Abelianisation of $\mathsf{SL}_2(\mathbb{C})$-Connections and Darboux Coordinates on their Moduli Spaces

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Overview

We describe a new approach (initiated in [GMN13] and developed in [Nik18]) to studying flat connections on holomorphic vector bundles over Riemann surfaces. Roughly speaking, we put connections on rank 2 vector bundles in bijective correspondence with much simpler objects: connections on line bundles.

$$\left\{\begin{array}{c}\text{meromorphic}\\ \mathsf{SL}_2(\mathbb{C})\text{-connections}\\ \text{on a Riemann surface } \mathsf{X}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{meromorphic}\\ \mathbb{C}^\times\text{-connections}\\ \text{on a double cover } \Sigma \text{ of } \mathsf{X}\end{array}\right\}$$

Meromorphic $\mathsf{SL}_2(\mathbb{C})$-Connections

LET: $\mathsf{X} :=$ compact Riemann surface
$\mathsf{D} :=$ finite set of points on X

A **meromorphic** $\mathsf{SL}_2(\mathbb{C})$**-connection** on (X, D) is the data $(\mathcal{E}, \nabla, \mu)$:

- $\mathcal{E} :=$ (sheaf of sections of) holomorphic vector bundle of rank 2
- $\mu :=$ trivialisation $\det(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathsf{X}}$, called **volume form**
- $\nabla :=$ first order differential operator on sections

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{\mathsf{X}}(\mathsf{D})$$

such that $\mu : (\det \mathcal{E}, \operatorname{tr} \nabla) \xrightarrow{\sim} (\mathcal{O}_{\mathsf{X}}, \mathrm{d})$.

HERE: $\Omega^1_{\mathsf{X}}(\mathsf{D}) :=$ sheaf of meromorphic differential 1-forms on X with at most simple poles along D.

Generic Residues and Levelt Lines

The **residue** of ∇ at $\mathsf{p} \in \mathsf{D}$ is an endomorphism of the fibre $\mathcal{E}|_{\mathsf{p}}$:

$$\operatorname{Res}_{\mathsf{p}}(\nabla) \in \mathsf{End}(\mathcal{E}|_{\mathsf{p}}) \cong \mathfrak{sl}_2(\mathbb{C})$$

The residue $\operatorname{Res}_{\mathsf{p}}(\nabla)$ is **generic** if its eigenvalues $\{\lambda, -\lambda\}$ have distinct real parts and $\lambda \notin 2\mathbb{Z}$.

If $(\mathcal{E}, \nabla)$ has a generic residue at $\mathsf{p} \in \mathsf{D}$, there exists a distinguished ∇-invariant line subbundle $\mathcal{L}_{\mathsf{p}} \subseteq \mathcal{E}$ near p, called **Levelt line**. It is generated by a flat section that decays to 0 as it approaches p.

Double Cover

LET: $\pi : \Sigma \to \mathsf{X}$ be a branched double cover such that

- $\Sigma =$ compact Riemann surface (of sufficiently high genus)
- $\pi : \Sigma \to \mathsf{X}$ is *not* branched over D
- X is equipped with a triangulation $\mathbb{T}$ of X with vertices at D and faces enumerated by branch points

MAIN EXAMPLE: Σ is the spectral curve of a meromorphic quadratic differential with prescribed residues along D.

Transverse $\mathsf{SL}_2(\mathbb{C})$-Connections

Connections form a category:

$$\mathsf{Conn}_{\mathsf{X}} := \left\{\begin{array}{c}\text{meromorphic}\\ \mathsf{SL}_2(\mathbb{C})\text{-connections on } (\mathsf{X}, \mathsf{D})\\ \text{with fixed generic residues}\end{array}\right\}$$

A connection $(\mathcal{E}, \nabla, \mu) \in \mathsf{Conn}_{\mathsf{X}}$ is **transverse with respect to** $\mathbb{T}$ if for any triangle $\Delta \in \mathbb{T}$ with vertices $\mathsf{p}, \mathsf{q}, \mathsf{r} \in \mathsf{D}$, the three Levelt lines $\mathcal{L}_{\mathsf{p}}, \mathcal{L}_{\mathsf{q}}, \mathcal{L}_{\mathsf{r}}$ are distinct. This defines a full subcategory:

$$\mathsf{Conn}_{\mathsf{X}}(\mathbb{T}) := \left\{ (\mathcal{E}, \nabla, \mu) \in \mathsf{Conn}_{\mathsf{X}} \;\middle|\; \text{transverse wrt } \mathbb{T} \right\}$$

Odd $\mathbb{C}^\times$-Connections

LET: $\Sigma^\times := \Sigma \setminus \mathsf{Ram}(\pi)$; i.e., we puncture Σ at ramification points.

A $\mathbb{C}^\times$-connection $(\mathcal{L}, \nabla)$ on $\Sigma^\times$ is called **odd** if it is equipped with a skew-symmetric isomorphism $\mu : \mathcal{L} \otimes \sigma^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_\Sigma$, where $\sigma : \Sigma \to \Sigma$ is the canonical involution. Odd connections form a category:

$$\mathsf{Conn}_\Sigma := \left\{\begin{array}{c}\text{odd meromorphic}\\ \mathbb{C}^\times\text{-connections on } (\Sigma^\times, \pi^{-1}\mathsf{D})\\ \text{with fixed residues}\\ (\mathcal{L}, \nabla, \mu)\end{array}\right\}$$

Abelianisation

Abelianisation of $(\mathcal{E}, \nabla, \mu) \in \mathsf{Conn}_{\mathsf{X}}(\mathbb{T})$ proceeds in three steps:

1. Extract all the Levelt lines $\{\mathcal{L}_{\mathsf{p}}\}_{\mathsf{p} \in \mathsf{D}}$
2. Pull each $\mathcal{L}_{\mathsf{p}}$ up to the double cover Σ
3. Use transversality wrt $\mathbb{T}$ to deduce canonical isomorphisms to glue $\{\mathcal{L}_{\mathsf{p}}\}$ into a single odd $\mathbb{C}^\times$-connection $(\mathcal{L}, \nabla^{\mathrm{ab}}, \mu^{\mathrm{ab}})$ on $\Sigma^\times$.

The main result is that this operation is functorial and invertible.

Theorem:
There is an equivalence of categories

$$\begin{aligned} \pi^{\mathrm{ab}} : \mathsf{Conn}_{\mathsf{X}}(\mathbb{T}) &\xrightarrow{\sim} \mathsf{Conn}_\Sigma \\ (\mathcal{E}, \nabla, \mu) &\longmapsto (\mathcal{L}, \nabla^{\mathrm{ab}}, \mu^{\mathrm{ab}}) \end{aligned}$$

called the **abelianisation functor**.

Darboux Coordinates

LET: $\mathbb{M}_{\mathsf{X}} :=$ moduli space corresponding to $\mathsf{Conn}_{\mathsf{X}}$
$\mathbb{M}_{\mathsf{X}}(\mathbb{T}) :=$ moduli space corresponding to $\mathsf{Conn}_{\mathsf{X}}(\mathbb{T})$
$\mathbb{M}_\Sigma :=$ moduli space corresponding to Conn_Σ

FACTS: [AT83, Boa01]

1. $\mathbb{M}_{\mathsf{X}}, \mathbb{M}_{\mathsf{X}}(\mathbb{T}), \mathbb{M}_\Sigma$ are holomorphic symplectic manifolds (or stacks)
2. $\mathbb{M}_{\mathsf{X}}(\mathbb{T})$ is open dense subset of $\mathbb{M}_{\mathsf{X}}$
3. $\mathbb{M}_\Sigma$ is isomorphic to some algebraic torus $(\mathbb{C}^\times)^n$ with symplectic structure in the Darboux form:

$$\omega_\Sigma = \sum \operatorname{dlog} z_i \wedge \operatorname{dlog} z_j$$

Corollary:
The functor π^{ab} induces a holomorphic symplectomorphism

$$\pi^{\mathrm{ab}} : \big(\mathbb{M}_{\mathsf{X}}(\mathbb{T}), \omega_{\mathsf{X}}\big) \xrightarrow{\sim} \big(\mathbb{M}_\Sigma, \omega_\Sigma\big)$$

Since ω_Σ is in Darboux form, this isomorphism π^{ab} can be interpreted as a Darboux coordinate chart on a dense open subset $\mathbb{M}_{\mathsf{X}}(\mathbb{T})$ of the moduli space $\mathbb{M}_{\mathsf{X}}$. This corollary recovers similar considerations in [HN16] and [GMN13].

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