Lagrangian submanifolds in the nearly Kaehler $\mathbb{S}^3 \times \mathbb{S}^3$ from minimal surfaces in $\mathbb{S}^3$

Moruz Marilena
KU Leuven, Belgium

joint work with Burcu Bektas, Joeri Van der Veken and Luc Vrancken

Nearly Kähler manifolds are almost Hermitian manifolds with almost complex structure J satisfying that $\tilde{\nabla} J$ is skew-symmetric. Butruille [?] proved that the only homogeneous 6-dimensional nearly Kähler manifolds are $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, the complex projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. A natural and interesting question for the above four nearly Kähler manifolds is to investigate their almost complex submanifolds and their Lagrangian submanifolds. As nearly Kähler manifolds are an important class of Hermitian manifolds, we can consider Lagrangian submanifolds more generally in almost Hermitian manifolds. Therefore, we say that such a submanifold is Lagrangian if the almost complex structure J interchanges the tangent and the normal spaces and if the dimension of the submanifold is half the dimension of the ambiental manifold.
In the present talk, we refer to the study of minimal Lagrangian submanifolds M in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ described by $g \mapsto f(g) = (p(g), q(g))$ and angle functions (see [?]) $\theta_1, \theta_2, \theta_3$. We show that if all angle functions are constant, then the submanifold is either totally geodesic or has constant sectional curvature and there is a classification theorem that follows from [?]. Moreover, we show that if precisely one angle function is constant, then it must be equal to $0, \frac{\pi}{3}, \frac{2\pi}{3}$. Using then two remarkable constructions, we prove that it is sufficient to study the case when $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \Lambda + \frac{\pi}{3}$ and $\theta_3 = -\Lambda + \frac{\pi}{3}$, where Λ is a non constant function. It results into a special case for which p is not an immersion. We show that M is a frame bundle over a minimal surface N and is determined by a differential equation. Moreover, we study as well the reverse problem, in the cases when the surface is totally geodesic or not.

References

[1] J.-B. Butruille, Homogeneous nealy Kähler manifolds, in: Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc., Zürich, 2010. pp.399-423.

[2] B. Dioos, L. Vrancken and X.Wang, Lagrangian submanifolds in the Nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.