Killing vector field)

Theorem B. The rotational number is the same for all Liouville tori.

Proof. W.l.o.g. the Killing vector field is $\frac{\partial}{\partial \phi}$. Then, its flow Ψ acts by $\Psi_t(\phi, \theta) = (\phi + t, \theta)$.

As we proved above, $\psi(\theta, \phi) = (-\theta, \phi + \lambda\pi)$, where λ is a constant. Then, $\psi^2 \circ \Psi_{-2\lambda\pi}$ is the identity. But the mapping $\psi^2 \circ \Psi_{-2\lambda\pi}$ is just the projection of the time 2π Hamiltonian flow with the Hamiltonian $H_F + \lambda p_\theta$.

Thus, the orbits of the Hamiltonian $H_F + \lambda p_\theta$ are closed. Since the orbits of the Hamiltonian p_θ are also closed, the rotation number of $H_F + \lambda p_\theta$ is constant as we want.

Theorem B is proved.

Remark. We just proved that our Hamiltonian H_F is a linear combination with constant coefficient of two generators of a Hamiltonian torus action.

Proof of the remaining part of Theorem C

Theorem C. The Lioville foliation has only two singular fiber which are circles and are ellipic singularities in the Morse-Bott classification.

Proof. In the case all geodesics are closed, it is nothing to prove. If not, we have shown that the Hamiltonian action is embedded in that of the torus. It is known (Delzant 1988? it is not directly in Delzant but rather a folklore) that Hamiltonian action of the torus has only Bott-Morse nondegenerate elliptic singularities.

Last part of Theorem C, and therefore Main Theorem are proved.

Multidimensional generalizations

Let M be a closed manifold of any dimension and F a Finsler metric of constant flag curvature on it.

- **Theorem (Folklore, e.g. Shen 2002).** M is covered by the sphere.
- **Theorem.** Geodesic flow of F is Liouville integrable and has zero topological entropy.
- **Theorem.** Either all geodesics are closed, or there exists a Killing vector field such that after Zermelo transform all geodesics are closed.
    - In dimension 2, one can even make all geodesics to be of the same length 2π

Examples of metrics of constant curvature on S^2 (most results and techniques are due to R. Bryant; possibly an ongoing project with Bryant)

Def. A Riemannian metric on the (2-) sphere is Zoll, if all geodesics are closed (which automatically imply by that they are of the same length by Wadsley 1975 and Raymond 1968)

Fact (Bryant 2002). Given a Zoll Riemannian metric g on S^2 satisfying $\underbrace{\text{certain condition}}_{\text{to be commented}}$, we can construct a Finsler metric of constant curvature $+1$ on S^2
$\underline{\text{such that all its geodesics are closed and of the lenght } 2\pi}$.

Comment on "certain conditions":

- if the construction works for a Zoll metric, then it works for its small perturbation
- the construction works for the standard sphere
- In fact the construction is more general: one starts with *Zoll Weyl structures* (Riemannian structures are special case of Weyl structures).

Duality of the construction:

- The geodesics of the Zoll metrics correspond to the points of the Finsler sphere (actually, to the circles $\{\xi \in T_x S^2 \mid F(x, \xi) = 1\}$)
- the circles $\{\xi \in T_x S^2 \mid g_x(\xi, \xi) = 1\}$ correspond to geodesics of the Finsler metric.

Zoll metrics were intensively studied

- There exist a lot of explicit examples: Tannery 1886, Zoll 1906, Blaschke 1924, Kiyohara 2001, Matveev & Shevchishin 2009
- The space of Zoll metrics is quite big:

  Theorem of Guillemin (1974), informal version. The tangent space to the set of Zoll metrics at the round metric can be identified with the space of odd functions on the sphere.

  In particular, "most" Zoll metrics do not have any symmetry – first examples with no symmetry are due to Blaschke 1924, I explain them if I have time.

But how to construct Finsler metrics of constant curvature such that not all geodesics are of the same length? Are there examples different from Katok?

Fact. The construction of Bryant survives if we start not from S^2 but from a Riemannian orbifold such that all geodesics are closed and of the same length (I call it Zoll orbifold). Our main theorem says that the orbifolds interesting for us have at most two orbital points and prescribes the type of these points.

- **Example.** If we start from weighted projective space, we obtain the Katok metric.
- The construction of Tannery 1886 and Zoll 1906, survives for orbifolds and produces rotationally symmetric Finsler metrics different from Katok one.
- **Example:** Starting from integrals quadratic in momenta, we constructed an example of a Zoll orbifold with no symmetries
- Goal of the project with R. Bryant: does Theorem of Guillemin survive as well? Positive answer to this question would possibly be a final result in the direction of description of Finsler metrics of constant curvature.

Conclusion: what we did and what we can not do

- **Main Theorem.** Geodesic flow of metrics of constant flag curvature on the 2-sphere is conjugate to that of a Katok metric.
  - This reduces the problem of describing metrics of constant curvature to discussion of Zoll Weyl orbifolds with two orbifold points.
- It is not clear what happens in higher dimensions.
  - In dimension 3 there still are chances that one can solve the problem by the same circle of ideas, but in higher dimensions isometry group of a riemannian sphere is not necessary conjugate (at least it does not follow from known general results) to a linear subgroup so we possibly need additional circle of ideas