A Short Research Overview

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Helmholtz equation and preconditioning

\[-[\nabla^2 + k(x)^2] u(x) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^d.\]  

[H. von Helmholtz, 19th century]

- Applications: scattering models, inverse problems (forward problem)
- Numerical discretization using FDM, FVM, FEM, BEM, . . .
- Iterative solution: MG-preconditioned Krylov methods (CG, GMRES)
- Complex shifted Laplacian (CSL) preconditioner [Erlangga et al. 2004]

<table>
<thead>
<tr>
<th>$k_0$</th>
<th>$n_x \times n_y$</th>
<th>128$^2$</th>
<th>256$^2$</th>
<th>512$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MG-FGMRES</td>
<td>51 (5.40)</td>
<td>92 (46.8)</td>
<td>174 (813)</td>
<td></td>
</tr>
<tr>
<td>MG-FGMRES(10)</td>
<td>66 (5.52)</td>
<td>140 (46.7)</td>
<td>250 (421)</td>
<td></td>
</tr>
<tr>
<td>LVL-MG</td>
<td>44 (3.42)</td>
<td>83 (25.3)</td>
<td>149 (215)</td>
<td></td>
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2D ionization problem - iterations (CPU time in s.) [Cools et al. 2013 (NLAA)]

Homogeneous (radar/sonar)  Wedge problem (seismics)  Coulomb problem (quantum break-up)
Far field map for Helmholtz (HH) & Schrödinger (SCH)

\[ F(\alpha) = \int_{\Omega} G(x, \alpha) \left( k^2(x) - k_0^2 \right) \left( u_{sc}(x) + u_{in}(x) \right) \, dx. \] (2)

- Oscillatory integral [Huybrechts et al. 2006]
- Computational bottleneck: solve HH or SCH system for \( u_{sc}(x) \)
- Reformulation to complex integral
- Advantage: fast iterative computation of HH or SCH system (CSL)

3D object of interest Far field map

3D Helmholtz scattering problem (256\(^3\)-pt. FDM)
- Previously: hard to compute supercomputing infrastructure required
- Now: efficient iterative computation solvable in ± 8 min. on laptop

Schematic representation: 1D complex contour

[Colton et al. 1998]

[Cools et al. 2014 (SISC)]
Wavelet-MG preconditioner for ATR

ATR = algebraic tomographic reconstruction - regularized linear system

\[ \min_x \{ \| Wx - b \|_2 + \lambda \| Lx \|_2 \}. \] (3)

- Classic reconstruction algorithms (ART, SIRT, etc.) converge poorly
  - Slow resolvance of low-frequency modes! (sharp edges)

- Krylov methods require suitable preconditioner: WMG-preconditioner
  1. wavelet-projection on low-, semi- & high-oscillatory coarse subspaces → resolve all modes
  2. multigrid-type error correction scheme → coarse grid representations correct initial guess

<table>
<thead>
<tr>
<th>method</th>
<th>iter</th>
<th>CPU</th>
<th>error $L_2$</th>
</tr>
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<tbody>
<tr>
<td>SIRT</td>
<td>1000</td>
<td>81.1 s.</td>
<td>0.1385</td>
</tr>
<tr>
<td>BiCGStab</td>
<td>100</td>
<td>9.35 s.</td>
<td>0.1074</td>
</tr>
<tr>
<td>WMG-BiCGStab</td>
<td>14</td>
<td>5.44 s.</td>
<td>0.1083</td>
</tr>
</tbody>
</table>

[Cools et al. 2014 (under review)]