Momentum maps and integrability in noncommutative (quasi-)Poisson geometry

MAXIME FAIRON   UNIVERSITY OF LEEDS, SCHOOL OF MATHEMATICS

UNIVERSITY OF LEEDS

OVERVIEW

One aspect of non-commutative algebraic geometry consists in following Kontsevich-Rosenberg's philosophy, which states that the non-commutative version of a structure defined on a non-commutative algebra should yield the corresponding classical structure on the representation spaces of this algebra. The work of Van den Bergh [VdB] deals with the introduction of non-commutative (quasi-)Poisson geometry, which contains the necessary objects to perform a (quasi-)Hamiltonian reduction on the corresponding representation spaces. In particular, Van den Bergh applies this formalism to quivers.

We explain how to search for integrable systems in this context, and why extended cyclic quivers are a good starting point to study systems in the Calogero-Moser family.

1. REPRESENTATION SPACES

Given a unital associative algebra A over $\mathbb{C}$ and $N \in \mathbb{N}^\times$, the representation space $\operatorname{Rep}(A,N)$ is the affine scheme defined by the coordinate ring generated by symbols a_{ij} for $a \in A$, $1 \leq i,j \leq N$, $\mathbb{C}$-linear in a and satisfying

$$\textstyle\sum_j a_{ij} b_{jk} = (ab)_{ik}\,, \quad 1_{ij} = \delta_{ij}\,.$$

If we write $\mathcal{X}(a)$ for the $N \times N$ matrix (a_{ij}) representing a, we get the rules $\mathcal{X}(a)\mathcal{X}(b) = \mathcal{X}(ab)$ and $\mathcal{X}(1) = \operatorname{Id}_N$.

In the finitely generated case, we see this association as

$$\begin{aligned} A &= \mathbb{C}\langle a_m \rangle / \langle\langle F_l(a_1,\ldots,a_M) = 0 \rangle\rangle \\ &\downarrow A_m := \mathcal{X}(a_m) \\ \operatorname{Rep}(A,N) &= \{A_m \in \mathfrak{gl}_N(\mathbb{C}) \mid F_l(A_1,\ldots,A_M) = 0_N\} \end{aligned}$$

with $1 \leq l \leq L$, $1 \leq m \leq M$ for some $L, M \in \mathbb{N}$. There is a natural $\operatorname{GL}_n(\mathbb{C})$ action by simultaneous conjugation.

2. THE QUIVER CASE

Consider a quiver Q with a finite set of arrows $a \in Q$ between a finite set of vertices I. We define $h,t : Q \to I$ as the maps assigning to an arrow a its head $h(a)$ or its tail $t(a)$. Given $a,b \in Q$, denote by ab the path given by following a then b. If $h(a) \neq t(b)$, $ab = 0$. A general path is a word whose letters are the arrows of Q.

To each vertex $s \in I$, we associate an element e_s such that $e_r a e_s = a\,\delta_{r,t(a)}\delta_{s,h(a)}$. We form the path algebra $\mathbb{C}Q$ generated by all paths and we decompose its unit as $1 = \sum_s e_s$ since $e_r e_s = \delta_{rs} e_s$.

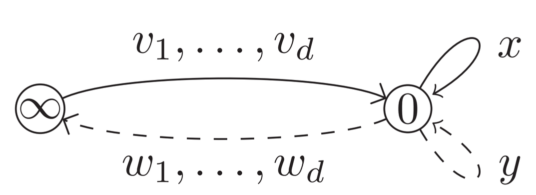
Fix $\alpha = (\alpha_s) \in \mathbb{N}^I$ and set $N = \sum_s \alpha_s$. A representation of Q of dimension α consists in attaching the vector space $\mathbb{C}^{\alpha_s}$ at each $s \in I$ and see the arrows as linear maps between them (acting from the right!)

$$\operatorname{Rep}(\mathbb{C}Q,\alpha) := \textstyle\prod_{a\in Q} \{\mathcal{X}(a) \in \operatorname{Mat}(\alpha_{t(a)} \times \alpha_{h(a)}, \mathbb{C})\}$$

We see the representation space $\operatorname{Rep}(\mathbb{C}Q,\alpha)$ as the subset of $\operatorname{Rep}(\mathbb{C}Q,N)$ such that each e_s is represented by the s-th diagonal identity block of size α_s. We get an action of the algebraic group $\prod_s \operatorname{GL}_{\alpha_s}(\mathbb{C})$ on $\operatorname{Rep}(\mathbb{C}Q,\alpha)$ if we embed it diagonally inside $\operatorname{GL}_n(\mathbb{C})$, which acts by conjugation.

Example :

quiver $\bar{\mathbf{Q}}(d)$



For $n,d \in \mathbb{N}^\times$, $\alpha_0 = n$ and $\alpha_\infty = 1$, we obtain the space $\mathcal{M}_d := \operatorname{Rep}(\mathbb{C}\bar{Q}(d),\alpha)$ characterised by the data

$$\begin{aligned} &X, Y \in \mathfrak{gl}_n(\mathbb{C}), \\ &V_\alpha \in \operatorname{Mat}(1 \times n, \mathbb{C}),\ \alpha = 1,\ldots,d \\ &W_\alpha \in \operatorname{Mat}(n \times 1, \mathbb{C}),\ \alpha = 1,\ldots,d \end{aligned} \tag{1}$$

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Email : mmmfai@leeds.ac.uk - Supported by a *University of Leeds 110 Anniversary Research Scholarship*.

3. DICTIONARY FOR THE POISSON CASE

Assume that $1 \in A$ can be written in terms of orthogonal idempotents $1 = \sum_s e_s$, so A is a $B = \oplus_s \mathbb{C}e_s$-algebra. As in the quiver case, take $N = \sum_s \alpha_s$ and consider that in $\operatorname{Rep}(A,N)$ each e_s is represented by the the s-th diagonal identity block of $\mathcal{X}(1) = \operatorname{Id}_N$ with size α_s. Recall that $\mathcal{O}(\operatorname{Rep}(A,N)) = \{a_{ij} \mid a \in A,\ 1 \leqslant i,j \leqslant N\}$.

We want to define a Poisson structure on $\operatorname{Rep}(A,N)$ completely determined on A. Following [VdB], we put

$$\{a_{ij}, b_{kl}\} := \{\!\{a,b\}\!\}'_{kj}\, \{\!\{a,b\}\!\}''_{il} \quad \text{for } \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}'' \in A \otimes A \ \ \text{a **double bracket**.} \tag{2}$$

A **double bracket** is a B-linear map $\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}$ which is cyclically antisymmetric and a derivation for the outer bimodule structure in its second argument.

To any double bracket, we can naturally associate a triple bracket $Jac_{\{\!\{-,-\}\!\}} : A^{\otimes 3} \to A^{\otimes 3}$. A double bracket is said to be **Poisson** if $Jac_{\{\!\{-,-\}\!\}} = 0$.

Using the idempotent decomposition of unity, we define the s-th **gauge element** $E_s : A \to A^{\otimes 2}$, by

$$E_s(a) = ae_s \otimes e_s - e_s \otimes e_s a\,. \tag{3}$$

We say that $\mu = \sum_s \mu_s$ for $\mu_s \in e_s A e_s$ is a **momentum map** if $\{\!\{\mu_s, -\}\!\} = E_s$. Such a triple $(A, \{\!\{-,-\}\!\}, \mu)$ is called a **Hamiltonian algebra**.

Theorem 1 *[VdB] If A has a double bracket $\{\!\{-,-\}\!\}$, then (2) defines an antisymmetric biderivation $\{-,-\}$ on $\operatorname{Rep}(A,N)$. If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}$ is a Poisson bracket.*

Theorem 2 *[VdB] If $(A, \{\!\{-,-\}\!\}, \mu)$, is a Hamiltonian algebra, then $\mathcal{X}(\mu)$ is a momentum map for the action of $\prod_s \operatorname{GL}_{\alpha_s}(\mathbb{C})$ on $(\operatorname{Rep}(A,N), \{-,-\})$ by conjugation.*

Van den Bergh constructed a Hamiltonian algebra for any double quiver, and this yields usual quiver varieties.

Theorem 3 *If $\{\!\{a,a\}\!\} = \sum_s (e_s \otimes b_s - b_s \otimes e_s)$ for some $a, b_s \in A$, the element a is said **involutive** and the functions $(\operatorname{tr}(\mathcal{X}(a)^k))_k$ are in involution.*

3.1. RATIONAL CM MODEL

Take $A = \mathbb{C}\bar{Q}(1)$. Following Van den Bergh, this algebra is Hamiltonian and such that, setting $v = v_1$ and $w = w_1$,

$$\{\!\{x,x\}\!\} = \{\!\{y,y\}\!\} = 0\,, \quad \mu = e_0([x,y] - wv)e_0 + e_\infty vwe_\infty$$

Consider the representation space $\mathcal{M}_1$ given by matrices X, Y, V, W as in (1). For generic λ, put $\Lambda = (\lambda \operatorname{Id}_n, -n\lambda)$ and the level set $\{\mathcal{X}(\mu) = \Lambda\}$ is smooth so that for $G = \operatorname{GL}_n(\mathbb{C})$, under $\operatorname{GL}_n \cong (\operatorname{GL}_n \times \mathbb{C}^\times)/\mathbb{C}^\times$, the space $\mathcal{X}(\mu)^{-1}(\Lambda)/G$ given by

$$\{X, Y, V, W \mid XY - YX = \lambda \operatorname{Id}_n + WV\} / \operatorname{GL}_n(\mathbb{C})$$

is a complex Poisson manifold denoted $\mathcal{C}_{n,\lambda}$. The case $\lambda = 1$ appears as the n-th Calogero-Moser space in [W]. When X is in diagonal form, Y is the Calogero-Moser Lax matrix and we get the Poisson commutativity of its symmetric functions because $y \in A$ is involutive. The flow of any $\operatorname{tr} Y^k$ can be integrated either explicitly, or obtained by remarking that the $(\operatorname{tr} Y^k)_{k=1}^n$ form an integrable system.

3.2. SPIN RATIONAL CM MODEL

Take $A = \mathbb{C}\bar{Q}(d)$, $d \geq 2$. This algebra is also Hamiltonian, and the moment map now reads

$$\mu = e_0([x,y] - \Sigma_\alpha w_\alpha v_\alpha)e_0 + e_\infty(\Sigma_\alpha v_\alpha w_\alpha)e_\infty$$

Consider the representation space $\mathcal{M}_d$. Again, for generic λ, the space $\mathcal{X}(\mu)^{-1}(\Lambda)/G$ is a complex Poisson manifold denoted $\mathcal{C}^d_{n,\lambda}$. We can write it as

$$\{X, Y, V_\alpha, W_\alpha \mid XY - YX = \lambda \operatorname{Id}_n + \Sigma_\alpha W_\alpha V_\alpha\} / \operatorname{GL}_n(\mathbb{C})$$

We can recognise the space for the Calogero-Moser system with d spins introduced in [GH]. Indeed, we have generically when $X = \operatorname{diag}(x_1,\ldots,x_n)$ that

$$Y_{ij} = \delta_{ij} p_i + \delta_{(i\neq j)} \frac{\sum_\alpha a_i^\alpha c_j^\alpha}{x_i - x_j},\ (W_\alpha)_i = a_i^\alpha,\ (V_\alpha)_j = c_j^\alpha,$$

with $\sum_\alpha a_i^\alpha c_i^\alpha = -\lambda$. We get the Poisson commutativity of the $(\operatorname{tr} Y^k)_k$ because $y \in A$ is involutive. In fact, we can check the involutivity of the nd elements forming the integrable system containg $\operatorname{tr} Y^2$ already in A.

4. DICTIONARY FOR THE QUASI-POISSON CASE

With the notations of the Poisson case, we want to determine $\{\!\{-,-\}\!\}$ on A to define a quasi-Poisson bracket by (2).

A double bracket is **quasi-Poisson** if $Jac_{\{\!\{-,-\}\!\}}$ equals $\frac{1}{12}\sum_s \{\!\{-,-,-\}\!\}_{E_s^3}$. Here, the triple bracket $\{\!\{-,-,-\}\!\}_{E_s^3}$ is naturally defined from the gauge element E_s, which is the double derivation given by (3).

We say that $\Phi = \sum_s \Phi_s$, $\Phi_s \in e_s A e_s$, is a **multiplicative momentum map** if $\{\!\{\Phi_s, -\}\!\} = \frac{1}{2}(\Phi_s E_s + E_s \Phi_s)$. Such a triple $(A, \{\!\{-,-\}\!\}, \Phi)$ is called a **quasi-Hamiltonian algebra**.

Van den Bergh defined a quasi-Hamiltonian algebra from any double quiver. The structure gives rise to a Poisson bracket on multiplicative quiver varieties.

Theorem 4 *Let $(A, \{\!\{-,-\}\!\}, \Phi)$ be a quasi-Hamiltonian algebra, and consider $G := \prod_s \operatorname{GL}_{\alpha_s}(\mathbb{C})$ acting on $\operatorname{Rep}(A,N)$ by conjugation. Then $\{\!\{-,-\}\!\}$ defines a quasi-Poisson bracket $\{-,-\}$ on $\operatorname{Rep}(A,N)$ by (2), and $\mathcal{X}(\Phi)$ is a corresponding (multiplicative) momentum map.*

Theorem 4 is proved in [VdB] for a *differential* double quasi-Poisson bracket, but holds in any case [F2].

Theorem 5 *If $a \in A$ is involutive (see Theorem 3), then the functions $(\operatorname{tr}(\mathcal{X}(a)^k))_k$ are in involution in the Poisson algebras $\mathcal{O}(\operatorname{Rep}(A,N))^G$, or $\mathcal{O}(\mathcal{X}(\Phi)^{-1}(g))^G$ for any $g \in G$.*

4.1. TRIGONOMETRIC RS MODEL

A localisation of $A = \mathbb{C}\bar{Q}(1)$ has a quasi-Hamiltonian algebra structure such that $\{\!\{y,y\}\!\} = \frac{1}{2}(e_0 \otimes y^2 - y^2 \otimes e_0)$ and $\Phi = e_0 xyx^{-1}y^{-1}(1+wv)^{-1}e_0 + e_\infty(1+vw)e_\infty$.

For generic $q \in \mathbb{C}^\times$, we obtain by quasi-Hamiltonian reduction of $\mathcal{M}_1^\circ \subset \mathcal{M}_1$, where $X, Y, \operatorname{Id}_n + WV$ are invertible, the complex Poisson manifold $\mathcal{R}_{n,q}$

$$\{X, Y, V, W \mid XYX^{-1}Y^{-1} = q(\operatorname{Id}_n + WV)\} / \operatorname{GL}_n(\mathbb{C}).$$

A review of this simple case explaining how to see Y as the Lax matrix for the RS system can be found in [CF1], together with generalisations. Here, the symmetric functions $(\operatorname{tr} Y^k)_k$ Poisson commute because $y \in A$ is involutive. Moreover, flows can be precisely integrated.

4.2. SPIN TRIGO. RS MODEL

By looking at $A = \mathbb{C}\bar{Q}(d)$, $d \geq 2$, we can obtain in a way similar to the non-spin case a Poisson manifold $\mathcal{R}^d_{n,q}$ given by $\operatorname{GL}_n(\mathbb{C})$-orbits of generic elements $(X, Y, V_\alpha, W_\alpha) \in \mathcal{M}_d$ such that

$$XYX^{-1}Y^{-1} = q(\operatorname{Id}_n + W_d V_d)\ldots(\operatorname{Id}_n + W_1 V_1)\,.$$

Generically, for some spin variables (a_i^α, c_j^α),

$$X = \operatorname{diag}(x_1,\ldots,x_n)\,, \quad Y_{ij} = q\frac{\sum_\alpha a_i^\alpha c_j^\alpha\, x_j}{x_i - qx_j}\,.$$

This is the Lax matrix for the spin trigonometric RS system, and the construction is given in [CF2]. It can also be obtained by looking at cyclic quivers [F1].

5. COMMENTS AND OPEN PROBLEMS

Let's emphasize that we have developped a straightforward method to obtain integrable systems in the Calogero-Moser family from quivers, and that we can check involutivity of functions already at the algebra level.

Furthermore, we can precisely **write the Poisson bracket** in local coordinates in each case, which was an open problem for the spin trigonometric RS model. Also, the **interesting flows can be explicitly integrated**. In each case, we can find **generalisations** with $S_n \ltimes \mathbb{Z}_m^n$ symmetry by looking at extended cyclic quivers.

We know which Hamiltonian algebra yields the spin trigonometric CM case in the same manner. At the moment, we can obtain the elliptic CM system from a Hamiltonian algebra only for specific spectral parameters, and the general case constitutes an interesting problem. A more challenging question consists in finding the algebra that would lead to the elliptic RS system. Even more difficult : **which integrable systems come from a (quasi-)Hamiltonian algebra?**