Semiglobal symplectic invariants of focus-focus singular fibers with multiple pinched points

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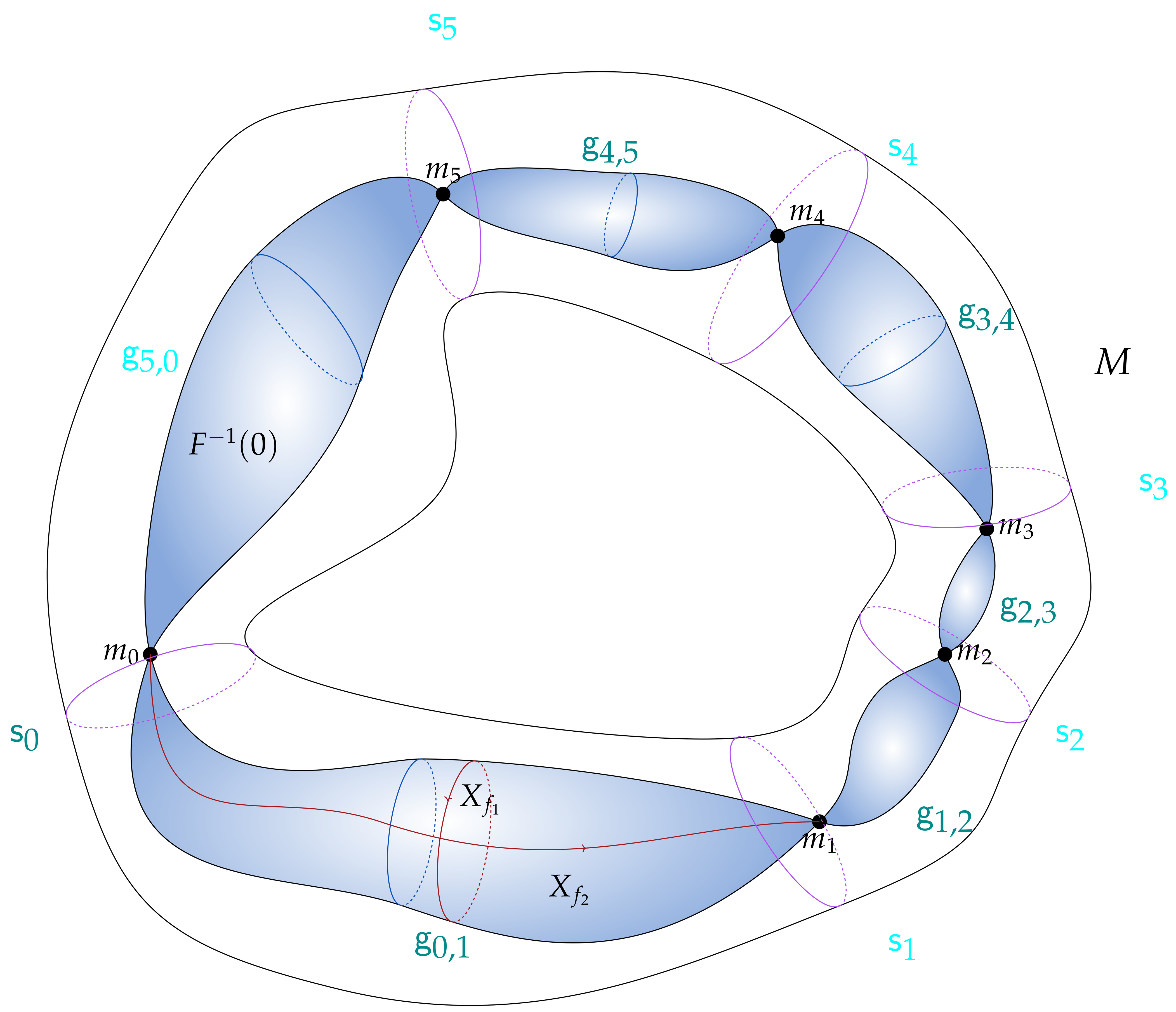


Figure 1: Fibers and invariants

Abstract

We classify, up to symplectomorphisms, a neighborhood of a singular fiber of an integrable system (which is proper and has connected fibers) containing $k > 1$ non-degenerate focus-focus critical points. Our result shows that there is a one-to-one correspondence between such neighborhoods and k formal power series, up to a $(\mathbb{Z}_2 \times D_k)$-action, where D_k is the k-th dihedral group. This proves a conjecture of San Vũ Ngọc from 2002.

Integrable systems

An *integrable system* of two dimensions of freedom is a 3-tuple (M, ω, F), where

- (M, ω) is a 4-dimensional symplectic manifold, and
- $F = (f_1, f_2)\colon M \to \mathbb{R}^2$ is a smooth map such that the Poisson bracket $\{f_1, f_2\}$ vanishes and df_1, df_2 are linearly independent almost everywhere.

In the poster we assume F is proper and has connected fibers.

Local normal form

Eliasson gave a normal form of an integrable system in a neighborhood of a nondegenerate singular point.

Lemma (Eliasson)**.** *If $p \in M$ is a nondegenerate singular point of an integrable system (M, ω, F) of focus-focus type, then there exists local complex coordinates $(z, \zeta) \in \mathbb{C}^2$ about p, and $c \in \mathbb{C}^2$ about $F(p)$ such that $\omega = \mathrm{Im}(dz \wedge d\zeta)$ and $c(F(p')) = z(p')\zeta(p')$ for p' in a neighborhood of p in M.*

Singular and regular fibers

- The singular fiber of F is a torus pinched at each focus-focus point.
- The regular fibers of F are Lagrangian tori.

Invariants

Theorem (Pelayo-T. 2018 [1])**.** *Germs of integrable systems at a focus-focus fiber with k singular points, one of which labeled, is 1-1 correspondent to the k-tuple*

$$(s_0, g_{0,1}, g_{1,2}, \dots, g_{k-2,k-1})$$

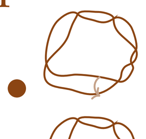
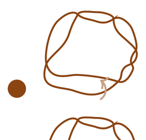
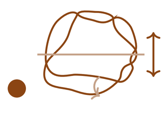
up to the action by $\mathbb{Z}_2 \times D_k$, where

- $s_0 = \sum_{i,j=0}^{\infty} a_{ij} X^i Y^j$ *with* $a_{00} = 0$, $a_{10} \in \mathbb{R}/(2\pi\mathbb{Z})$, *other* $a_{ij} \in \mathbb{R}$;
- $g_{\ell,\ell+1} = \sum_{i,j=0}^{\infty} a_{ij} X^i Y^j$ *with* $a_{00} = 0$, $a_{01} > 0$, *other* $a_{ij} \in \mathbb{R}$, $(\ell = 0, \dots, k-2)$.

Our theorem generalized San Vũ Ngọc's result in [2] where he proved the classification in the case $k = 1$.

The $\mathbb{Z}_2 \times D_k$-action

The power series are subject to an action by the symmetry group of k-pinched torus and the direction of the periodic Hamiltonian flow.

-  : $(s_0(X, Y), \dots, g_{\ell,\ell+1}(X, Y), \dots)$;
-  : $(-s_0(-X, Y) + k\pi X, \dots, g_{\ell,\ell+1}(X, Y), \dots)$;
-  : $\left(-s_0(X, -Y), \dots, -g_{k-\ell-1,k-\ell}^{-1}(X, -Y), \dots\right)$;
-  : $(s_0(X, g_{0,1}^{-1}(X, Y)), \dots, g_{\ell+1,\ell+2}(X, Y), \dots)$.

References

[1] Á. Pelayo and X. Tang. Vu Ngoc's Conjecture on focus-focus singular fibers with multiple pinched points. *ArXiv e-prints*, March 2018.

[2] San Vũ Ngọc. On semi-global invariants for focus-focus singularities. *Topology*, 42(2):365–380, 2003.