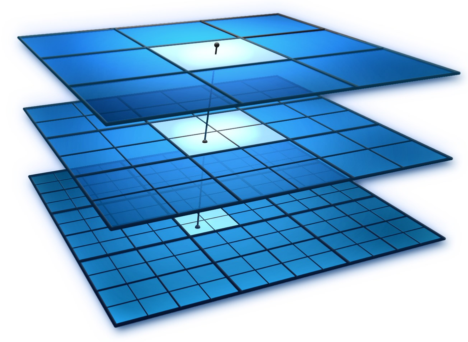
Complex Shifted Multigrid on the indefinite Helmholtz equation



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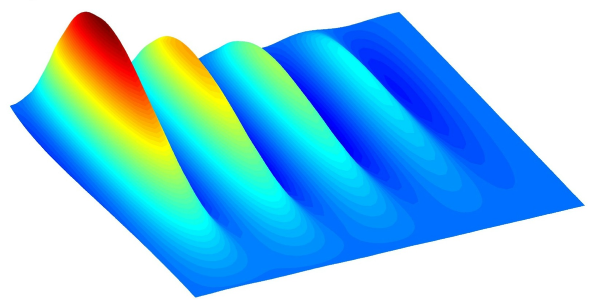
The Helmholtz Equation

The focus of the research lies on solving the **Helmholtz equation** **(indefinite)** on a domain Ω with given boundary conditions on $\partial\Omega$

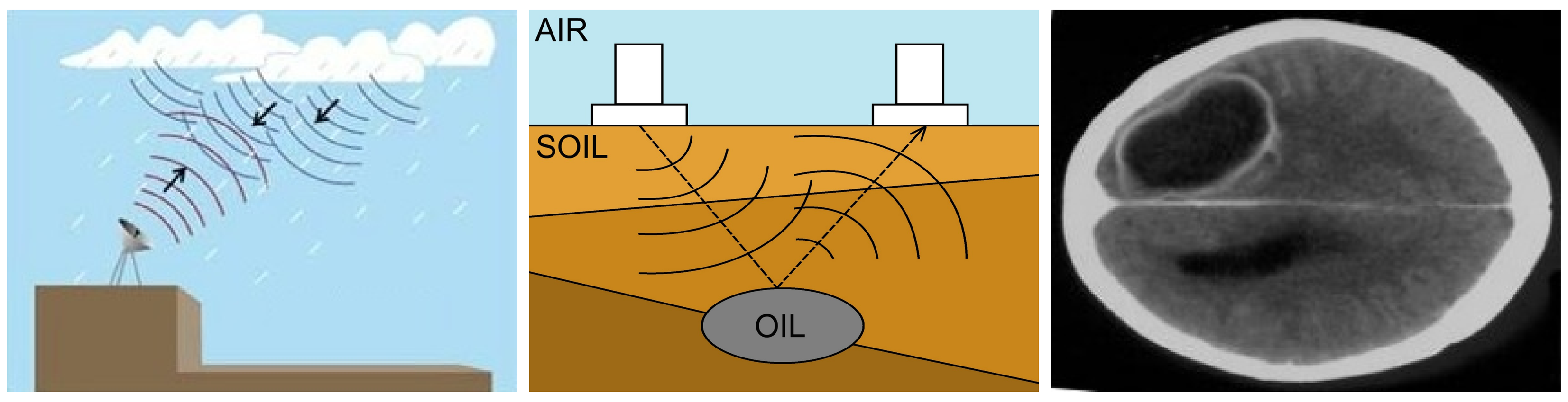
$$\begin{aligned} -\Delta u + \sigma u &= f \qquad \text{on } \Omega \subset \mathbb{R}^d, \qquad (\sigma < 0) \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Model problem is the 2D Helmholtz equation on $[0,1]^2$ with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta u(x,y) + \sigma u(x,y) &= f(x,y), \\ u(x,y) &= 0 \qquad (x,y) \in \partial\Omega. \end{aligned}$$



A variety of industrial **applications** include Electromagnetic Scattering (radar), Seismic Imaging (petrol exploration), Medical Imaging (PCT), and many more.



The equation is typically discretised using second-order central differences, yielding a matrix-vector equation of the form

$$A\mathbf{v} = \mathbf{f}$$

where $\mathbf{v}$ contains the solution in each of the n^d interior grid points, and A is a $n^d \times n^d$-discretisation matrix, which in 2D is defined by the stencil

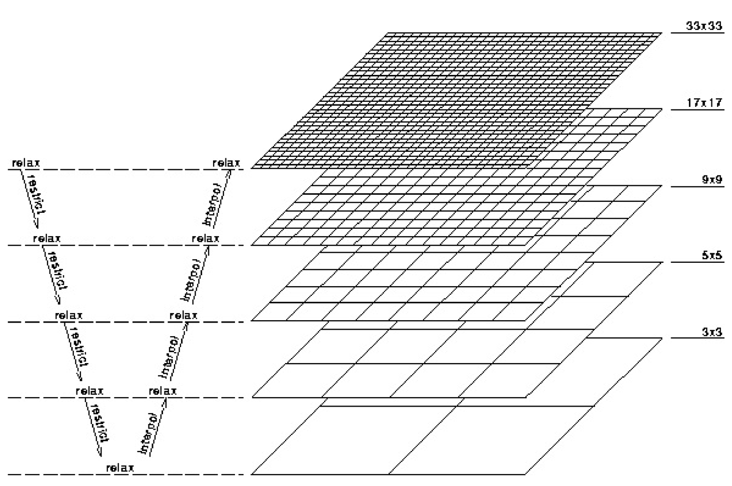
$$A(\sigma) = \frac{1}{h^2}\begin{pmatrix} & -1 & \\ -1 & 4+\sigma h^2 & -1 \\ & -1 & \end{pmatrix}$$

Wave numbers $\sigma < 0$ with $|\sigma| > 1/h^2$ undermine the diagonal dominance of A, rendering the matrix indefinite.

The Multigrid Method

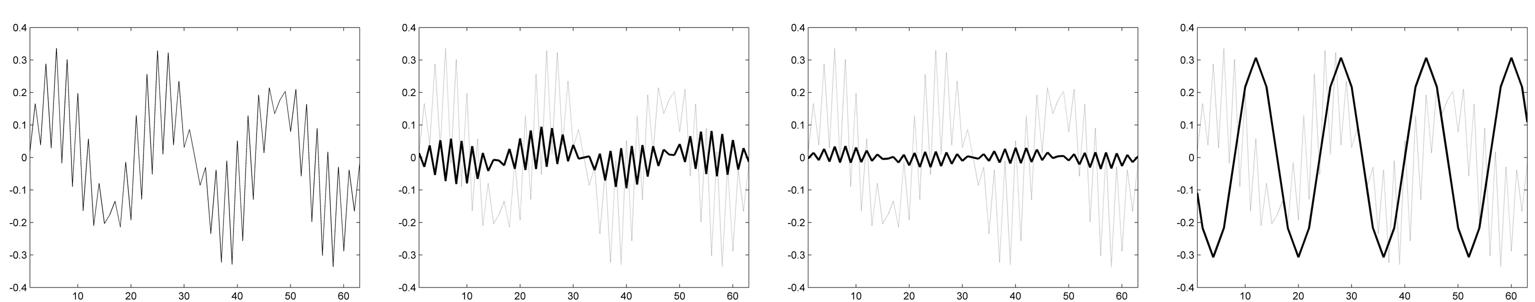
It is our aim to solve this system of equations numerically using the advanced iterative method known as Multigrid. A Multigrid cycle is the nested version of the so-called **Two-Grid correction scheme**

$$\mathbf{e}^h \leftarrow [I - I_{2h}^h (A^{2h})^{-1} I_h^{2h} A^h] R^\nu \mathbf{e}^h := TGR\, \mathbf{e}^h.$$



A basic iterative solver R called the **smoother** eliminates the high oscillatory error components in each intergrid step. Part of the efficiency of the Multigrid solver lies in its **low computational cost**: whereas direct methods use $\mathcal{O}(N^p)$ flops to compute a solution, Multigrid typically requires only $\mathcal{O}(N)$.

Although Multigrid provides excellent results on the definite Helmholtz problem, a direct application of Multigrid to the indefinite variant is inadvisable.



An explanation for this failure can be found by studying the eigenvalues of TGR, which can be written approximately as

$$\lambda_k(TGR) \approx \lambda_k(R)^\nu \left[1 - \frac{\lambda_k(A^h)}{\lambda_k(A^{2h})}\right], \qquad 1 \leqslant k < \frac{N}{2}.$$

In situations where $\lambda_k(A^{2h})$ is both relatively small (≈ 0) and/or reversely signed w.r.t. $\lambda_k(A^h)$, one observes that $\rho(TGR) > 1$, implying convergence to the solution is not guaranteed.

Solutions & Research

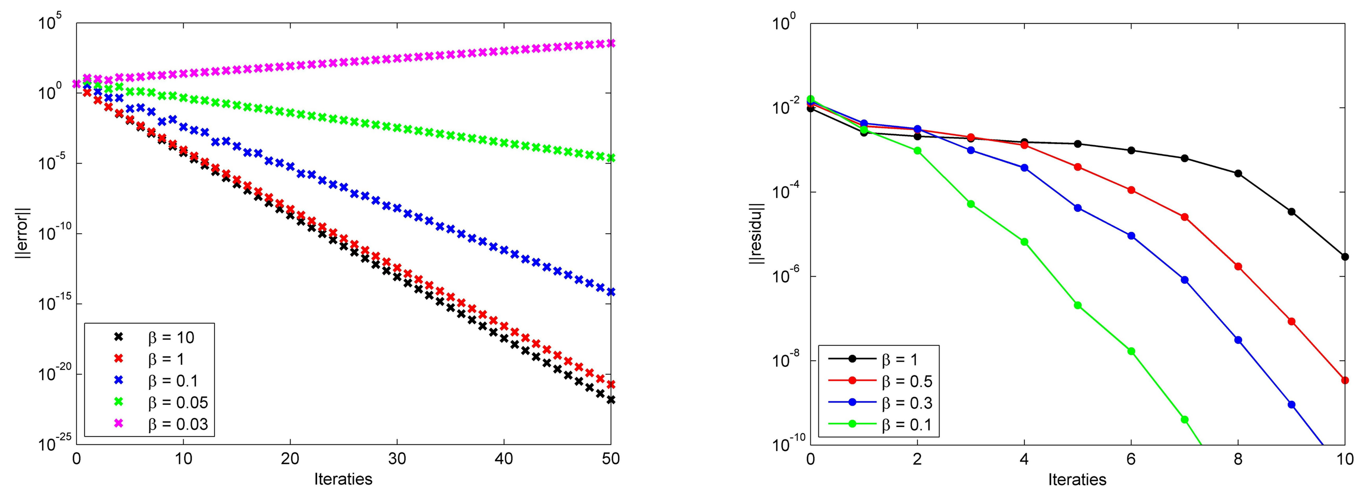
A Multigrid preconditioned Krylov solver is applied to the Helmholtz problem, intrinsically solving the system

$$M^{-1}A\mathbf{v} = M^{-1}\mathbf{f}$$

where M is designated to be a **Complex Shifted Laplacian** preconditioner

$$M(\sigma,\beta) = -\Delta + \sigma(1+\beta i).$$

This implies an inner loop of Multigrid iterations, constructing the Krylov subspace base vectors, is nested within the outer loop of Krylov iterations.



A contrariety rises from the fact that the Multigrid method requires β to be sufficiently large, while the envelopping Krylov solver benefits from $\beta \to 0$.

A joint **Local Fourier Analysis** of the smoother and Two-Grid operator provides

$$U^{(m+1)} = G(\theta_1,\theta_2,\sigma,\beta)\, U^{(m)}, \qquad m \geqslant 1, \quad \theta_i \in (-\pi,\pi],$$

with amplification factor $G(\theta_1,\theta_2,\sigma,\beta)$ describing the evolution of the error's amplitude $U^{(\cdot)}$ through consecutive iterations. Separate analysis shows

$$G_R(\theta_1,\theta_2,\sigma,\beta) = 1 - \omega + \frac{2\omega}{4+\tilde{\sigma}h^2}(\cos\theta_1 + \cos\theta_2),$$

$$G_{TG}(\theta_1,\theta_2,\sigma,\beta) = 1 - \frac{1}{16}(\cos\theta_1+1)^2(\cos\theta_2+1)^2\left(\frac{-2\cos\theta_1 - 2\cos\theta_2 + 4 + \tilde{\sigma}h^2}{\sin^2\theta_1 + \sin^2\theta_2 + \tilde{\sigma}h^2}\right)$$

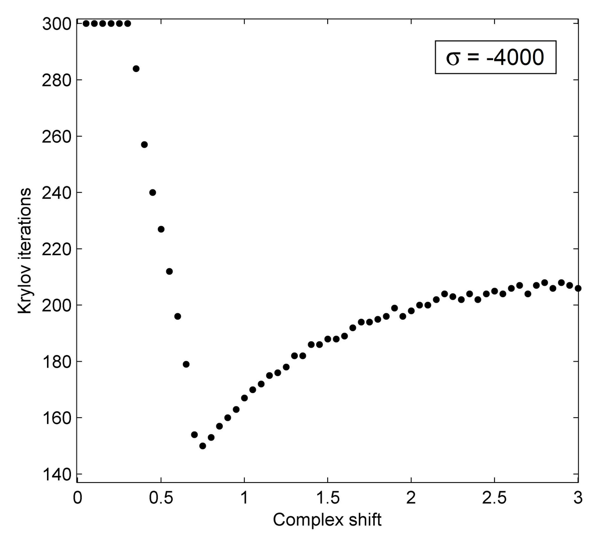
which can be combined into

$$G_{TGR}(\theta_1,\theta_2,\sigma,\beta) \qquad \text{for } \theta_i \in (-\pi,\pi].$$

The **Minimal Complex Shift Parameter**

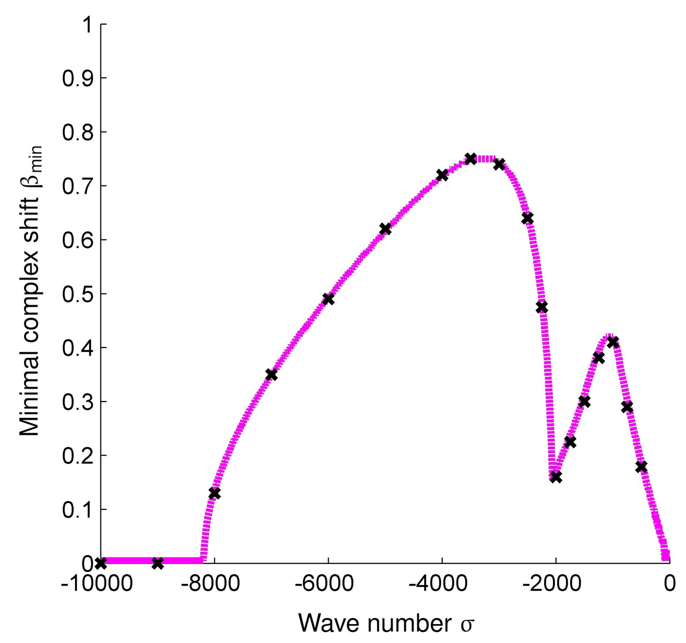
$$\beta_{min} := \underset{\beta \geqslant 0}{\operatorname{argmin}} \left\{ \max_{-\pi < \theta_1, \theta_2 \leqslant \pi} |G_{TGR}(\theta_1,\theta_2,\sigma,\beta)| \leqslant 1 \right\}$$

can be interpreted as both the smallest possible shift for Multigrid to converge and is, under this condition, optimal in view of Krylov convergence. It is intrinsically a function of σ.

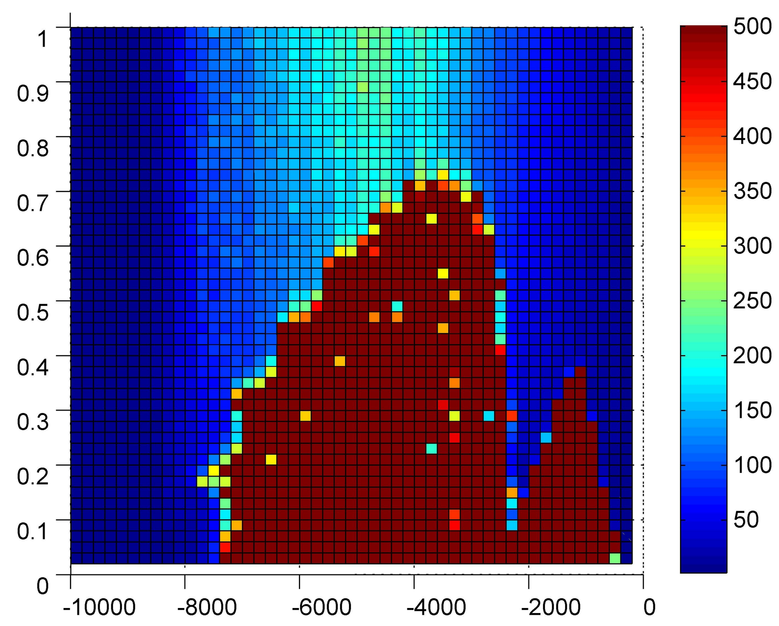


Numerical verification

Theoretical
Local Fourier Analysis



Experimental
TGR-BiCGStab



References

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