Poisson-Lie dual and Langlands dual

via Cluster Theory and Tropicalization

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Introduction

Goals: For compact real form K of semisimple Lie group G, we introduce the 'tropical' version of dual Poisson-Lie group K^*, and show that 'tropicalization' preserves symplectic leaves and their symplectic volume.

Tools: Cluster algebras on the double Bruhat cells G^{e,w_0} and $G^{\vee;e,w_0}$; Potential; Langlands dual $G^\vee$ of G.

Positivity Theory and Tropicalization

A toric chart of variety X is an open embedding $\theta\colon S \to X$ of a split torus S. The tropicalization of X:

$$(X,\theta)^t := \mathsf{Hom}(\mathbb{C}^\times, S).$$

Tropicalization of a positive functions: replacing $\cdot$ by $+$, $+$ by $\max$. **Ex:**

$$f(x_1,x_2,x_3) = \frac{x_1^2}{x_2x_3+x_1} \Rightarrow f^t(\xi_1,\xi_2,\xi_3) = 2\xi_1 - \max\{\xi_2+\xi_3, \xi_1\}.$$

A positive function f is dominated by g if their tropical cones satisfy:

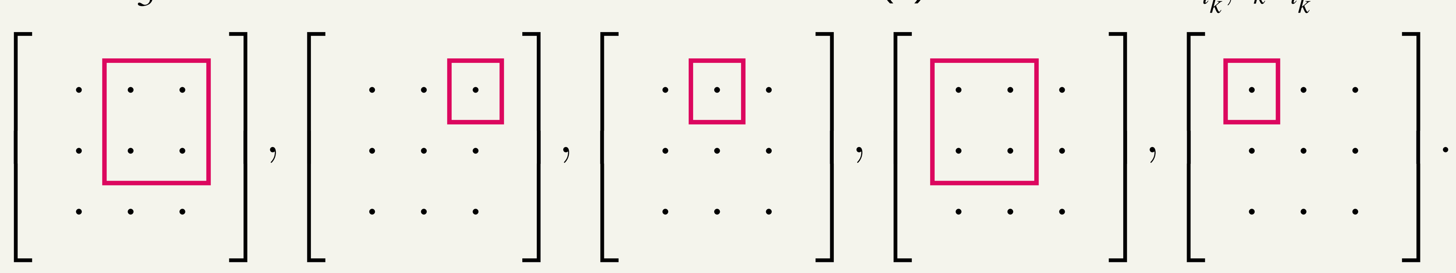
$$\mathcal{C}_g := \{\xi \in (X,\theta)^t \mid g^t(\xi) \leqslant 0\} \subset \mathcal{C}_f.$$

Toric Charts on G^{e,w_0}

Two kinds of charts on $G^{e,w_0} = B \cap B_- w_0 B_-$ for a reduced word $\boldsymbol{i}$ of w_0:

- seed $\sigma(\boldsymbol{i})$ of cluster algebra $\mathbb{C}[G^{e,w_0}]$ — Using for 'duality' to $G^\vee$;
- factorization $z_{\boldsymbol{i}}$ — Using for the crystal structure.

Ex: For SL_3^{e,w_0}, 1) Cluster variables of seed $\sigma(\boldsymbol{i})$: Minors $\Delta_{\omega_{i_k},u_k\omega_{i_k}}$



2) Factorization: Let $L^{e,w_0} = B \cap U_- w_0 U_-$. Then $z_{\boldsymbol{i}}(a_i,t_i) =$

$$\begin{bmatrix} a_1 & 0 & 0 \\ & a_2 & 0 \\ & & a_3 \end{bmatrix} \cdot \begin{bmatrix} t_1^{-1} & 1 & 0 \\ & t_1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & t_2^{-1} & 1 \\ & & t_2 \end{bmatrix} \begin{bmatrix} t_3^{-1} & 1 & 0 \\ & t_3 & 0 \\ & & 1 \end{bmatrix} \in H \times L^{e,w_0}.$$

Potentials? Weakly log-canonical Poisson Brackets!

Given the following Poisson bracket:

$$\{x_1,x_2\} = x_1x_2 + x_1^2 + x_1x_2^2 = x_1x_2(1 + \frac{x_1}{x_2} + x_2).$$

Changing the variables: $x_i \mapsto e^{t\xi_i}$,

$$\{\xi_1,\xi_2\}_t := t^2\{\xi_1,\xi_2\} = 1 + e^{t(\xi_1-\xi_2)} + e^{t\xi_2}.$$

As $t \to +\infty$, it has limit 1 on $\mathcal{C}_f = \{\xi_1 < \xi_2 < 0\}$ where

$$f = \frac{x_1}{x_2} + x_2 \text{ and } f^t = \max\{\xi_1 - \xi_2, \xi_2\}.$$

On G^{e,w_0}, the BK potential is:

$$\Phi_{BK} = \frac{E \cdot \Delta_{w_0\rho,\rho} + \Delta_{w_0\rho,\rho} \cdot E}{\Delta_{w_0\rho,\rho}}.$$

Ex: For GL_n $\qquad \mathcal{C}_{BK} \cong$ Gelfand-Zeitlin cone.

Theorem: Weakly log-canonical brackets on G^*

Let $G^* \subset G \times G$ be the dual Poisson Lie group of G. On the positive variety $(G^*, \sigma(\boldsymbol{i}))$, the standard Poisson structure is weakly log-canonical, i.e. for cluster variables x_i, x_j, we have:

$$\{x_i,x_j\}_{\pi_{G^*}} = x_ix_j(\pi_{ij} + f_{ij}),$$

where π_{ij} is a constant and f_{ij} is dominated by Φ_{BK}.

References

[1] A. Berenstein, D. Kazhdan, *Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases quantum groups*, 2007.

Partial Tropicalization $PT(K^*)$ of Real Form K^*

View $K^* \subset B$. Define $PT(K^*) := \mathcal{C}_{BK} \otimes \mathbb{R} \times (S^1)^m$ with a constant Poisson bracket π_{PT} from π_{ij}. Let hw be the projection of $G^{e,w_0} = H \times L^{e,w_0} \to H$.

Proposition

- *$rank(\pi_{PT}) = rank(\pi_{K^*})$;*
- *The symplectic leaves of $PT(K^*)$ are the fibers of* $\mathsf{hw}^t \circ \mathsf{pr}_1$.

$$(k^*, \pi_{KKS}) \xrightarrow{\text{GW iso}} (K^*, \pi_{K^*}) \xRightarrow{\text{Partial Trop}} (\mathcal{C}_{BK} \times (S^1)^m, \pi_{PT})$$

$$\mathcal{O}_{\lambda^\vee} \qquad\qquad \mathcal{O}_{\exp \lambda^\vee} \qquad\qquad \mathsf{hw}^{-t}(\lambda^\vee)$$

Theorem [1]: Crystal structure on BK cone

To chart $z_{\boldsymbol{i}}$, $\exists$ a direct decomposition over dominant weight of $G^\vee$:

$$\mathcal{C}_{BK}^G = \bigsqcup \mathsf{hw}^{-t}(\lambda^\vee), \text{ and } \mathsf{hw}^{-t}(\lambda^\vee) \cong B_{\lambda^\vee},$$

where $B_{\lambda^\vee}$ is the crystal of the irreducible rep of $G^\vee$ with highest weight $\lambda^\vee$.

Duality of BK cone for G and $G^\vee$: Example

Let $G = \mathsf{SL}_2$ and H be the subgroup of diagonal matrices, then

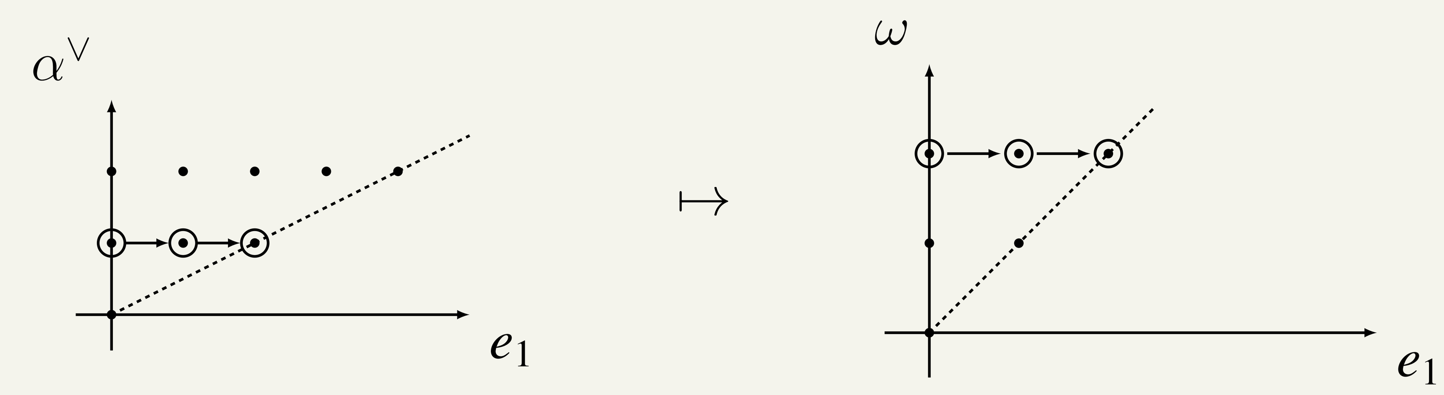
$$x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} = \begin{bmatrix} at^{-1} & a \\ 0 & a^{-1}t \end{bmatrix}; \quad \Phi_{BK} = \frac{1}{a^2}t + t^{-1}.$$

Then

$$\mathcal{C}_{BK}^{\mathsf{SL}_2} = \{(x\alpha^\vee, \xi_1 e_1) \in X_*(H) \times \mathbb{Z} \mid 2x \geqslant \xi_1 \geqslant 0\}.$$

For $G^\vee = \mathsf{PSL}_2$,

$$\mathcal{C}_{BK}^{\mathsf{PSL}_2} = \{(x\omega, \xi_1 e_1) \in X^*(H) \times \mathbb{Z} \mid x \geqslant \xi_1 \geqslant 0\}.$$



The encircled points in $\mathcal{C}_{BK}^{\mathsf{SL}_2}$ are sent to the one in $\mathcal{C}_{BK}^{\mathsf{PSL}_2}$, and arrows to arrows.

Poisson-Lie Dual and Langlands Dual

Let $A = [a_{ij}]_r$ be a Cartan matrix, with symmetrizer $\boldsymbol{d} = \{d_1, \ldots, d_r\} \in \mathbb{Z}_+^r$. A choice of $\boldsymbol{d}$ defines a bilinear form on Lie algebra $\mathsf{Lie}(G)$, and induces an isomorphism:

$$\psi\colon h \to h^* \ :\ \alpha_i^\vee \mapsto d_i\alpha_i.$$

which extends to $\mathcal{L} := (G^{w_0,e}, z_{\boldsymbol{i}})^t = X_*(H) \times \mathbb{Z}^m$:

$$\psi_{\boldsymbol{i}}\colon \mathcal{L} \to \mathcal{L}^\vee \ :\ (\lambda^\vee, v_1, \ldots, v_m) \mapsto (\psi(\lambda^\vee), d_{i_1}v_1, \ldots, d_{i_m}v_m),$$

The symmetrizer $\boldsymbol{d}$ induced a duality of the cluster algebra on G and $G^\vee$:

$$\Psi_{\boldsymbol{i}}^*\colon \mathbb{C}[G^{\vee;e,w_0}] \to \mathbb{C}[G^{e,w_0}] \ :\ \Delta_{\omega_{i_k}^\vee, u_k\omega_{i_k}^\vee} \to \Delta_{\omega_{i_k},u_k\omega_{i_k}}^{d_{i_k}}.$$

Let ν be the tropical change of coordinates, we have

$$\psi_{\boldsymbol{i}} = (\nu^\vee)^{-1} \circ \Psi_{\boldsymbol{i}}^\vee \circ \nu.$$

On the space $\mathsf{hw}^{-t}(\lambda^\vee)$, there are two natural lattices:

- lattice Λ given by the symplectic structure on $\mathsf{hw}^{-t}(\lambda^\vee)$;
- crystal structure on $B_{\lambda^\vee}$.

Theorem

The map $(\Psi_{\boldsymbol{i}}^t)_{\mathbb{R}}\colon \mathcal{C}_{BK}^G(\mathbb{R}) \to \mathcal{C}_{BK}^{G^\vee}(\mathbb{R})$ gives $(\Psi_{\boldsymbol{i}}^t)_{\mathbb{R}}(\Lambda) = (\mathsf{hw}^\vee)^{-t}(\psi(\lambda^\vee))$.

Symplectic Volume

The symplectic volume of the symplectic leaf $\mathcal{O}_{\psi^{-1}(\lambda)}$ is equal to the dimension of the highest weight representation of G with highest weight $\lambda - \rho$.