Stacky Hamiltonian Actions and their Moment Polytopes

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Introduction

Given a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), you can understand M via its image under the moment map $\mu : M \to \mathfrak{g}^*$, which is sometimes a rational convex polytope.

A natural question is to what spaces can be associated non-rational moment polytopes [3, 4, 5]. We answer this by describing Hamiltonian stacks, which are built by taking the stacky quotient of a presymplectic manifold by its null foliation. Hamiltonian stacks come with an action of a Lie group stack $\mathcal{G}$, and a moment map taking values in the dual of the lie algebra of $\mathcal{G}$.

After developing the basic theory we:

- **Construct the symplectic reduction of a Hamiltonian stack.**
- **Extend the Duistermaat-Heckman theorem.**

Symplectic Stacks

We study differentiable étale stacks, which are stacks presented by *foliation groupoids*. A Lie groupoid $X_\bullet = (X_1 \rightrightarrows X_0)$ is a foliation groupoid if it has discrete isotropy groups.

One can define the complex of differential forms of a foliation groupoid:

$$\Omega^k(X_\bullet) = \{\omega_0 \in \Omega^k(X_0) | s^*\omega_0 = t^*\omega_0\}$$

This notion is invariant under Morita equivalence [2] and so describes differential forms on the stack $BX_\bullet$.

Given a manifold $X = X_0$ with a constant rank foliation $\mathcal{F}$, there are many different foliation groupoids which integrate $\mathcal{F}$. For example there is the monodromy groupoid $\mathrm{Mon}(X, \mathcal{F})$ and the holonomy groupoid $\mathrm{Hol}(X, \mathcal{F})$.

In our case: $\mathcal{F}$ is the null foliation $\ker(\omega)$ of a presymplectic form ω. There are then many ways to construct a stacky quotient $X/\mathcal{F}$. The form ω descends to a symplectic form on each of these stacky quotients.

Lie Group Stacks

A group object $\mathcal{G}$ in the category of étale differentiable stacks is a Lie group stack. It can be presented by a Lie 2-group $G_\bullet$, or equivalently a crossed Lie module $\partial : H \to G$. If G_0 is compact, one can define the maximal stacky torus $\mathcal{T}$ of $\mathcal{G}$, with presentation

$$\partial : Q \to \mathbb{R}^n$$

where Q is a finitely generated group and $\mathbb{Z}^n \subset \partial(Q)$.

The infinitesimal version of $\mathcal{G} \cong BG_\bullet$ is the Lie algebra $\mathrm{Lie}(\mathcal{G}) = \mathfrak{g}_1/\mathfrak{g}_0$, which is a Morita invariant.

Example

The following extends an example of Prato [4]. Let $\mathbb{T} \cong \mathbb{R}^n/\mathbb{Z}^n$, and let $N \subset \mathbb{T}$ be an immersed subgroup. Consider a Hamiltonian $\mathbb{T}$-manifold $(M, \omega, \mathbb{T}, \mu)$, for instance

$$M = \mathbb{C}^n, \quad \omega = \frac{1}{2\pi i}\sum dz_j \wedge d\bar{z}_j, \quad \mu(z) = \sum |z_j|^2 e_j^* + \lambda$$

Let $\iota : \mathfrak{n} \to \mathfrak{t}$ be the inclusion of Lie algebras and $\iota^* : \mathfrak{t}^* \to \mathfrak{n}^*$ the dual projection. Then

$$(Z := (\iota^* \circ \mu)^{-1}(0), \omega|_Z, \mathbb{T}, \mu|_Z)$$

is a presymplectic Hamiltonian $\mathbb{T}$-manifold. Taking the stacky quotient,

$$(Z/N, \omega|_Z, \mathbb{T}/N, \mu|_Z)$$

is a Hamiltonian $\mathbb{T}/N$-stack. Notice that $\mathbb{T}/N$ is a Lie group stack (a *stacky torus*). We could have chosen to divide by any covering group of N, and doing so would have given a different Hamiltonian stack.

Main Definition

Let $(\mathcal{X}, \omega)$ be a symplectic stack, and let $\mathcal{G}$ be a Lie group stack. An action $a : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ is *Hamiltonian* if there is a *moment map* $\mu : \mathcal{X} \to (\mathrm{Lie}(\mathcal{G}))^*$. We require that (1) $d\mu^\xi = \iota_{\xi_\mathcal{X}}\omega$ for all $\xi \in \mathrm{Lie}(\mathcal{G})$, and (2) μ is $\mathcal{G}$-equivariant with respect to the coadjoint action of $\mathcal{G}$ on $\mathrm{Lie}(\mathcal{G})^*$. In this case $(\mathcal{X}, \omega, \mathcal{G}, \mu)$ is a *Hamiltonian $\mathcal{G}$-stack.*

Assume there is a presentation $\mathcal{G} \cong BG_\bullet$, where G_0 is compact. Choose a maximal stacky torus $\mathcal{T}$ of $\mathcal{G}$ and a Weyl chamber C of $\mathrm{Lie}(\mathcal{T})^*$. Define the *stacky moment body* as the pair $(\Delta(\mathcal{X}) = C \cap \mu(\mathcal{X}), \mathcal{T})$. In [3] the authors give conditions for $\Delta(\mathcal{X})$ to be convex. Even when these conditions hold, $\Delta(\mathcal{X})$ **is not in general rational.** The pair $(\Delta(\mathcal{X}), \mathcal{T})$ does not depend on the choices involved, up to isomorphism.

Symplectic Reduction

Let $(\mathcal{X}, \omega, \mathcal{G}, \mu)$ be a Hamiltonian stack. Assume $0 \in \mathrm{Lie}(\mathcal{G})$ is a regular value of μ. A stack $\mathcal{Y}$ is the *symplectic reduction at* 0 of $\mathcal{X}$ if there is a map of stacks $p : \mu^{-1}(0) \to \mathcal{Y}$ which is a principal $\mathcal{G}$-bundle, and if there is a symplectic form $\omega^{red} \in \Omega(\mathcal{Y})$ so that $p^*\omega^{red} = \omega|_{\mu^{-1}(0)}$.

Under mild conditions one can find a presentation of $\mathcal{G}$ by a crossed module $H \to G$ and construct an étale presentation $R_\bullet$ of $\mu^{-1}(0)$ so that G acts freely on R_0.

Theorem ([1]) *In this situation, the symplectic reduction of $\mathcal{X}$ at 0 exists if and only if the action of H on R_1 is free. If the reduction exists, it is presented by the Lie groupoid $G \times^H R_1 \rightrightarrows R_0$.*

Duistermaat-Heckman Theorem

Theorem ([1]) *Let $(\mathcal{X}, \omega, \mathcal{G}, \mu)$ be a Hamiltonian stack presented by $(X_\bullet, \omega, G_\bullet, \mu)$. Assuming:*

- *G_0 is a torus;*
- *The action of $G_\bullet$ on $X_\bullet$ is leafwise transitive (i.e., orbits of H are leaves of $\ker(\omega_0)$);*
- *The moment map μ is proper and 0 is a regular value.*

If the symplectic reduction of $\mathcal{X}$ exists at 0, then it exists at all points in a neighborhood U. For $u \in U$, there is an equivalence of symplectic stacks of the reduced spaces

$$(\mu^{-1}(u)/\mathcal{G}, \omega^{red(u)}) \cong (\mu^{-1}(0)/\mathcal{G}, \omega^{red(0)} + \Gamma)$$

where $\omega^{red(u)}, \omega^{red(0)}$ denote the symplectic forms on the reduced spaces at u and 0, respectively, and $\Gamma \in \Omega^2(\mu^{-1}(0)/\mathcal{G})$ varies linearly with $u \in \mathrm{Lie}(\mathcal{G})^*$. After fixing the presentation $X_\bullet$, the cohomology of Γ with respect to the complex $\Omega^\bullet(\mu^{-1}(0)/\mathcal{G})$ does not depend on the choices involved.

References

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