The Helmholtz Equation

The focus of the research lies on solving the Helmholtz equation (indefinite) on a domain Ω with given boundary conditions on ∂Ω

\[-Δu + αu = f \quad \text{on} \; Ω \subseteq \mathbb{R}^d, \quad u = g \quad \text{on} \; ∂Ω.\]

Model problem is the 2D Helmholtz equation on \((0,1)^2\) with homogeneous Dirichlet boundary conditions

\[-Δu(x, y) + αu(x, y) = f(x, y), \quad u(x, y) = 0 \quad (x, y) \in ∂Ω.\]

A variety of industrial applications include Electromagnetic Scattering (radar), Seismic Imaging (petrol exploration), Medical Imaging (PCT), and many more.

The equation is typically discretised using second-order central differences, yielding a matrix-vector equation of the form

\[A\nu = f\]

where \(\nu\) contains the solution in each of the \(n^d\) interior grid points, and \(A\) is an \(n^d \times n^d\)-discretisation matrix, which in 2D is defined by the stencil

\[A(α) = \frac{1}{h^2} \begin{pmatrix} -1 & -1 \\ -1 & 4 + 6h^2 & -1 \end{pmatrix} \]

Wave numbers \(α < 0\) with \(|α| > 1/h^2\) undermine the diagonal dominance of \(A\), rendering the matrix indefinite.

The Multigrid Method

It is our aim to solve this system of equations numerically using the advanced iterative method known as Multigrid. A Multigrid cycle is the nested version of the so-called Two-Grid correction scheme

\[\nu^{(k)} : \left[1 - \frac{1}{h^2} (A^2)\right]^{-1} \frac{1}{h^2} A^2 R \nu^{(k)} := \text{TGR} \; \nu^{(k)}\]

A basic iterative solver it called the smoother eliminates the high oscillatory error components in each grid step. Part of the efficiency of the Multigrid solver lies in its low computational cost: whereas direct methods use \(O(N^3)\) flops to compute a solution, Multigrid typically requires only \(O(N)\).

Although Multigrid provides excellent results on the definite Helmholtz problem, a direct application of Multigrid to the indefinite variant is inadvisable.

An explanation for this failure can be found by studying the eigenvalues of TGR, which can be written approximately as

\[λ_k(\text{TGR}) ≈ λ_k^2(R) \left(1 - \frac{λ_k^2(A)}{λ_k^2(A^2)}\right), \quad 1 ≤ k < \frac{N}{2}\]

In situations where \(λ_k(A^2)\) is both relatively small \((= 0)\) and/or reversely signed w.r.t. \(λ_k(A^2)\), one observes that \(ρ(\text{TGR}) > 1\), implying convergence to the solution is not guaranteed.

Solutions & Research

A Multigrid preconditioned Krylov solver is applied to the Helmholtz problem, intrinsically solving the system

\[M^{-1}A\nu = M^{-1}f\]

where \(M\) is designated to be a Complex Shifted Laplacian preconditioner

\[M(α, β) = -Δ + α(1 + β)i\]

This implies an inner loop of Multigrid iterations, constructing the Krylov subspace base vectors, is nested within the outer loop of Krylov iterations.

A contrariwise rises from the fact that the Multigrid method requires \(β\) to be sufficiently large, while the enveloping Krylov solver benefits from \(β → 0\).

A joint Local Fourier Analysis of the smoother and Two-Grid operator provides

\[U^{m+1} = G(θ_1, θ_2, α, β) U^m, \quad m ≥ 1, \quad θ_1, θ_2 ∈ (-π, π)\]

with amplification factor \(G(θ_1, θ_2, α, β)\) describing the evolution of the error’s amplitude \(U^{m+1}\) through consecutive iterations. Separate analysis shows

\[G_{θ_1}(θ_1, θ_2, α, β) = 1 - α + \frac{2αω}{4 + αh^2} \left(\cos θ_1 + \cos θ_2\right),\]

\[G_{θ_2}(θ_1, θ_2, α, β) = 1 - α + \frac{1}{2} \left(\cos θ_1 + 1\right)^2 \left(\cos θ_2 + 1\right)^2 \left(2\cos θ_1 - 2\cos θ_2 + 4 + αh^2\right) \left(\sin θ_1 + \sin θ_2 + αh^2\right)^2\]

which can be combined into

\[G_{θ_1θ_2}(θ_1, θ_2, α, β) \quad \text{for} \; θ_1, θ_2 ∈ (-π, π).\]

The Minimal Complex Shift Parameter

\[β_{min} := \arg \min_{β > 0} \max \left\{G_{θ_1θ_2}(θ_1, θ_2, α, β) : \; G_{θ_1θ_2}(θ_1, θ_2, α, β) ≤ 1\right\}\]

can be interpreted as both the smallest possible shift for Multigrid to converge and is, under this condition, optimal in view of Krylov convergence. It is intrinsically a function of \(α\).

Numerical verification

Theoretical Local Fourier Analysis

Experimental TGR-BICGStab

References

