# The Taylor series invariant of semitoric systems

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## I'm just passing by...

The **coupled spin-oscillator** and the **coupled angular momenta** are completely integrable systems with two degrees of freedom. They are fundamental examples of so-called **semitoric systems** and can be classified using the five symplectic invariants introduced by Á. Pelayo & S. Vũ Ngọc. One of this invariants consists of **Taylor series associated to the focus-focus singularities** of the system, which until recently had only been calculated up to linear order and for a specific choice of parameters. In this project we compute higher order terms of the invariant and show their dependency on parameters. Together with the calculation of the twisting-index invariant, this makes these two examples the first semitoric systems for which the symplectic classification is complete.

## Semitoric systems

Let $(M, \omega)$ be a 4-dimensional connected symplectic manifold. Consider a completely integrable system $(M, \omega, (L, H))$, where the functions $L, H : M \to \mathbb{R}$ are smooth. The points where $(DL, DH)$ fail to be linearly independent are **singularities**. If $m$ is a non-degenerate singularity, there exist local symplectic coordinates $(x_1, y_1, x_2, y_2)$ and functions $(Q_1, Q_2)$ satisfying $\{L, Q_i\} = \{H, Q_i\} = 0$ for $i = 1, 2$ of the following types:

a) **Elliptic component:** $Q_i = \dfrac{{x_i}^2 + {y_i}^2}{2}$.

b) **Hyperbolic component:** $Q_i = x_i y_i$.

c) **Focus-focus component:** $\begin{cases} Q_1 = x_1 y_2 - x_2 y_1 \\ Q_2 = x_1 y_1 + x_2 y_2. \end{cases}$

d) **Regular component:** $Q_i = y_i$.

We say that the system is **semitoric** if all singularities are non-degenerate, there are no hyperbolic components, the map $L$ is proper (i.e. preimage of compact sets is compact) and $L$ induces a Hamilton $\mathbb{S}^1$-action on $M$. In particular, the flow of $L$ is $2\pi$-periodic.

## Coupled spin-oscillator

The **coupled spin-oscillator** is a semitoric system consisting of the coupling of a classical spin on a 2-sphere with a harmonic oscillator in the Euclidean plane. Consider the product manifold $M = \mathbb{S}^2 \times \mathbb{R}^2$ with symplectic form $\omega = \lambda\, \omega_{\mathbb{S}^2} \oplus \mu\, \omega_{\mathbb{R}^2}$. Let $(x, y, z)$ be Cartesian coordinates on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and $(u, v)$ Cartesian coordinates on the plane $\mathbb{R}^2$. The smooth maps $L, H : M \to \mathbb{R}$ are given by

$$L := \mu \frac{u^2 + v^2}{2} + \lambda(z - 1), \quad H := \frac{xu + yv}{2}.$$

This system has a single focus-focus singularity at the point $m = (0, 0, 1, 0, 0)$. After a coordinate transformation, we can write the system as

$$L = p_1, \quad H = \sqrt{\frac{-p_2(p_2 - p_1)(p_2 - p_1 - 2)}{2\lambda^2 \mu}} \cos(q_2) \qquad (1)$$

with $\omega = dq_1 \wedge dp_1 \oplus dq_2 \wedge dp_2$. Since $H$ is independent of $q_1$, the variable $p_1 = L$ becomes a constant of motion and we can reduce the system using coordinates $(q_2, p_2)$.

## Coupled angular momenta

The **coupled angular momenta** is a semitoric system consisting of the classical version of a coupling of two quantum spins. Let $R_1, R_2$ be two positive constants with $R_2 > R_1$ and $t \in [0, 1]$ a parameter. Consider the product manifold $M = \mathbb{S}^2 \times \mathbb{S}^2$ with $\omega = -(R_1\, \omega_{\mathbb{S}^2} \oplus R_2\, \omega_{\mathbb{S}^2})$ as symplectic form. Let $(x_1, y_1, z_1, x_2, y_2, z_2)$ be coordinates in $M$. The smooth maps $L, H : M \to \mathbb{R}$ are given by

$$\begin{cases} L := R_1(z_1 - 1) + R_2(z_2 + 1) \\ H := (1 - t) z_1 + t(x_1 x_2 + y_1 y_2 + z_1 z_2) + 2t - 1. \end{cases}$$

The system has exactly one singularity of focus-focus type at $m = (0, 0, 1, 0, 0, -1)$ for $t^- < t < t^+$, where

$$t^{\pm} = \frac{R_2}{2R_2 + R_1 \mp 2\sqrt{R_1 R_2}}, \qquad (2)$$

and none otherwise. After a coordinate transformation, we can write the system as

$$L = p_1, \quad H = A(p_1, p_2) + \sqrt{B(p_1, p_2)} \cos(q_2) \qquad (3)$$

for some polynomials $A(p_1, p_2), B(p_1, p_2)$ and $\omega = dq_1 \wedge dp_1 \oplus dq_2 \wedge dp_2$. Again $H$ is independent of $q_1$, so $p_1 = L$ is a constant of motion and we can perform a symplectic reduction.

## Symplectic invariants of semitoric systems

Á. Pelayo & S. Vũ Ngọc [4] have given a **symplectic classification** of semitoric systems in terms of the following five symplectic invariants:

- **Number of singularities invariant:** the number of focus-focus singularities $n_{\text{FF}}$.
- **Polygon invariant:** an equivalence class of weighted collections of rational convex polygons and vertical lines crossing them.
- **Height invariant:** $n_{\text{FF}}$ numbers corresponding to the volume of certain symplectic submanifolds.
- **Taylor series invariant:** a collection of $n_{\text{FF}}$ formal Taylor series in two variables describing the foliation around each focus-focus singularity.
- **Twisting-index invariant:** An equivalence class of $n_{\text{FF}}$ integers measuring the twisting of the system around the focus-focus singularities.

This means on the one hand that two semitoric systems are **isomorphic** if and only if they have the same list of invariants and on the other hand that given a list of invariants, a semitoric system with those invariants can be constructed [3].

However, the explicit calculation of these invariants can be difficult. Until now only the first three invariants had been calculated for the two examples on the right, together with the linear terms of the Taylor series invariant for specific choices of the parameters. We have completed the classification of these two examples by calculating the Taylor series invariant with parameters and higher order terms, together with their twisting-index invariants.

## Want to know more?

### References

[1] J. Alonso, H. R. Dullin and S. Hohloch, *Completing the symplectic classification of the coupled angular momenta*, (to appear).

[2] J. Alonso, H. R. Dullin and S. Hohloch, *Taylor series and twisting-index invariants of coupled spin-oscillators*, arXiv:1712.06402, (2017).

[3] Á. Pelayo and S. Vũ Ngọc, *Symplectic theory of completely integrable Hamiltonian systems*, Bull. Amer. Math. Soc. (N.S.), (2011).

[4] Á. Pelayo and S. Vũ Ngọc, *Semitoric integrable systems on symplectic 4-manifolds*, Invent. Math. (2009).

## Taylor series invariant of the coupled spin-oscillator

If we isolate $q_2 = q_2(p_2; l, h)$ from equation (1), we can define the **real action** of the reduced system as

$$I_{so}(l, h) := \frac{1}{2\pi} \oint_\beta q_2(p_2; l, h)\, dp_2. \qquad (4)$$

This integral becomes a real elliptic integral after integration by parts and therefore it can be expressed in terms of Legendre's standard elliptic integrals:

**Theorem 1.** *The action integral of the spin-oscillator* (4) *is given by*

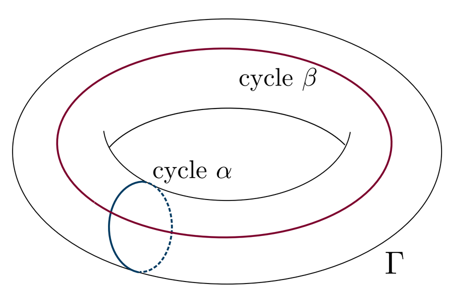
$$I_{so}(l, h) = c_1 K(k) + c_2 \Pi(n_2, k) + c_3 \Pi(n_3, k), \qquad (5)$$

*where $K$ and $\Pi$ are Legendre's complete elliptic integrals of first and third kind respectively.*

Equation (5) can be expanded around the focus-focus point, which corresponds to the singular limit $k \to 1$. We now define the **imaginary action** as

$$J_{so}(l, h) := \frac{1}{2\pi i} \oint_\alpha q_2(p_2; l, h)\, dp_2, \qquad (6)$$

where $\alpha$ is the vanishing cycle of the elliptic curve $\Gamma$ in which $I_{so}$ is defined.




By computing the series expansion of $J_{so}$ and inverting it, we can express the energy $h$ as a 'Birkhoff normal form' $h = B(l, j)$ depending on the value $l$ of the angular momentum $L$ and the value $j$ of the imaginary action $J_{so}$:

$$B(l, j) = \frac{j}{2\sqrt{\lambda\mu}} + \frac{lj}{16\sqrt{\lambda^3 \mu}} - \frac{5j\,(l^2 + j^2)}{512\sqrt{\lambda^5 \mu}} + \frac{lj\,(11l^2 + 15j^2)}{4096\sqrt{\lambda^7 \mu}} + \ldots$$

Substituting $h = B(l, j)$ in the expansion of (5) and knowing the structure of the foliation around the focus-focus point, we obtain the final result:

**Theorem 2.** *The expansion of the action of the spin-oscillator as a function of the angular momentum and the imaginary action is*

$$2\pi I_{so}(l, j) = 2\pi\lambda - l \arg(w) + j - j \ln|w| + S(l, j)$$

*where $w := l + ij$ and $S(l, j)$ is the Taylor series symplectic invariant*

$$\begin{aligned} S(l, j) = {} & \frac{\pi}{2} l + (5 \ln 2 + \ln \lambda)\, j + \frac{1}{4\lambda} lj - \frac{1}{768\lambda^2} j(39l^2 + 34j^2) \\ & + \frac{1}{1536\lambda^3} j(34lj^2 + 23l^3) \\ & - \frac{1}{2621440\lambda^4} j(13505l^4 + 30620l^2 j^2 + 10727j^4) + \ldots \end{aligned}$$

## Taylor series invariant of the coupled angular momenta

By isolating $q_2 = q_2(p_2; l, h)$ from equation (3) we can define the real action $I_{am}$ and the imaginary action $J_{am}$ using (4) and (6). The modified Birkhoff normal form of the focus-focus singularity of the coupled angular momenta is

$$\begin{aligned} B(l, j) = {} & \frac{1}{2R_1 R}(a_{10} l + a_{01} j) + \frac{t}{8{R_1}^2 R {r_A}^2}(a_{20} l^2 + a_{11} lj + a_{02} j^2) \\ & + \frac{t^2(-1 + t)}{8{R_1}^3 R {r_A}^6}(a_{30} l^3 + a_{21} l^2 j + a_{12} l j^2 + a_{03} j^3) + \ldots \qquad (7) \end{aligned}$$

where $R := R_1/R_2$, $r_A$ is the constant

$$r_A := \sqrt{-R^2(1 - 2t)^2 + 2Rt - t^2} \qquad (8)$$

and $a_{mn}$ are polynomial coefficients in $r_A$, $R$ and $t$. One option is to now follow the method used for the coupled spin-oscillator. Another approach is to define the **reduced period**, i.e. the period of the system after symplectic reduction, and the **rotation number**:

$$T_{am}(l, h) := 2\pi \frac{\partial I_{am}}{\partial h}(l, h), \qquad W_{am}(l, h) := -\frac{\partial I_{am}}{\partial l}(l, h),$$

together with their imaginary counterparts

$$T^\alpha_{am}(l, h) := 2\pi \frac{\partial J_{am}}{\partial h}(l, h), \qquad W^\alpha_{am}(l, h) := -\frac{\partial J_{am}}{\partial l}(l, h).$$

These are related to the Taylor series invariant as follows:

**Theorem 3.** *The partial derivatives of the Taylor series invariant $S(l, j)$ are given by*

$$\begin{aligned} \frac{\partial S}{\partial l} &= 2\pi \left( W^\alpha_{am}(l, h) \frac{T_{am}(l, h)}{T^\alpha_{am}(l, h)} - W_{am}(l, h) \right)\Bigg|_{h = B(l, j)} + \arg(w), \\ \frac{\partial S}{\partial j} &= 2\pi \frac{T_{am}(l, h)}{T^\alpha_{am}(l, h)}\Bigg|_{h = B(l, j)} + \ln|w|. \qquad (9) \end{aligned}$$

This allows us to compute the Taylor series invariant:

**Theorem 4.** *The Taylor series symplectic invariant of the coupled angular momenta is given by*

$$\begin{aligned} S(l, j) = {} & \arctan\left( \frac{t - R(1 + t) - R^2(1 - 2t)}{(1 - R) r_A} \right) l + \ln\left( \frac{4R_1 {r_A}^3}{R^{3/2}(1 - t)t^2} \right) j \\ & + \frac{l^2}{16 R_2 {r_A}^3} \Big( R(3 - 5t)t^2 - t^3 - R^4(-1 + 2t)^3 - 3R^2 t(1 - 7t + 4t^2) \\ & \qquad + R^3(1 - 17t + 46t^2 - 32t^3) \Big) \\ & + \frac{lj}{8 R_2 {r_A}^2}(-1 + R) \Big( R^2(1 - 2t)^2 + t^2 + 2Rt(-1 + 6t) \Big) \\ & + \frac{j^2}{16 R_2 {r_A}^3} \Big( t^3 + R^4(-1 + 2t)^3 + Rt^2(-3 + 13t) + R^2 t(3 + 3t - 28t^2) \\ & \qquad - R^3(1 + 15t - 42t^2 + 16t^3) \Big) + \ldots \end{aligned}$$

In particular, in the limit of the region where the singularity $m$ is of focus-focus type, $t \to t^{\pm}$, the coefficient $r_A \to 0$. As a consequence, the linear terms become ill-defined and the higher order terms diverge.