The Gelfand-Zeitlin system as a tropical limit of Ginzburg-Weinstein diffeomorphisms

joint with A Alekseev and Y Li (arxiv: 1804.01504)
and ongoing work also including B Hoffman.

§1 GZ systems

Setup: $\mathcal{H} = n \times n$ Hermitian matrices $(\cong u(n)^*)$

GZ functions: λ_i^k = ith ordered eig. of $\left(\boxed{k \times k}\right)$

$$\mathcal{H}_0 = \{\lambda_i^k \text{ all distinct}\}$$

The fn's λ_i^k are smooth on $\mathcal{H}_0$.

[Guillemin-Sternberg, '82] Facts about these fn's:

- λ_i^n are complete sfe of Casimir fn's on $\mathcal{H}_0$
- λ_i^k, $k<n$, generate $m = n(n-1)/2$ indep. comm. Ham. S^1-actions

} integrable sys $\underline{w}$ action coords on $\mathcal{H}_0$.

- Image of $\mathcal{H}$ = C_{GZ} Gelfand-Zeitlin cone (convex, polyhedral cone)

- $\mathcal{H}_0 \underset{\substack{\text{Poisson}\\ \text{mfld}}}{\cong} C^{\circ}_{GZ} \times T^m$ with a-a coords (λ, φ)

$\uparrow$ interior of C_{GZ}

i.e. fibers of GZ system are connected and we can ~~to~~ make a global choice of angle coords φ.

A different system:

$$B = \{ b \in M_n\mathbb{C} : b \text{ upper-}\Delta,\ b_{ii} > 0 \}$$

has a Poisson structure (π_B) coming from identification of B with the dual Poisson-Lie group $U(n)^*$ (cf. STS)

$$\mu_i^k = \ln(\lambda_i^k(bb^*))$$

pos. def Hermitian $\Rightarrow \lambda_i^k(bb^*) > 0$

[Flaschka-Ratiu, '95] Repeat items above for GZ system, replacing $\lambda_i^k \longmapsto \mu_i^k$.

These facts tell us there is a Poisson $\cong$

$$\mathcal{H}_0 \underset{\substack{\uparrow \\ \text{a-a for} \\ \text{GZ system}}}{\cong} C^{\circ}_{GZ} \times T^m \xrightarrow{\ id\ } C^{\circ}_{GZ} \times T^m \underset{\substack{\uparrow \\ \text{a-a for} \\ \text{FR system}}}{\cong} B_0$$

($B_0 \subseteq B$ the open dense set where μ_i^k are distinct).

There is one example where we can write this $\cong$ down explicitly in standard mtx coordinates (for larger n, this looks impossible)

<u>Example</u> ($n=2$)

$$\begin{bmatrix} x+y & z \\ \bar{z} & x-y \end{bmatrix} \mapsto \begin{bmatrix} e^{\frac{x+y}{2}} & e^{i\theta}\left(e^{x+r}+e^{x-r}-e^{x+y}-e^{x-y}\right)^{1/2} \\ 0 & e^{\frac{x-y}{2}} \end{bmatrix}$$

where $r = \sqrt{y^2+|z|^2} \approx$ radius of orbit,

$\theta = \operatorname{Arg}(z)$.

This map is an example of a Ginzburg-Weinstein diffeomorphism:

[Alekseev-Meinrenken, '05] The map described above $\mathcal{H}_0 \to B_0$, extends to a smooth Poisson $\cong$ $\mathcal{H} \to B$.

This is an instance of the following general theorem

[Ginzburg-Weinstein, '92] (K, π_K) compact, connected, simplyconnected Poisson-Lie group (i.e. π_K is "multiplicative"), then

$\exists$ Poisson diffeomorphisms

$$gw : (\mathfrak{k}^*, \pi_{\mathfrak{k}^*}) \xrightarrow{\cong} (K^*, \pi_{K^*})$$

linear Lie-Poisson structure

dual Poisson-Lie group for (K, π_K).

§2 Partial Tropicalization of π_{K^*}

This section is a summary of [Alekseev-Davydenkova, '14]. More general version of this story was recently found by [ABHL, '17].

Define fn's on B:

$$\Delta_i^k = \left(\boxed{i \times i} \right)^{-n-k+1}_{-n-k+i}$$

$$\Delta_i^k = e^{t\xi_i^k + \sqrt{-1}\varphi_i^k}$$

i.e. $\xi_i^k = \frac{1}{t}\ln|\Delta_i^k|$, $\varphi_i^k = \operatorname{Arg}\Delta_i^k$, only defined on the open dense subset where $\Delta_i^k \neq 0$. This defines charts (dep. on t)

$$\Delta_t : B \longrightarrow \mathbb{R}^{n+m} \times T \leftarrow \overset{\text{coords}}{(\xi, \varphi)}$$

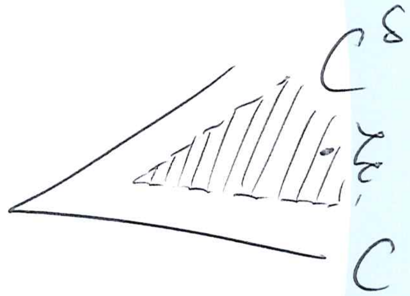
$$\pi_t := (\Delta_t)_*(t\pi_B)$$

Define also a cone C to be the image of C_{GZ} under the map $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ (linear $\cong$) given in coords by

$$\xi_i^k = \lambda_1^k + \cdots + \lambda_i^k .$$

The results of [AD, '14]:

- For all $\xi \in C^\delta$ =



$$\{\xi_i^k, \varphi_q^p\}_t = \pi_{ikpq} + O(e^{-t\delta})$$

and other brackets of coords are $O(e^{-t\delta})$.

i.e. as $t \to \infty$, $\pi_t \to \pi_\infty$, a constant Poisson structure, and from the formula for π_{ikpq} (which we omit here) one sees that wrt π_∞,

$$\left.\begin{array}{l} - \ \xi_i^n \text{ are complete set of Casimir fns} \\ - \ \xi_i^k,\ k<n, \text{ are action coords} \end{array}\right\} \text{int sys on } C^\circ \times T^m, \pi_\infty$$

- φ_i^k are <u>not</u> angle coords (instead, some lin. comb. thereof)

§3 Gelfand-Zeitlin as a tropical limit

Thus, we have the following composition, which is a sequence of Poisson $\cong$:

$$\underset{\pi_{k^*}}{k^*} \xrightarrow{\times t} \underset{t\pi_{k^*}}{k^*} \xrightarrow{gw} \underset{t\pi_{K^*}}{K^*} \xrightarrow{\Delta_t} \underset{\pi_t}{\mathbb{R}^{n+m} \times T^m}$$

Goal: Show that as $t \to \infty$, this converges to a Poisson $\cong$

$$gw_\infty : k^* \longrightarrow C^0 \times T^m, \ \pi_\infty,$$

or, in other words, an integrable system on k^* with global a-a coords.

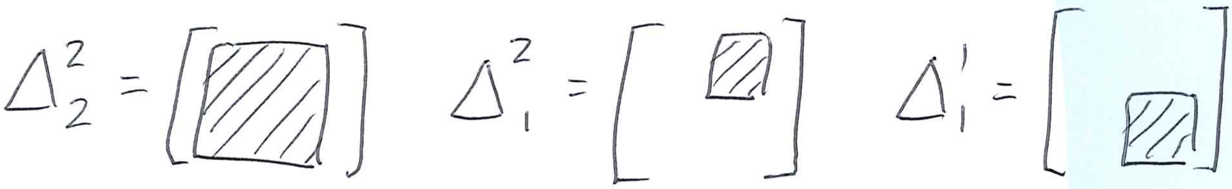
- Already exist many integrable systems on (k^*, π_{k^*}) (e.g. Mischenko-Fomenko, limits of MF), but none appear to have natural global aa coords,

- We really, really like aa coords!

We can write down this composition for our ex:

Example revisited:

$$gw\left(t\begin{pmatrix} x+y & z \\ \bar{z} & x-y\end{pmatrix}\right) = \begin{bmatrix} e^{\frac{t(x+y)}{2}} & e^{i\theta}\left(e^{t(x+r)} + e^{t(x-r)} - e^{t(x+y)} - e^{t(x-y)}\right)^{\frac{1}{2}} \\ 0 & e^{\frac{t(x-y)}{2}} \end{bmatrix}$$



Several examples of convergence of action coords:

$$\zeta_2^2 = \frac{1}{t}\ln\left|\Delta_2^2(gw(tA))\right|$$

$$= \frac{1}{t}\ln(e^{tx}) = \frac{1}{2}\left(\lambda_1^2(A) + \lambda_2^2(A)\right)$$

coords ζ_k^k are constantly equal to $\frac{1}{2}(\lambda_1^k + \dots + \lambda_k^k)$ since det = prod of eigs.

$$\zeta_1^2 = \frac{1}{t}\ln\left|\Delta_1^2\right|$$

$$= \frac{1}{t}\ln\left(e^{t(r+x)} + e^{t(x-r)} - e^{t(x+y)} - e^{t(x-y)}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\lambda_1^2(A)$$

by ineq. defining C_{GZ} (interlacing), $x+r > x-r, x+y, x-y$ when $A \in \mathcal{H}_0$

<u>Theorem:</u> [ALL, '18] For all ~~x~~ $A \in \mathcal{H}_0$,

$$\lim_{t \to \infty} \Delta_t \circ gw(tA) \text{ exists.}$$

Moreover, the fn $gw_\infty : \mathcal{H}_0 \to C^0 \times T^m$ defined by this limit is a Poisson $\cong$, with

$$\xi_i^k = \frac{1}{2}(\lambda_1^k + \cdots + \lambda_i^k).$$

<u>Sketch of proof</u>: Main lemmas come from [APS, '17] papers on planar networks and the Horn problem.

Two parts: $\begin{cases} \text{convergence of } \xi_i^k\text{'s} \leftarrow \text{actions} \\ \text{convergence of } \varphi_i^k\text{'s} \leftarrow \text{angles.} \end{cases}$

Convergence of actions comes from the Cauchy-Binet formula + "tropical analysis" from APS:

e.g. for $1 \le i \le n$, $I \subseteq \{1, \dots, n\}$, $b_t = gw(tA)$,

$$\sum_{|I|=i} e^{t \sum_{j \in I} \lambda_j(A)} \overset{GZ=FR}{=} \sum_{|I|=i} \prod_{j \in I} \lambda_j(b_t b_t^*)$$

$$\overset{\substack{\text{Char}\\\text{poly}}}{=} \sum_{|I|=i} \Delta_{II}(b_t b_t^*)$$

$$\overset{\substack{\text{Cauchy}\\\text{-Binet}}}{=} \sum_{|I|=|J|=i} |\Delta_{I,J}(b_t)|^2$$

Δ_{IJ} = minor w/ rows I, columns J.

$$= \sum_{IJ} \left(\text{Laurent polynomials in } e^{2t\xi_j^k + \sqrt{-1}\varphi_j^k} \right)$$

Laurent phenomenon / generalized Plücker relations

The tropical analysis results from APS tell us that for $t \gg 0$ and $GZ(A) \in C^{\delta}_{GZ}$, the eqn of the previous page looks like:

$$e^{t(\lambda_1^k + \cdots + \lambda_i^k)}\left(1 + O(e^{-t\delta})\right)$$

$$= e^{2t\xi_i^k}\left(1 + O(e^{-t\delta})\right)$$

so $\quad \xi_i^k(gw(tA)) \to \frac{1}{2}(\lambda_1^k + \cdots + \lambda_i^k).$

Convergence of angles is a bit more tricky. Two parts:

- convergence of Ham. vf +
- choice of ~~action~~ angle coords on $\mathcal{H}_0$ such that

$$\{\psi = 0\} \longrightarrow \{\varphi = 0\}$$

Final remarks: – We recover a system we already had. The point is that unlike previous construction of G2 system (Thimm trick) which is doomed to fail for more general k (at least, if you try the naive thing), every part of the composition

$$k^* \xrightarrow{gw_t} K^* \xrightarrow{\Delta_t} C \times T$$

works in more generality. Hope that we can show the limit exists in these cases too.

– Why is it a "tropical" limit?

Usual tropical limit: $p(x)$ a positive polynom $\rightsquigarrow \lim\limits_{t \to \infty} \frac{1}{t} \ln\left(p(e^{tw})\right)$

Our limit: Δ_i^k are positive poly in coords on B & gw_t grows like $\exp(t-)$ $\rightsquigarrow \lim\limits_{t \to \infty} \frac{1}{t} \ln\left(\Delta(gw_t(A))\right)$

(plus, tropical geometry is hot!)