Peter-Weyl's theorem as a bridge between representation theory and harmonical analysis

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Abstract

Classical Fourier analysis and representation theory are at first glance two separate areas in mathematics which don't have much in common. The connection becomes clear when one recalls what the Hilbert space $L_2([-\pi,\pi])$ is. Instead of working with a 2π periodic interval, one can look at functions on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The fact that the irreducible representations of the torus are pairwise orthogonal can be translated in terms of Fourier analysis to the property that de sine and cosine are mutually orthogonal, i.e. $\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\sin(nx)dx = 0$. Secondly, the separability of L^2 can be reformulated to the property that all linear combinations of the irreducible representations are dense in $L^2(T)$. The connection between the classical Fourier theory and representation theory can be encapsulated in the following formula:

$$L^2(\mathbb{T}) \cong \hat{\bigoplus}_{(\pi,V)} V^* \otimes V \tag{1}$$

Whereas for the torus T the underlying structure of the Lebesgue space is given by the know Lebesgue-measure, one needs to use the so called *Haar measure* in the general context. The use of this powerful tool will come with the fact that we'll need to impose some restrictions to the underlying group structure. Having this machinery at our disposal, the goal of the talk will be to generalize formula (1) for arbitrary groups. This result is known as the Peter-Weyl theorem and is one of the basic tools of Harmonic Analysis, which can be viewed as a bridge between analysis, representation theory and Lie algebras.