Tetrahedron equation, totally positive matrices, and (quantum) dilogarithm identities

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1. q–exponential function / quantum dilogarithm identity

$$\langle x \rangle_q = \sum_{n\geq 0} (-1)^n \frac{x^n}{(q)_n}, \qquad (q)_n \equiv (1-q)\dots(1-q^n)$$

functional properties: $\langle qx \rangle_q = (1+x)\,\langle x \rangle_q\,, \qquad \langle 0 \rangle_q = 1$

If $\mathsf{Y}\,\mathsf{X} = q\,\mathsf{X}\,\mathsf{Y}$, then

$$(*) \qquad \langle \mathsf{X} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{X} + \mathsf{Y} \rangle_q$$

$(*)$ – Schützenberger ('53)

$$(**) \qquad \langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q = \langle \mathsf{X} \rangle_q \, \langle \mathsf{X}\,\mathsf{Y} \rangle_q \, \langle \mathsf{Y} \rangle_q$$

$(**)$ – "quantum pentagon" relation / quantum dilogarithm identity – Faddeev, Kashaev, Volkov ('93–'94)

2. q–exponential function and R-matrix

Yang–Baxter equation: $R_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}$$

$$R_{12}\, L_{13}\, L_{23} = L_{23}\, L_{13}\, R_{12}, \qquad \mathsf{R}_{23}\, L_{12}\, L_{13} = L_{13}\, L_{12}\, \mathsf{R}_{23}$$

$$L_{01}\, L_{02}\, L_{03} \leftrightarrow L_{03}\, L_{02}\, L_{01} \quad \Rightarrow \quad \mathsf{R}_{12}\, \mathsf{R}_{13}\, \mathsf{R}_{23} = \mathsf{R}_{23}\, \mathsf{R}_{13}\, \mathsf{R}_{12}$$

$$L^{+} = \begin{pmatrix} \mathsf{K} & 0 \\ \mathsf{E} & \mathsf{K}^{-1} \end{pmatrix},\ L^{-} = \begin{pmatrix} \mathsf{K}^{-1} & \mathsf{F} \\ 0 & \mathsf{K} \end{pmatrix}, \quad \mathsf{E}, \mathsf{F}, \mathsf{K} \sim U_q(\mathfrak{sl}_2)$$

Drinfeld ('86): $\mathsf{R} = q^{\frac{1}{4}\,\mathsf{H}\otimes\mathsf{H}}\, \langle \mathsf{E} \otimes \mathsf{F} \rangle_q\, q^{\frac{1}{4}\mathsf{H}\otimes\mathsf{H}}, \qquad \mathsf{K} = q^{\mathsf{H}}$

3. Tetrahedron equation and the symmetric group

A. Zamolodchikov ('81): $R_{123} : V_1 \otimes V_2 \otimes V_3 \to V_3 \otimes V_2 \otimes V_1$

Tetrahedron equation :

$$R_{123}\, R_{145}\, R_{246}\, R_{356} = R_{356}\, R_{246}\, R_{135}\, R_{123}$$

an analogue of the RLL approach ?

S_n - the symmetric group:

$s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$ and $s_i s_j = s_j s_i$ if $|i-j| > 1$

$$R : s_i s_{i+1} s_i \to s_{i+1} s_i s_{i+1}$$

$$\overbrace{s_1 s_2 s_1}^{R_{123}} s_3 s_2 s_1 = s_2 s_1 \overbrace{s_2 s_3 s_2} s_1 = s_2 s_3 \overbrace{s_1 s_2 s_1} s_3 = \overbrace{s_2 s_3 s_2} s_1 s_2 s_3 = s_3 s_2 s_1 s_3 s_2 s_3$$

$$||\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad||$$

$$s_1 s_2 s_3 \underbrace{s_1 s_2 s_1}_{R_{456}} = s_1 \underbrace{s_2 s_3 s_2} s_1 s_2 = s_3 \underbrace{s_1 s_2 s_1} s_3 s_2 = s_3 s_2 s_1 \underbrace{s_2 s_3 s_2} = s_3 s_2 s_1 s_3 s_2 s_3$$

4. An upper triangular quantum group

The quantum group $GL^+_{q,p}(n)$, $p = q^r$

$$R^{(n)} = q^{\theta_{ij}} p^{\theta_{ji}} \sum_{1\le i,j\le n} E_{ii} \otimes E_{ii} + (qp-1) \sum_{1\le i<j\le n} E_{ij} \otimes E_{ji}, \quad \theta_{ij} = \begin{cases} 1 \text{ if } i \le j \\ 0 \text{ if } i > j \end{cases}$$

$$R^{(2)}_{12}\, L^+_{13}\, L^+_{23} = L^+_{23}\, L^+_{13}\, R^{(2)}_{12}, \qquad L^+ = \begin{pmatrix} a & b \\ 0 & a^{-r} \end{pmatrix}, \quad a\, b = q\, b\, a$$

$$R^{(3)}_{12}\, T_{13}\, T_{23} = T_{23}\, T_{13}\, R^{(3)}_{12}, \quad R^{(3)}_{12}\, T'_{13}\, T'_{23} = T'_{23}\, T'_{13}\, R^{(3)}_{12}$$

$$T = \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & a_1^{-r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & 0 & a_2^{-r} \end{pmatrix} \begin{pmatrix} a_3 & b_3 & 0 \\ 0 & a_3^{-r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim s_1 s_2 s_1$$

$$T' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & 0 & a_3^{-r} \end{pmatrix} \begin{pmatrix} a_2 & b_2 & 0 \\ 0 & a_2^{-r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & 0 & a_1^{-r} \end{pmatrix} \sim s_2 s_1 s_2$$

5. An upper triangular quantum group and TE

$$\mathsf{R}\, T = T'\, \mathsf{R}$$

R : $s_i s_{i+1} s_i \to s_{i+1} s_i s_{i+1} \to$ a solution to TE:

AB-Volkov ('14) (a generalization of Sergeev ('97), Kashaev-Volkov ('98)):

$$\mathsf{R} = \langle W \rangle_{q^{-(r+1)}}\, \mathsf{F}\, \langle W \rangle_{q^{r+1}}\,, \qquad W = a_1^r a_2^{-r} b_1 b_3^{-1} a_3$$

$$\mathsf{F}\, a_1 = a_1\, \mathsf{F}, \qquad \mathsf{F}\, a_2 = a_1 a_3\, \mathsf{F}, \quad \mathsf{F}\, a_3 = a_1^{-1} a_2$$
$$\mathsf{F}\, b_1 = b_1 b_2 b_3^{-1}\, \mathsf{F}, \quad \mathsf{F}\, b_2 = b_3\, \mathsf{F}, \qquad \mathsf{F}\, b_3 = b_2$$

F is a counterpart of the flip P in YB: $\mathsf{F}^2 = id$

$$\mathsf{F}_{123}\, \mathsf{F}_{145}\, \mathsf{F}_{246}\, \mathsf{F}_{356} = \mathsf{F}_{356}\, \mathsf{F}_{246}\, \mathsf{F}_{135}\, \mathsf{F}_{123}$$

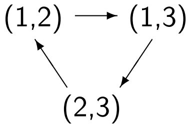
6. TE, quantum dilogarithm identities, S_3 symmetry

YB: $R = P\check{R}$, $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \to \ \check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$

TE: $\mathsf{R} = \mathsf{F}\,\langle W\rangle_{q^{r+1}}$, $\mathsf{R}_{123}\,\mathsf{R}_{145}\,\mathsf{R}_{246}\,\mathsf{R}_{356} = \mathsf{R}_{356}\,\mathsf{R}_{246}\,\mathsf{R}_{135}\,\mathsf{R}_{123} \ \to$

$$\langle \mathsf{Z}_{12}\rangle_q \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\rangle_q \langle \mathsf{Z}_{13}\rangle_q \langle \mathsf{Z}_{23}\rangle_q = \langle \mathsf{Z}_{13}\rangle_q \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \langle \mathsf{Z}_{23}\rangle_q \langle \mathsf{Z}_{12}\rangle_q \quad (*)$$

$$\mathsf{Z}_{12}\mathsf{Z}_{13} = q\,\mathsf{Z}_{13}\mathsf{Z}_{12}, \quad \mathsf{Z}_{13}\mathsf{Z}_{23} = q\,\mathsf{Z}_{23}\mathsf{Z}_{13}, \quad \mathsf{Z}_{23}\mathsf{Z}_{12} = q\,\mathsf{Z}_{12}\mathsf{Z}_{23}$$



automorphism ρ: $\mathsf{Z}_{12} \to \mathsf{Z}_{13}$, $\mathsf{Z}_{13} \to \mathsf{Z}_{23}$, $\mathsf{Z}_{23} \to \mathsf{Z}_{12}$

anti-automorphisms μ_i:

μ_1: $\mathsf{Z}_{12} \leftrightarrow \mathsf{Z}_{13}$, μ_2: $\mathsf{Z}_{13} \leftrightarrow \mathsf{Z}_{23}$, μ_3: $\mathsf{Z}_{23} \leftrightarrow \mathsf{Z}_{12}$

$\mathcal{Z} = \langle \mathsf{Z}_{12}\rangle_q \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\rangle_q \langle \mathsf{Z}_{13}\rangle_q \langle \mathsf{Z}_{23}\rangle_q$

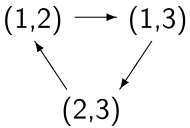
$(*) \quad \Leftrightarrow \quad \mathcal{Z} = \rho(\mathcal{Z}) = \rho(\rho(\mathcal{Z}))$ and $\mu_i(\mathcal{Z}) = \mathcal{Z}$ – S_3 symmetry

AB-Volkov ('15):

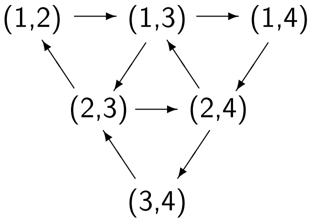
$N \geq 2$, a quiver Q_N with $\binom{N}{2}$ vertices (i,j), $1 \leq i < j \leq N$

torus algebra with $\binom{N}{2}$ generators Z_{ij}, $1 \leq i < j \leq N$

Q_3 :



Q_4 :



S_3 symmetry: $\rho(\mathsf{Z}_{ij}) = \mathsf{Z}_{j-i,N+1-i}$,

$\mu_1(\mathsf{Z}_{ij}) = \mathsf{Z}_{j-i,j}$, $\mu_2(\mathsf{Z}_{ij}) = \mathsf{Z}_{N+1-j,N+1-i}$, $\mu_3(\mathsf{Z}_{ij}) = \mathsf{Z}_{i,N+1+i-j}$

<u>Theorem</u>: $$\mathcal{Z} = \prod_{1 \leq i < j < k \leq N}^{\rightarrow} \Big\langle \prod_{0 \leq m \leq k-j-i}^{\rightarrow} \mathsf{Z}_{i+m,j+m} \Big\rangle_q$$

$\mathcal{Z} = \rho(\mathcal{Z}) = \rho(\rho(\mathcal{Z}))$, $\mu_i(\mathcal{Z}) = \mathcal{Z}$ – quantum dilogarithm identities

8. Classical dilogarithm

$$\langle x \rangle_q = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(q)_n} \ \sim \ \exp\Big(\frac{1}{\log q} Li_2(-x)\Big) \quad \text{as } q \to 1$$

dilogarithm : $$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Rogers dilogarithm : $$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \log(x) \ \log(1-x), \quad 0 < x < 1$$

$$l(x) = \frac{6}{\pi^2} L\Big(\frac{x}{1+x}\Big), \ \ x > 0$$

$l(0) = 0, \ \ l(\infty) = 1, \quad l(1/x) + l(x) = 1$

pentagon identity : $$l(x) + l(y) = l\Big(\frac{x}{1+y}\Big) + l\Big(\frac{xy}{1+x+y}\Big) + l\Big(\frac{y}{1+x}\Big)$$

9. Quasi-classical limit

$\mathsf{X} = \exp(\hat{Q})$, $\mathsf{Y} = \exp(\hat{P})$, $[\hat{P}, \hat{Q}] = -i\hbar \;\rightarrow\; \mathsf{Y}\,\mathsf{X} = q\,\mathsf{X}\,\mathsf{Y}$, $q = e^{i\hbar}$

symbol of an operator: $\langle\!\langle \hat{O} \rangle\!\rangle(P, Q) = \dfrac{\langle Q|\hat{O}|P\rangle}{\langle Q|P\rangle}$

$|P\rangle$ and $|Q\rangle$ – eigenstates of $\hat{P}$ and $\hat{Q}$

Faddeev–Kashaev ('94): $\langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q = \langle \mathsf{X} \rangle_q \, \langle \mathsf{X}\,\mathsf{Y} \rangle_q \, \langle \mathsf{Y} \rangle_q$

$$\langle\!\langle \, \langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q \, \rangle\!\rangle(P, Q; \hbar) = \langle\!\langle \, \langle \mathsf{X} \rangle_q \, \langle \mathsf{X}\,\mathsf{Y} \rangle_q \, \langle \mathsf{Y} \rangle_q \, \rangle\!\rangle(P, Q; \hbar)$$

asymptotics in the quasi-classical limit, $\hbar \to 0$: ($x = e^Q$, $y = e^P$)

$$l(x) + l(y) = l\Big(\frac{x}{1+y}\Big) + l\Big(\frac{xy}{1+x+y}\Big) + l\Big(\frac{y}{1+x}\Big)$$

10. S_3-invariant dilogarithm identity

$\mathsf{Z}_{12}\mathsf{Z}_{13} = q\,\mathsf{Z}_{13}\mathsf{Z}_{12}, \quad \mathsf{Z}_{13}\mathsf{Z}_{23} = q\,\mathsf{Z}_{23}\mathsf{Z}_{13}, \quad \mathsf{Z}_{23}\mathsf{Z}_{12} = q\,\mathsf{Z}_{12}\mathsf{Z}_{23}$

the center: $\mathsf{C} = \mathsf{Z}_{12}\mathsf{Z}_{13}\mathsf{Z}_{23}$, $\quad \mathscr{Z} = \langle \mathsf{Z}_{12} \rangle_q \langle \mathsf{Z}_{12}\mathsf{Z}_{23} \rangle_q \langle \mathsf{Z}_{13} \rangle_q \langle \mathsf{Z}_{23} \rangle_q$

$$\mathscr{Z} = \rho\big(\mathscr{Z}\big) = \rho\big(\rho\big(\mathscr{Z}\big)\big), \;\; \mu_i\big(\mathscr{Z}\big) = \mathscr{Z}$$

asymptotics of $\mathscr{Z}$ in the quasi-classical limit, $q \to 1$:

$$F(x,y,z) = l\Big(\frac{x}{1+y}\Big) + l\Big(\frac{(1+x+y)z}{(1+x)(1+y)}\Big) + l\Big(\frac{xy}{(1+x+y)(1+z)}\Big) + l\Big(\frac{y}{1+x}\Big)$$

<u>Theorem</u>:

$$F(x,y,z) = F(y,x,z) = F(x,z,y) = F(z,y,x) = F(y,z,x) = F(z,x,y)$$

– an S_3-invariant dilogarithm identity

$$F(x,y,z) = l(x) + l(y) + l(z) - l\Big(\frac{xyz}{1+x+y+z+xy+xz+yz}\Big)$$

11. Totally positive matrices

N_n^+ – the variety of $n \times n$ real unipotent upper triangular matrices that are totally positive, i.e., every minor of $M \in N_n^+$ which does not vanish identically is positive

$J_k(x) = I_n + x\, E_{k,k+1}$

Berenstein–Fomin–Zelevinsky ('96): $M \in N_n^+$ iff there exist positive x_{ij}, $1 \leq i < j \leq n$ (the Jacobi coordinates) such that

$$M = \overrightarrow{\prod_{1 \leq k \leq n-1}} \big(J_k(x_{1,k+1}) \dots J_1(x_{k,k+1}) \big)$$

$n = 4$: $M = J_1(x_{12}) J_2(x_{13}) J_1(x_{23}) J_3(x_{14}) J_2(x_{24}) J_1(x_{34}) \sim s_1 s_2 s_1 s_3 s_2 s_1$

12. Involutions M', M'' on N_n^+

P – the permutation matrix such that $P_{ij} = \delta_{i+j,n+1}$

Theorem: (AB-Volkov ('17))

1) for every $M \in N_n^+$, there exist unique $M', M'' \in N_n^+$ and a unique diagonal matrix D_M such that

$$PMP = M' P D_M M''$$

2) $(M')' = M, \quad (M'')'' = M$ – involutions $(s_1 s_1 = s_2 s_2 = id)$

3) $\big((M')''\big)' = \big((M'')'\big)''$ $\qquad (s_1 s_2 s_1 = s_2 s_1 s_2)$

if $\displaystyle\prod_{k=i+1}^{n} \frac{x_{ik}(M)}{x_{n+1-k,n+1-i}(M)} = \prod_{k=1}^{i-1} \frac{x_{ki}(M)}{x_{n+1-i,n+1-k}(M)}$ for all $1 \le i < n/2$

13. Involutions M', M'', and TE

an involution $M \to \bar{M}$: $x_{ij}(\bar{M}) = \dfrac{1}{x_{ij}(M)}$
Local changes of the Jacobi coordinates that affect only x_{ij}, x_{ik}, x_{jk}:

$$L_{ijk}(x_{ij}) = x_{ik}, \quad L_{ijk}(x_{ik}) = x_{ij}, \quad L_{ijk}(x_{jk}) = \frac{x_{ik}x_{jk}}{x_{ik}},$$
$$R_{ijk}(x_{ij}) = \frac{x_{ij}x_{ik}}{x_{jk}}, \quad R_{ijk}(x_{ik}) = x_{jk}, \quad R_{ijk}(x_{jk}) = x_{ik}.$$

4) $n = 3$: $\qquad \bar{M}' = L_{123}(M), \quad \bar{M}'' = R_{123}(M)$

$n = 4$:

$$L_{123}(L_{124}(L_{134}(L_{234}(M)))) = L_{234}(L_{134}(L_{124}(L_{123}(M)))) = \bar{M}',$$
$$R_{123}(R_{124}(R_{134}(R_{234}(M)))) = R_{234}(R_{134}(R_{124}(R_{123}(M)))) = \bar{M}''$$

14. S_3-invariant dilogarithm identity

$n = 4$: $\quad M = J_1(x_{12})J_2(x_{13})J_1(x_{23})J_3(x_{14})J_2(x_{24})J_1(x_{34})$

$\Delta_I(M)$ – the right flag minor of M (intersection of the last $|I|$ columns with the rows labeled by the set I)

$$\mathcal{L}(M) = l\Big(\frac{\Delta_{14}\Delta_{234}}{\Delta_{34}\Delta_{124}}\Big) + l\Big(\frac{\Delta_{1}\Delta_{24}}{\Delta_{4}\Delta_{12}}\Big) + l\Big(\frac{\Delta_{12}\Delta_{234}}{\Delta_{24}\Delta_{123}}\Big) + l\Big(\frac{\Delta_{2}\Delta_{34}}{\Delta_{4}\Delta_{23}}\Big)$$

$$\frac{z}{1+y} = \frac{x_{12}}{x_{23}}, \qquad \frac{y}{1+x} = \frac{x_{13}}{x_{24}}, \qquad \frac{x}{1+z} = \frac{x_{23}}{x_{34}}$$

Theorem:

$$\mathcal{L}(M) = F(x,y,z), \quad \mathcal{L}\big(M'\big) = 4 - F(z,y,x), \quad \mathcal{L}\big(M''\big) = 4 - F(x,z,y)$$

$F(x,y,z)$ is S_3-invariant $\;\Leftrightarrow\; \mathcal{L}(M) + \mathcal{L}(M') = \mathcal{L}(M) + \mathcal{L}(M'') = 4$

15. a generalization for N_n^+

$M \in N_n^+$:

$$Y_{ijk}(M) = \frac{\Delta_{[i,j-1]\cup[k+1,n]}\Delta_{[i+1,b]\cup[k,n]}}{\Delta_{[i,j]\cup[k+1,n]}\Delta_{[i+1,j-1]\cup[k,n]}}, \quad \mathcal{L}(M) = \sum_{1\leq i<j<k\leq n} l\Big(Y_{ijk}(M)\Big)$$

<u>Theorem</u>: the dilogarithm identities:

$$\mathcal{L}(M) + \mathcal{L}(M') = \mathcal{L}(M) + \mathcal{L}(M'') = \frac{1}{6}n(n-1)(n-2)$$

$$\sum_{1\leq i<j<k\leq n} l\Big(\big(Y_{ijk}(s(M))\big)^{\operatorname{sgn} s}\Big) = \text{const}, \quad s \in S_3$$

(?) the quasi-classical limit of $\mathcal{Z} = \rho\big(\mathcal{Z}\big) = \rho\big(\rho\big(\mathcal{Z}\big)\big)$

Relation to cluster algebras, Y-systems, ...

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