Scattering invariants of integrable systems

Nikolay Martynchuk*

joint with H.R. Dullin, K. Efstathiou and H. Waalkens

* *Address:* Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

E-mail: N.Martynchuk@rug.nl

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Introduction

1. Outline

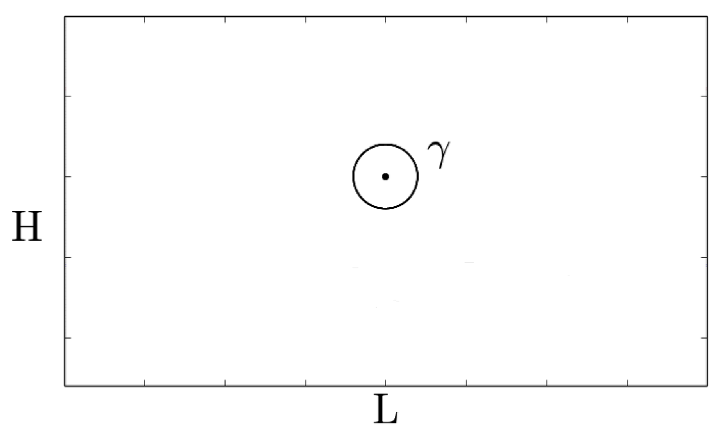
It is known ([1]) that focus-focus singularities do not imply non-trivial Hamiltonian monodromy [2] in the case of integrable systems with non-compact fibers. On the other hand, in the case of the so-called scattering systems, there are other invariants, such as Knauf's index [7], which are non-trivial. We show that in the case of systems that are both scattering and integrable, one can define invariants that are similar to both Hamiltonian monodromy and Knauf's index. Our main example is *scattering monodromy*, which was originally introduced in [1] for a planar hyperbolic oscillator and in [3] for scattering systems in the plane (with a repulsive rotationally symmetric potential).

2. Deflection angle

Let us recall the deflection angle definition of scattering monodromy given in [1, 3].
Consider a repulsive rotationally symmetric potential $V\colon \mathbb{R}^2 \to \mathbb{R}$ that decays at infinity sufficiently fast. For instance, one can take

$$V = W(\|q\|),\ W(r) = 1/(r^2+1).$$

Observe that the corresponding Hamiltonian system is integrable since the angular momentum $L = xp_y - yp_x$ is conserved.



Definition 1. The *deflection angle* of a scattering orbit $q(t)$ is defined by $\delta = \int_{-\infty}^{+\infty} \dot\varphi dt - \pi$, where $\varphi = \mathrm{Arg}(q_1 + iq_2)$ is the polar angle.

Definition 2. ([1, 3]) *Scattering monodromy* along the path γ is defined as the variation of the deflection angle

$$\delta = 2\int_{r_{min}}^{\infty} \frac{l dr}{r^2\sqrt{2(E - l^2/2r^2 - W(r))}} - \pi$$

along γ. Here E is the energy, l is the momentum and r_{min} is the turning point.

Theorem 1. ([3]) *Scattering monodromy along γ is given by*

$$\frac{1}{2\pi}\int_\gamma d\delta = 1.$$

3. Non-compact monodromy

In [5] it was observed that rotational symmetry allows to compactify the non-compact fibration in a neighborhood of a focus-focus fiber. The notion of *non-compact monodromy* was then defined in [5] as the standard monodromy of any such $\mathbb{S}^1$-invariant compactification. (Here the scattering assumption is not important.)
The fact that the result is well-defined does not directly follow from the classical result on focus-focus singularities since the compactified fibration is not Lagrangian. One way that allows to prove this is to apply the topological results of [6, 8].
We present a new approach to scattering monodromy, which does not assume the existence of a circle action and which is based on scattering theory.

Scattering invariants

4. Scattering map

We consider a pair of Hamiltonians on $P = T^*\mathbb{R}^n$ given by

$$H = \frac{1}{2}\|p\|^2 + V(q) \quad \text{and} \quad H_r = \frac{1}{2}\|p\|^2 + V_r(q),$$

where the (singular) potentials V and V_r are assumed to decay at infinity sufficiently fast.
Consider the set of the scattering states

$$s = \{x \mid H(x) > 0,\ \sup_{t\in\mathbb{R}}\|g^t_H(x)\| = \infty\}.$$

For $x \in s$, the limits $\hat p^{\pm}(x) = \lim_{t\to\pm\infty} p(t,x)$ and

$$q^{\pm}_{\perp}(x) = \lim_{t\to\pm\infty} q(t,x) - \langle q(t,x), \hat p^{\pm}(x)\rangle \frac{\hat p^{\pm}(x)}{2h},$$

are defined and depend continuously on $x \in s$. Due to the g^t_H-invariance of $\hat p^{\pm}$ and $q^{\pm}_{\perp}$, we have the maps

$$A^{\pm} = (\hat p^{\pm}, q^{\pm}_{\perp})\colon s/g^t_H \to \mathbb{R}^n \times \mathbb{R}^n.$$

Similarly, one constructs the maps $A^{\pm}_r$ for the 'reference' Hamiltonian $H_r = \frac{1}{2}p^2 + V_r(q)$.

Definition 3. Let R be a g^t_H-invariant submanifold of s and $B = R/g^t_H$. Assume that the composition map

$$S = (A^-)^{-1} \circ A^-_r \circ (A^+_r)^{-1} \circ A^+$$

is well defined and maps B to itself. The map S is called the *scattering map* (w.r.t. H, H_r and B).

Definition 4. ([7]) Let $V_r = 0$ and E be such that $H^{-1}(E) \subset s$. Then $H^{-1}(E)/g^t_H$ can be identified with the cotangent bundle T^*S^n. Knauf's index $\deg(E)$ is then defined as the topological degree of the map $\mathrm{Pr} \circ S\colon (T^*_pS^n) \cup \infty \to S^n$.

5. Scattering monodromy

Assume that the Hamiltonian system given by H is integrable. Let $F\colon P \to \mathbb{R}^n$ be the corresponding integral (also called the energy-momentum) map. The scattering map S gives rise to a new fibration

$$F_c\colon s_c \to \mathbb{R}^n.$$

Invariants of the fibration F_c contain essential information about the scattering dynamics; we get *scattering invariants*.

Definition 5. Assume that

$$F_c\colon s_c \to \mathbb{R}^n$$

is a torus bundle. The standard monodromy of this bundle will be called *scattering monodromy* of the fibration F.

Proposition 1. *In the case when $H_r = H$, the scattering and the Hamiltonian monodromy matrices of F coincide.*

Theorem 2. *In the case of planar systems with a rotational symmetry, scattering monodromy along γ is given by the matrix*

$$M_\gamma = \begin{pmatrix} 1 & m_{\gamma_\alpha} \\ 0 & 1 \end{pmatrix}.$$

In the case when the reference Hamiltonian $H_r = \frac{1}{2}\|p\|^2$, the following statements hold.

- *The scattering and the non-compact monodromy along γ coincide;*
- *The index m_γ is minus the variation of the deflection angle;*
- *The index $m_\gamma = \deg(E_{low}) - \deg(E_{high})$, where $\deg(E)$ denotes Knauf's index.*

Euler's two-center problem

Preliminaries

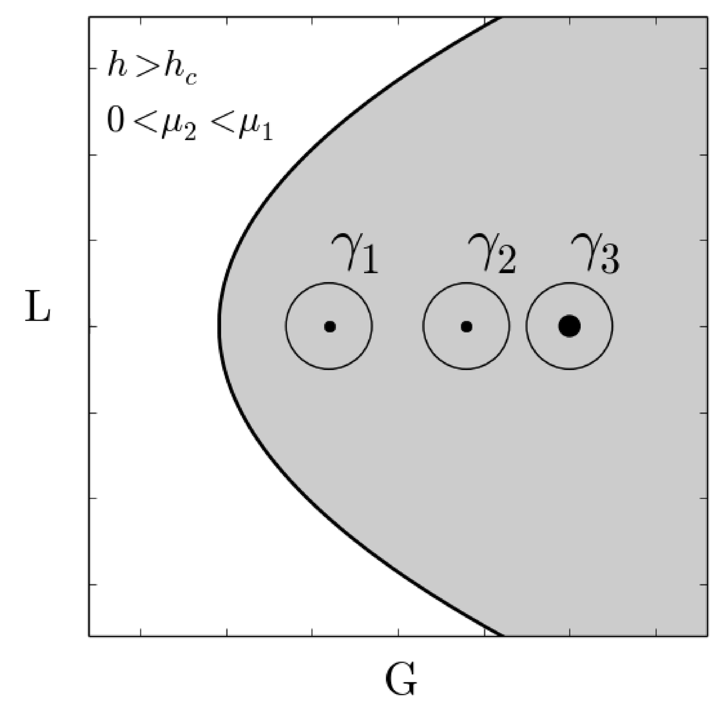
The *Euler two-center problem* can be defined as a Hamiltonian system on $T^*(\mathbb{R}^3 \setminus \{o_1, o_2\})$ with a Hamiltonian function H given by

$$H = \frac{\|p\|^2}{2} + V(q), \quad V(q) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2},$$

where $r_i\colon \mathbb{R}^3 \to \mathbb{R}$ is the distance to the center o_i.
In [9] it was shown that Hamiltonian monodromy is non-trivial in the gravitational problem in case of negative energies (the motion is then bounded). In [4] it was shown that non-trivial Hamiltonian monodromy is present also in the Kepler problem.
We consider the problem in the case of positive energies (the motion is scattering in this case). We show that the problem has scattering monodromy of two different types: purely scattering monodromy and another type, where both scattering and Hamiltonian monodromy are non-trivial. The latter type appears only if the number of degrees of freedom $n \geq 3$. We note that it is present also in the Kepler problem.

Bifurcation diagram

The Euler problem is Liouville integrable. The corresponding integral map $F = (H, L, G)$ comes from the separation in elliptic coordinates. A positive energy slice of the bifurcation diagram is shown in the following figure.



Following the construction given in Sections 5, we can define scattering monodromy of F with respect to the reference Hamiltonian

$$H_r = \frac{\|p\|^2}{2} - \frac{\mu_1 - \mu_2}{r_1}$$

Let γ_i be the path shown in the bifurcation diagram.

Scattering monodromy

Theorem 3. *The monodromy matrices M_i along γ_i (with respect to the natural basis) have the form*

$$M_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the limiting cases, as the Kepler problem, the scattering monodromy is given by the product of some of these matrices.

Theorem 4. *The scattering map $S\colon B_3 \to B_3$, $B_3 = F^{-1}(\gamma_3)/g^t_H$, is a Dehn twist. The push-forward map of S is conjugate in $SL(3,\mathbb{Z})$ to*

$$S_\star = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

References

[1] L. Bates and R. Cushman, *Scattering monodromy and the A1 singularity*, Central European Journal of Mathematics **5** (2007), no. 3, 429–451.

[2] J. J. Duistermaat, *On global action-angle coordinates*, Communications on Pure and Applied Mathematics **33** (1980), no. 6, 687–706.

[3] H. Dullin and H. Waalkens, *Nonuniqueness of the phase shift in central scattering due to monodromy*, Phys. Rev. Lett. **101** (2008).

[4] H.R. Dullin and H. Waalkens, *Defect in the joint spectrum of hydrogen due to monodromy*, Phys. Rev. Lett. **120** (2018), 020507.

[5] K Efstathiou, A Giacobbe, Pavao Mardešić, and D Sugny, *Rotation forms and local hamiltonian monodromy*, Journal of Mathematical Physics **58** (2017), no. 2, 022902.

[6] K. Efstathiou and N. Martynchuk, *Monodromy of Hamiltonian systems with complexity-1 torus actions*, Geometry and Physics **115** (2017), 104–115.

[7] A. Knauf, *Qualitative aspects of classical potential scattering*, Regul. Chaotic Dyn. **4** (1999), no. 1, 3–22.

[8] N. Martynchuk and K. Efstathiou, *Parallel transport along Seifert manifolds and fractional monodromy*, Communications in Mathematical Physics **356** (2017), no. 2, 427–449.

[9] H. Waalkens, H.R. Dullin, and P.H. Richter, *The problem of two fixed centers: bifurcations, actions, monodromy*, Physica D: Nonlinear Phenomena **196** (2004), no. 3-4, 265–310.