ABELIANISATION OF $\mathsf{SL}_2(\mathbb{C})$-CONNECTIONS AND DARBOUX COORDINATES ON THEIR MODULI SPACES

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We describe a new approach to studying flat connections on holomorphic vector bundles over Riemann surfaces. Such connections are put in correspondence with much simpler objects: flat connections on line bundles. This approach is called *abelianisation*, because the nonabelian monodromy data of a connection on a vector bundle is put in equivalence with abelian monodromy data of a connection on a line bundle (a.k.a., abelian connection). An elementary analogy is the approach in linear algebra to describe a linear transformation equivalently in terms of its eigenvalues and eigenspaces.

Take a holomorphic $\mathsf{SL}_2(\mathbb{C})$-vector bundle with a flat connection $(\mathcal{E}, \nabla)$ with simple singularities over a Riemann surface X. By studying the behaviour of flat sections of ∇ at the singularities, we can extract certain asymptotic data characterising the connection. We then encode this asymptotic data in a holomorphic line bundle with a flat connection $(\mathcal{E}^{\mathrm{ab}}, \nabla^{\mathrm{ab}})$ with simple singularities over another Riemann surface Σ which is a double cover $\pi : \Sigma \to \mathsf{X}$. The procedure $(\mathcal{E}, \nabla) \mapsto (\mathcal{E}^{\mathrm{ab}}, \nabla^{\mathrm{ab}})$ is a functor, called an *abelianisation functor* π^{ab}. Moreover, the abelianisation functor is an equivalence of categories from flat $\mathsf{SL}_2(\mathbb{C})$-connections to connections on line bundles.

One significance of this equivalence is that the corresponding moduli spaces (i.e., sets of equivalence classes) of flat connections are holomorphic symplectic manifolds, and the abelianisation functor π^{ab} descends to a symplectomorphism. Furthermore, the moduli space of flat connections on line bundles is a very simple space: it is just an algebraic torus, and so the abelianisation functor can be interpreted as a Darboux coordinate chart on the moduli space of $\mathsf{SL}_2(\mathbb{C})$-connections. Such coordinates were first considered by Fock-Goncharov in 2006 and later in 2013 by Gaiotto-Moore-Neitzke.