Symplectic invariants of integrable Hamiltonian systems: the case of degenerate singularities

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Geometric aspects of momentum maps and integrability
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Integrable systems and their symplectic invariants

An *integrable system* on a symplectic manifold (M^{2n}, ω) is defined by n functions $f_1, \ldots, f_n$ satisfying two properties:

- they Poisson commute;
- they are functionally independent on M almost everywhere.

These functions define the momentum map $\mathcal{F} = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$.
If $a \in \mathbb{R}^n$ is a regular value of $\mathcal{F}$, then connected components of $\mathcal{F}^{-1}(a)$ are Lagrangian submanifolds treated as fibers of the singular Lagrangian fibration on M^{2n} associated with our integrable system.
A fiber is called *singular*, if it contains a point from the singular set

$$\mathrm{Sing} = \{x \in M^{2n} \mid \operatorname{rank} dF(x) < n\} \subset M^{2n}.$$

We assume that all the fibers are compact.

Singular Lagrangian fibrations

We are interested in the properties of the momentum map and, in some sense, "ignore" the dynamics. In particular,

- we are not going to solve this Hamiltonian system;
- we do not choose any distinguished Hamiltonian function among $f_1, \dots, f_n$;
- we do not fix these functions $f_1, \dots, f_n$ either allowing any kind of invertible transformations $(f_1, \dots, f_n) \mapsto (\tilde{f}_1, \dots, \tilde{f}_n)$.

In this view, the object we want to study is just a singular Lagrangian fibration

$$M^{2n} \to B^n,$$

which locally can be given by commuting functions.

It is more convenient to replace the image $\mathcal{F}(M^{2n})$ of the momentum map by the set of fibers B which, in general, is not a smooth manifold. However in all interesting examples, B has a structure of a stratified manifold with good topological properties (bifurcation complex).

Equivalent integrable systems

Given two integrable systems $F : M^{2n} \to B$ and $\widetilde{F} : \widetilde{M}^{2n} \to \widetilde{B}$ (singular Lagrangian fibrations), we want to find/discuss/study conditions for the existence of fiberwise maps between them:

$$\begin{array}{ccc} \mathrm{M}^{2n} & \xrightarrow{\Phi} & \tilde{\mathrm{M}}^{2n} \\ \downarrow \mathrm{F} & & \downarrow \tilde{\mathrm{F}} \\ \mathrm{B} & \xrightarrow{\phi} & \tilde{\mathrm{B}}. \end{array}$$

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- local (neighborhood of a singular point);
- semilocal (neighborhood of a singular fiber);
- global (whole manifold M).

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Three options for Φ (fiberwise map between M and $\widetilde{M}$):

- topological;
- smooth;
- symplectic.

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  - S. Vũ Ngọc:
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- Global
  - J. Duistermaat: Regular case (no singular fibers)
  - T. Delzant: Toric actions
  - S. Vũ Ngọc, À. Pelayo: Semitoric manifolds (two degrees of freedom)
  - N.T. Zung: Very general case

Actions and integer affine structure on B

Theorem (Liouville theorem)

Let $\mathcal{L}$ be a regular compact fiber of a Lagrangian fibration. Then in a suitable neighborhood $U(\mathcal{L})$ (i.e., semilocally) this fibration is symplectically equivalent to the following standard model:

$$T^n \times D^n \to D^n, \qquad (\textit{here } T^n \textit{ is a torus and } D^n \textit{ is a disc})$$

and $\omega = \sum_{i=1}^n dI_i \wedge d\varphi_i$, where $\varphi_1, \ldots, \varphi_n$ (angles) are 2π-periodic coordinates on T^n (fiber) and $I_1, \ldots, I_n$ (actions) are coordinates on D^n (base).

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Three important properties:

(i) explicit formula for action variables: $I_i = \frac{1}{2\pi} \oint_{\gamma_i} \alpha$, where $d\alpha = \omega$;

(ii) the actions are defined modulo $GL(n, \mathbb{Z}) \times \mathbb{R}^n$;

(iii) when the actions are chosen, the angles are defined uniquely after fixing "initial condition", a transversal Lagrangian section $N = \{\varphi_1 = 0, \dots, \varphi_n = 0\} \subset T^n \times D^n$.

Conclusion: action variables $=$ integer affine structure on B_{reg}.
They are the most natural symplectic invariants of integrable systems.

From "affine" to "symplectic"

Question. Let $\phi : B \to \widetilde{B}$ be an affine equivalence. Can ϕ be lifted up to a fiberwise symplectomorphism $\Phi : M \to \widetilde{M}$? If not, what are additional symplectic invariants?

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The answer to the first question is known to be positive in many important cases:

Liouville:	neighborhoods of Liouville tori
Delzant:	toric manifolds
Dufour–Toulet:	neighborhoods of hyperbolic singular fibers
S.Vũ Ngọc:	neighborhoods of pinched tori (focus fibers)
S.Vũ Ngọc, Dullin:	saddle-saddle singularities
S.Vũ Ngọc, Pelayo:	semitoric manifolds (?)

For any affine equivalence between the corresponding bases $\phi : B \to \tilde{B}$ there is a symplectic map $\Phi : M \to \tilde{M}$ such that the following diagram is commutative:

$$\begin{array}{ccc} M^{2n} & \xrightarrow{\Phi} & \tilde{M}^{2n} \\ \downarrow F & & \downarrow \tilde{F} \\ B & \xrightarrow{\phi} & \tilde{B}. \end{array}$$

Non-triviality of actions as symplectic invariants

Except for the Liouville theorem, the existence of an affine equivalence between B and $\widetilde{B}$ is a non-trivial condition.

Example

Hyperbolic case, one degree of freedom: the action is defined on the Reeb graph of the Hamiltonian as just a function of t, parameter on the edge. At the vertex of this graph, we have:

$$I(t) = a(t) \ln t + b(t)$$

with $a(t)$ and $b(t)$ being smooth at zero and $a(0) = 0$, $a'(0) \neq 0$. We can use reparametrisation $\tau = \tau(t)$ to reduce I to a canonical form, e.g.

$$I(\tau) = \tau \ln \tau + c(\tau)$$

Here the function $c(\tau)$ (more precisely, its Taylor expansion at zero) is well defined and can be understood as a non-trivial semilocal symplectic invariant.

Degenerate singularities

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Because:

- They appear in many integrable systems in classical mechanics, geometry and mathematical physics
- They naturally occur as "transition states" between non-degenerate singularities (cusps, Hamiltonian Hopf bifurcation, etc.)
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One degree of freedom:

The simplest case: $H = x^2 - y^3$.
Or something more complicated, e.g., $H = x^p \pm y^q$, $H = \mathrm{Re}\,(x + \mathrm{i}\,y)^k$.

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Main idea is to complexify everything.

In the complex world...

Theorem (well known fact?)

Let $H(x, y) : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *be an complex analytic function with an isolated singularity,* ω *and* $\widetilde{\omega}$ *be two symplectic forms and* $\mathcal{L}_\varepsilon = \{H = \varepsilon\}$ *denote a local fiber of* H*. The following two conditions are equivalent*

- *there exist a (germ of) complex analytic map* $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *such that* $\Phi^* H = H$ *and* $\Phi^* \widetilde{\omega} = \omega$*;*
- *the complex actions coincide, i.e., for any vanishing cycle* $\gamma \in H_1(\mathcal{L}_\varepsilon, \mathbb{Z})$ *we have*

$$\oint_\gamma \alpha = \oint_\gamma \widetilde{\alpha}.$$

Conceptually this means that as symplectic invariants we should consider m functions of one variable $I_1(H), \dots, I_m(H)$ where m is the Milnor number of a singularity, i.e., the number of independent vanishing cycles.

Conclusion: In the complex case,

local symplectic invariants are complex actions.

No other invariants needed!

Uniqueness and initial conditions. From *real* to *complex* and back

Theorem

Assume that a symplectic equivalence map Φ from the above statement exists. Consider two arbitrary transversal sections N_0 and N to the singular fiber $\mathcal{L}_0$ (the same irreducible component of singular fiber). Then there exists a symplectic equivalence map Φ' that sends N_0 to N and such an Φ' is unique.

In other words, we may consider the condition $\Phi(N_0) = N$ as a kind of natural initial condition for Φ. In particular, one may assume that $\Phi(N_0) = N_0$, or equivalently, $\Phi|_{N_0} = \mathrm{id}$.

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"From": a real analytic map $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ satisfying $\Phi^* H = H$ and $\Phi^* \widetilde{\omega} = \omega$ naturally induces the complexified map $\Phi^{\mathbb{C}} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ satisfying the same property for the complexified functions and form.

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"Back" is not so obvious. However, it is definitely true if the real singular fiber $\mathcal{L}_0$ is one-dimensional (we apply the uniqueness theorem). If $\mathcal{L}_0 = \{0\}$, i.e. just a single point, then we do not have any analog for "initial conditions" ... There should be something else.

Idea of the proof (why "complex" is important?)

In the complex world:

Consider an arbitrary section N_0 and try to construct Φ with the initial condition $\Phi|_{N_0} = \mathrm{id}$. For each point $x \in U(0)$ there exists $x_0 \in N$ such that x and x_0 belong to the same fiber $\mathcal{L}_\varepsilon$ that can also be viewed as an orbit of the complex Hamiltonian vector field generated by H, i.e., there exists $t = t(x) \in \mathbb{C}$ such that $x = \sigma^t(x_0)$. Then, we simply have to set

$$\Phi(x) = \widetilde{\sigma}^{t(x)} \circ \sigma^{-t(x)}(x)$$

or, equivalently,

$$\Phi(x) = \widetilde{\sigma}^{t(x)-t'(x)}(x) = \widetilde{\sigma}^{r(x)}(x),$$

where $t'(x)$ is defined by $\widetilde{\sigma}^{t'(x)}(x_0) = x$ and $r(x) = t(x) - t'(x)$.

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The function $r(x)$ (and hence $\Phi(x)$) is:

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— well-defined (as the complex actions coincide!);

— defined everywhere except for the singular point 0.

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We now apply Hartog's theorem to conclude that $r(x)$ is bounded and holomorphic everywhere. Hence, $\Phi(x)$ is explicitly constructed.

Canonical forms for ω

For quasi-homogeneous singularities (at least?), we have the following analog of the isochore Morse lemma.
Consider the differential of H:

$$\mathrm{d}H = (H_x, H_y)$$

and the quotient space $\mathbb{R}[x, y]/\langle H_x, H_y\rangle$. This space is m-dimensional (where m is a Milnor number). Let us choose a basis in it:

$$f_1, \ldots, f_m.$$

We will say that two sympletic forms ω and $\widetilde{\omega}$ are equivalent, if there exists a map Φ with the above properties.

Theorem (Françoise)

Each 2-form is equivalent to one of the forms of the following kind:

$$\omega_{\mathrm{can}} = \alpha_1(H)\,\omega_1 + \alpha_2(H)\,\omega_2 + \cdots + \alpha_m(H)\,\omega_m$$

where $\omega_k = f_k(x, y)\mathrm{d}x \wedge \mathrm{d}y$*. The functions* $\alpha_i(\cdot)$ *are uniquely defined.*

Examples

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$H = x^2 - y^2, \quad \omega_{\mathrm{can}} = \alpha(H)\, dx \wedge dy, \quad I(H) = a(H) \ln H + \dots$

By reparametrising: $H \mapsto h = a(H) \;\Rightarrow\; I(h) = h \ln h + \dots$

Conclusion: No local symplectic invariants

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$H = x^2 - y^3,$
$\omega_{\mathrm{can}} = \alpha(H)\, dx \wedge dy + \beta(H)\, y\, dx \wedge dy,\ I(H) = a(H) \cdot H^{5/6} + b(H) \cdot H^{7/6}$
Reparametrisation $H \mapsto h = h(H)$ gives:
$\omega_{\mathrm{can}} = dx \wedge dy + \widehat{\beta}(h)\, y\, dx \wedge dy,\ I(h) = \mathrm{const} \cdot h^{5/6} + \widehat{b}(h) \cdot h^{7/6}$
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Example

$H = x^2 + y^4,\ \omega_{\mathrm{can}} = \alpha(H)\, dx \wedge dy + \beta(H)\, y\, dx \wedge dy + \gamma(H)\, y^2\, dx \wedge dy$
Does the action determine the symplectic structure? No, the forms

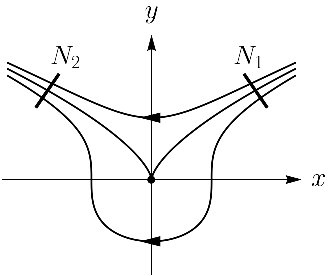
$$dx \wedge dy \quad \text{and} \quad dx \wedge dy + y\, dx \wedge dy$$

have the same actions but are not equivalent.
Conclusion: Sometimes, actions are not enough.

Can we see complex actions from real data?

Fact. Symplectic invariants are complex actions $\oint_{\gamma_i} \alpha$, where γ_i's are vanishing cycles.

Question. Can we see them from the real data, namely from $\Pi_{\tau_j}(H) = \int_{\tau_j} \frac{\omega}{\mathrm{d}H}$, where τ_j are real relative cycles?



$\Pi_{\tau_j}(H) = \int_{\tau_j} \frac{\omega}{\mathrm{d}H}$ (where $\frac{\omega}{\mathrm{d}H}$ is known as Gelfand-Leray form) is just the pasage time form N_1 to N_2 and we have the relation:

$$\int_{\tau_j} \frac{\omega}{\mathrm{d}H} = \frac{\mathrm{d}}{\mathrm{d}H} \int_{\tau_i} \alpha.$$

Can we see complex actions from real data?

If we think of H as a real variable, then $\Pi_\tau(H)$ is a single-valued function having perhaps certain singularity at zero (tends to infinity).

Let us now think of it as a complex function of a complex variable.

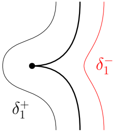
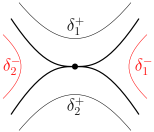
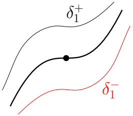
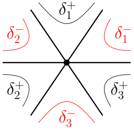
When the endpoints of τ go around a small loop and return to their initial positions, the relative cycle τ does not remain the same but changes $\tau \mapsto \tau + \mathrm{Var}(\tau)$ where $\mathrm{Var}(\tau)$ is certain cycle in $H_1(\mathcal{L}_H, \mathbb{Z})$ and therefore

$$\Pi(e^{i\phi}H)|_{\phi=2\pi} = \Pi(H) + \oint_{\mathrm{Var}(\tau)} \frac{\omega}{\mathrm{d}H}$$

If we iterate this procedure, then at the next step we will get an additional increment and so on:

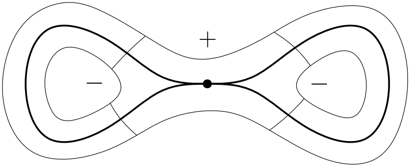
$$\oint_{\mathrm{Var}(\tau)} \frac{\omega}{\mathrm{d}H} + \oint_{M(\mathrm{Var}(\tau))} \frac{\omega}{\mathrm{d}H} + \oint_{M^2(\mathrm{Var}(\tau))} \frac{\omega}{\mathrm{d}H} + \dots$$

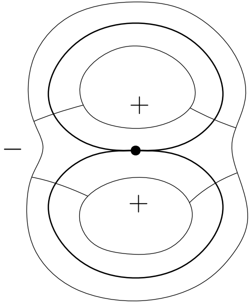
Conclusion. If $\mathrm{Var}(\tau)$, $M(\mathrm{Var}(\tau))$, $M^2(\mathrm{Var}(\tau))$, $\dots$ generate the whole homotopy group $H_1(\mathcal{L}_H, \mathbb{Z})$, then the real data is sufficient.

Singularity		Good δ	Bad δ
$y^2 - x^p$ p prime $p > 2$		$\{\delta_1^+\}$, $\{\delta_1^-\}$	
$y^2 - x^{2k}$ $k = 2, 3$		$\{\delta_1^+ + \delta_2^+,\ \delta_1^-, \delta_2^-\}$	$\{\delta_1^+, \delta_2^+,\ \delta_1^- + \delta_2^-\}$
$y^p - x^q$ p, q prime $p, q > 2, p \neq q$		$\{\delta_1^+\}$, $\{\delta_1^-\}$	
$y^3 - x^2 y$		$\{\delta_1^+, \delta_2^+, \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$, same with $+ \leftrightarrow -$	$\{\delta_1^+ + \delta_2^+ + \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$

Example 1

Consider the singularity of type $f(x, y) = y^2 - x^4$.

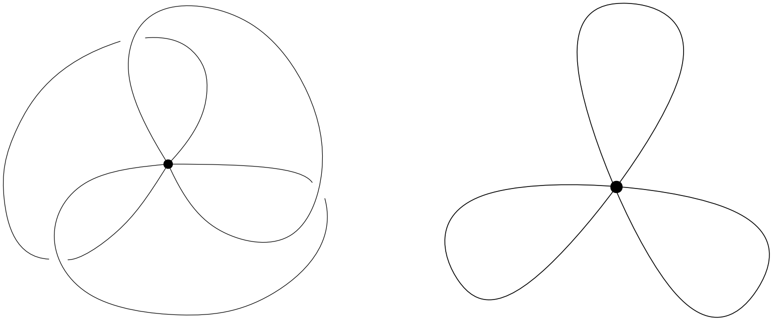




Case 1: the actions do not determine the symplectic structure.
Case 2: the actions determine the symplectic structure.

Example 2

Consider the singularity of type $f(x, y) = y^3 - x^2 y$.



Case 1: the actions do not determine the symplectic structure.
Case 2: the actions determine the symplectic structure.

Cusp type (parabolic) singularities in 2 degrees of freedom

Two degrees of freedom system $\mathcal{F} = (H, F) : M^4 \to \mathbb{R}^2$

Elliptic: $H = p_1^2 + q_1^2,\ F = p_2$

Hyperbolic: $H = p_1^2 - q_1^2,\ F = p_2$

In the both cases: No symplectic invariants (equivariant version of Eliasson theorem by Miranda, Zung)

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Parabolic: $H = p_1^2 + q_1^3 + q_1 p_2$, $F = p_2$

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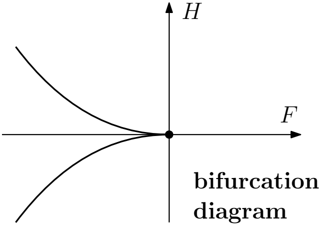
Parabolic (cusp) singularities in the context of integrable Hamiltonian systems have been studied by many authors:
L. Lerman, Ya. Umanskii (1987, 1994), V. Kalashnikov (1998), N. T. Zung (2000), Y. Colin de Verdière (2003), H. Dullin, A. Ivanov (2005) and K. Efstathiou, A. Giacobbe (2012).

Parabolic orbits in 2 degrees of freedom

Proposition ($\simeq$ Definition)

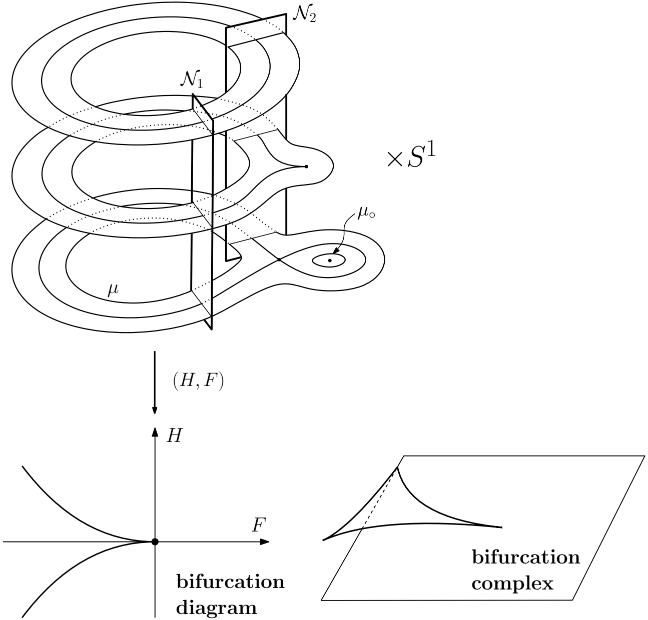
Let γ_0 be a parabolic orbit for an integrable Hamiltonian system with the momentum mapping $\mathcal{F} = (H, F) : M^4 \to \mathbb{R}^2$. Assume that F is a generator of the S^1-action and the local bifurcation diagram $\Sigma \subset \mathbb{R}^2(H, F)$ of $\mathcal{F}$ takes the standard form

$$\Sigma = \left\{ H^2 = -\tfrac{4}{27} F^3 \right\}$$



with $\Sigma_{\mathrm{ell}} = \Sigma \cap \{H < 0\}$, $\Sigma_{\mathrm{hyp}} = \Sigma \cap \{H > 0\}$.
Then in a neighborhood of a parabolic point there exists a local coordinate system (x, y, λ, φ) in which $H = x^2 + y^3 + \lambda y$ and $F = \lambda$ and

$$\Omega = f(x, y, \lambda)\, dx \wedge dy + d\lambda \wedge d\phi + \dots$$


$\mathcal{N}_2$
$\mathcal{N}_1$
$\times S^1$
$\mu_\circ$
μ
(H, F)
H
F
bifurcation
diagram
bifurcation
complex

Symplectic invariants of parabolic orbits (version 1)

Theorem

Consider a singular fibration in a neighbohood of a parabolic orbit γ_0 defined by H and F which is Lagrangian w.r.t. two symplectic forms Ω and $\widetilde{\Omega}$. Then the following two statements are equivalent.

(i) *There is a (real-analytic) diffeomorphism Φ such that*
   - *Φ preserves H and F;*
   - *$\Phi^*(\widetilde{\Omega}) = \Omega$.*

(ii) *These two integrable systems have common action variables*

$$I(H,F) = \widetilde{I}(H,F) + \text{const} \quad \textit{and} \quad I_\circ(H,F) = \widetilde{I}_\circ(H,F)$$

or, equivalently, for every closed cycle τ on any "narrow" torus we have

$$\oint_\tau \alpha = \oint_\tau \widetilde{\alpha}, \quad \textit{for } d\alpha = \Omega, \; d\widetilde{\alpha} = \widetilde{\Omega},$$

where α and $\widetilde{\alpha}$ are chosen in such a way that $\oint_{\gamma_0} \alpha = \oint_{\gamma_0} \widetilde{\alpha} = 0$.

Symplectic invariants of parabolic orbits (version 2)

Theorem

The necessary and sufficient condition for the existence of a real-analytic fiberwise symplectomorphism $\Phi : U(\gamma_0) \to \widetilde{U}(\widetilde{\gamma}_0)$ *between small tubular neighborhoods* $U(\gamma_0), \widetilde{U}(\widetilde{\gamma}_0)$ *of two parabolic orbits* $\gamma_0, \widetilde{\gamma}_0$ *is that these two systems have common action variables in the sense that there is a real-analytic diffeomorphism* $\phi : (H, F) \mapsto (\widetilde{H}, \widetilde{F})$ *which*

- *respects the bifurcation diagrams together with their partitions into hyperbolic and elliptic branch:*

$$\phi(\Sigma) = \widetilde{\Sigma}, \quad \textit{moreover} \quad \phi(\Sigma_{\rm ell}) = \widetilde{\Sigma}_{\rm ell} \quad \textit{and} \quad \phi(\Sigma_{\rm hyp}) = \widetilde{\Sigma}_{\rm hyp},$$

- *and preserves the action variables defined on the "swallow-tail domains":* $I = \widetilde{I} \circ \phi$ *and* $I_\circ = \widetilde{I}_\circ \circ \phi$.

Symplectic invariants of cuspidal tori

Consider two integrable systems defined in some neighborhoods of cuspidal tori $\mathcal{L}_0$ and $\widetilde{\mathcal{L}}_0$:

$$\mathcal{F} : U(\mathcal{L}_0) \to B \subset \mathbb{R}^2(H, F) \quad \text{and} \quad \widetilde{\mathcal{F}} : \widetilde{U}(\widetilde{\mathcal{L}}_0) \to \widetilde{B} \subset \mathbb{R}^2(\widetilde{H}, \widetilde{F}),$$

where B and $\widetilde{B}$ are some neighborhoods of the corresponding cusp points of the bifurcation diagrams. There are three action variables I, $I_\circ$ and I_μ (similarly for the second system). We think of them as functions on B and $\widetilde{B}$ (more precisely on the corresponding domains defined by Σ and $\widetilde{\Sigma}$).

Theorem

A necessary and sufficient condition for the existence of a semilocal fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$ is that the corresponding bases B and $\widetilde{B}$ are affinely equivalent. This means that there exists a local real-analytic diffeomorphism $\phi : B \to \widetilde{B}$ respecting the bifurcation diagrams Σ and $\widetilde{\Sigma}$ and such that $I = \widetilde{I} \circ \phi$, $I_\circ = \widetilde{I}_\circ \circ \phi$ and $I_\mu = \widetilde{I}_{\widetilde{\mu}} \circ \phi$.
Every affine equivalence $\phi : B \to \widetilde{B}$ can be lifted up to a fiberwise symplectomorphism Φ.