ALGEBRA AND ARITHMETIC (ALGAR) 2019 – ALGEBRAS WITH INVOLUTION

Contents

1. Main lectures 1
1.1. Quadratic forms (K.J. Becher) 1
1.2. Involutions on central simple algebras (J.-P. Tignol) 2
1.3. Hermitian forms and adjoint involutions (T. Unger) 3
1.4. Linear algebraic groups (A. Quéguiner-Mathieu) 4
1.5. Galois cohomology (R. Parimala) 4
1.6. Invariants of involutions (J.-P. Tignol) 4
1.7. Springer's theorem and function fields of conics (R. Parimala) 5
1.8. Classification of linear algebraic groups (A. Quéguiner-Mathieu) 5
1.9. Totally decomposable algebras with involution (K.J. Becher) 5
1.10. The real theory of forms and involutions (T. Unger) 6
1.11. Exceptional isomorphisms and triality (A. Quéguiner-Mathieu) 6
1.12. Serre's Conjecture II (R. Parimala) 6
1.13. Valuation theoretic methods (J.-P. Tignol) 7
2. Special lectures 7
2.1. Totally decomposable quadratic pairs (A. Dolphin) 7
2.2. Mixed Witt ring of an algebra with involution (N. Garrel) 8
References 8

1. Main lectures

1.1. **Quadratic forms (K.J. Becher).** In this lecture we recall the basics of the theory of quadratic forms over general fields, restricting mostly to fields of characteristic different from 2.

We start by relating the two crucial view points on quadratic forms, namely as polynomials and as symmetric bilinear forms, which in their combination make them accessible to such a exceptional variety of tools and techniques. We recall Witt's classification theorems, the unique decomposition of a regular quadratic form into an anisotropic and a hyperbolic part and the related construction of the Witt ring. From there we move on to Pfister forms and their particular properties, including Witt's proof that they are 'round'. In this context we mention Hurwitz'

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theorem on the existence of compositions of quadratic forms in certain dimension, which will be relevant in later lectures in the context of decomposability of involutions.

We further address the behaviour of quadratic forms under field extensions. In particular, we see the statement, independently proven by E. Artin and by T.A. Springer, that anisotropy is conserved under finite field extensions of odd degree, which is related to a challenging open problem for involutions.

We finally introduce function fields of quadratic forms. This tool, stemming from the classical algebraic geometry view point on a quadratic form as a quadric, is extremely useful in proving general structure theorems on quadratic forms. As an illustration, we look at function fields of Pfister forms and sketch the proof of the famous Arason-Pfister Hauptsatz, stating that the intersection of all powers of the fundamental ideal in the Witt ring is zero. This result together with the Milnor-Conjecture, proven by V. Voevodsky, yields that quadratic forms can in theory be classified by a filtrated sequence of natural invariants taking values in Galois cohomology groups over the base field.

Fortunately, quadratic form theory is meanwhile covered by numerous accessible textbooks, in particular [28], [18], [13]. The material of this lecture is for example covered by the first chapters of [28].

1.2. **Involutions on central simple algebras (J.-P. Tignol).** This lecture introduces the involutions of ring theory, which are ring anti-automorphisms of period 2. We first explore the various involutions on matrix algebras and discover that they come in two *kinds*, depending on their restriction to the center. We first approach the *first kind*, which consists of involutions that fix the center. Using the fact that linear automorphisms of matrix algebras are inner, we succeed in setting up a correspondence between these involutions and invertible matrices, so that each involution is adjoint to an invertible matrix that is symmetric or skew-symmetric and uniquely determined up to a central factor. Assuming that the characteristic is different from 2, we thus obtain a distinction among involutions of the first kind, discriminating those that are adjoint to symmetric matrices from those that correspond to skew-symmetric matrices. The former are called *orthogonal involutions*, or involutions of *orthogonal type*, while the latter are known as *symplectic involutions*, or involutions of *symplectic type*; they live only on matrices of even degree.

We next take a more intrinsic viewpoint on our observations by substituting endomorphism algebras of finite-dimensional vector spaces for matrix algebras. From this perspective we revisit the relation between involutions and matrices and give it a new interpretation as a correspondence between involutions on endomorphism algebras of a vector space and symmetric or skew-symmetric nondegenerate bilinear forms up to scalars on the space. In this correspondence, orthogonal involutions correspond to symmetric forms while symplectic involutions correspond to skew-symmetric forms. We also learn how to tell one type from the other by looking at the dimension of the space of elements that are fixed under the involution.

A dramatic new turn in our quest comes when we realize that matrix algebras are an instance of the more general class of central simple algebras, but that central simple algebras are all very close relatives of matrix algebras, insofar as each of them becomes a matrix algebra when scalars are extended to an algebraic closure of the base field. This observation leads us to view central simple algebras as twisted forms of matrix algebras, and to extend to central simple algebras the theory of involutions developed so far for matrix algebras. On a central simple algebra that is not *split*, i.e., which is not a matrix algebra over a field or—equivalently—an endomorphism algebra of a vector space, the correspondence between involutions and bilinear forms is lost, so that involutions may be thought of as adjoint to ghost bilinear forms, which hatch only when the field of scalars is sufficiently enlarged for the algebra to be split.

We inspect several specimen of central simple algebras, with a special emphasis on the celebrated *quaternion algebras* of Hamilton's fame and their tensor products, which provide an ample supply of involutions. We then embark on the project to distinguish among central simple algebras those that carry an involution of the first kind. In the course of our exploration, we discover the wondrous virtues of the unfairly obscure *Goldman element*, which holds the key to a simple proof of Albert's criterion: a central simple algebra carries an involution of the first kind if and only if its tensor product with itself is split. It is also revealed that central simple algebras of odd degree are split when they carry an involution of the first kind.

The lecture concludes with a discussion along similar lines of involutions of the second kind, also known as unitary involutions, which are those that restrict to a nontrivial automorphism of the center. On split algebras, those involutions are adjoint to hermitian forms. We define the *norm* (or *corestriction*) of a central simple algebra over a quadratic field extension and characterize the central simple algebras that carry an involution of the second kind as those whose norm is split.

1.3. **Hermitian forms and adjoint involutions (T. Unger).** Starting with central simple algebras, more general rings may naturally occur, for example after scalar extension of the centre, or in the context of *gauges* (discussed in another lecture). The goal of this lecture, therefore, is to introduce the theory of *ε-hermitian forms* over general rings with involution. Most of the time we will not assume that 2 is invertible. Instead of a Witt ring, as in quadratic form theory, there is a Witt group, as there is in general no suitable product of forms. The elements of this group are isometry classes of nonsingular forms modulo hyperbolic forms. Metabolic forms will be introduced. Their classes are also zero in the Witt group. If 2 is invertible, metabolic and hyperbolic forms are the same. To a form over a ring with involution we can associate an adjoint involution on a suitable endomorphism ring, whose Witt group is related to the Witt group of the original ring with involution via the categorical mechanism of *Morita equivalence*. In the final part of this lecture we will see how the theory applies to simple artinian rings and division rings with involution. We will also briefly discuss *isotropic*, *hyperbolic* and *metabolic involutions* on central simple algebras, cf. [6] and [9].

The main reference for this lecture is the exhaustive monograph of Knus [16].

1.4. **Linear algebraic groups (A. Quéguiner-Mathieu).** Starting from one of the algebraic structures introduced in the previous talks (quadratic forms, central simple algebras with involution, hermitian forms), one may define several groups, such as invertible elements in the algebra, and the groups of isometries and similitudes. Those groups actually are the groups of points of some *algebraic groups*. Based on a detailed study of some examples, we will introduce *algebraic groups*, viewed as representable functors. The main reference for this point of view is Waterhouse [30]. We also refer to The Book of Involution [17] for a brief and comprehensive overview, and to Milne's reference book [21].

1.5. **Galois cohomology (R. Parimala).** In this lecture we define the notion of Galois cohomology both in the abelian and nonabelian setting. We explain how classical invariants of quadratic forms can be interpreted in terms of Galois cohomology. We also explain the Milnor Conjecture (from [22]) on higher cohomological invariants for quadratic forms. The proof of this conjecture due to Orlov-Vishik–Voevodsky [23] leads to a classification of quadratic forms up to Witt equivalence by the Galois cohomology invariants. We also explain how twisted forms of algebraic structures like quadratic forms and central simple algebras, up to isomorphism, can be described by certain nonabelian cohomology sets. We interpret invariants of hermitian forms over central simple algebras with involution in terms of Galois cohomology.

1.6. **Invariants of involutions (J.-P. Tignol).** Using pfaffians of skew-symmetric matrices, we discover that the discriminant of a quadratic form of even dimension can be determined directly from its adjoint involution. This observation leads to the definition of a *discriminant* for orthogonal involutions on central simple algebras of even degree. Tempting the same approach for the *Clifford algebra* of quadratic forms, we hit a serious obstacle because the Clifford algebra of a quadratic form of even dimension changes when the form is multiplied by a scalar, whereas orthogonal involutions correspond to quadratic forms up to scalars. However, the *even* Clifford algebra does not change, and we succeed in defining an analogue of the *even Clifford algebra* for orthogonal involutions on central simple algebras of even degree, following ideas of Jacobson and Tits. This approach also leads to the definition of a *discriminant* for unitary involutions; in this case the discriminant turns out to be a central simple algebra. Time permitting, we will also make some remarks on symplectic involutions. For these there is no "classical" invariant, but in certain cases an invariant can be defined in a *Galois cohomology group* of degree 3.

1.7. **Springer's theorem and function fields of conics (R. Parimala).** Starting with the classical theorem of Springer stating that a quadratic form over a field isotropic over an odd degree extension isotropic over the base field, several questions have been raised in literature concerning behaviour of isotropy for hermitian forms over algebras with involution. A weak Springer analogue for hermitian forms due to Bayer–Lenstra states that hermitian forms over algebras with involution which are hyperbolic over an odd degree base change is hyperbolic.

There are strong connections between the following two conjectures. *Hermitian Springer Conjecture*: Hermitian forms over central simple algebras with involutions of orthogonal type are isotropic if they are isotropic in an odd degree base change. *Generic Isotropicy Conjecture*: Hermitian forms over central simple algebras with orthogonal involution which are isotropic over the function field of the Severi-Brauer variety of the algebra is already isotropic.

Recently Karpenko showed in [15] that these two conjectures are equivalent. Nevertheless, the Hermitian Springer Conjecture is wide open except in the case of hermitian forms over quaternion algebras with orthogonal involutions, where it was proven in [24]. We explain a result on excellence for hermitian forms over divison algebras with involution with respect to conic function field extensions and deduce the hermitian Springer conjecture in the case of hermitian forms over quaternion algebras with orthogonal involutions.

See also [10] for an overview of the results and problems discussed here.

1.8. **Classification of linear algebraic groups (A. Quéguiner-Mathieu).** Semi-simple algebraic groups can be classified in terms of a combinatorial data, called a root data. They split into two families : exceptional and classical groups. In the 60's, André Weil proved that all classical adjoint groups can be described in terms of some algebra with involution, providing some additional motivation for studying these algebraic structures. In this talk, we will introduce *root systems* and *Dynkin diagrams*, and sketch the proof of Weil's theorem. We will also explain the dictionnary between *types of involutions* and *types of groups*. The main reference for root systems is [11]; a description of root systems and their diagrams is provided in [17, §24]; see also [21, Appendix C] for a comprehensive survey.

1.9. **Totally decomposable algebras with involution (K.J. Becher).** Pfister forms are the quadratic forms which are obtained as tensor products of binary quadratic forms. Tensor products of quaternion algebras with involution are an analogue in the realm of algebras with involution. Such algebras with involution are called *totally decomposable*. We will look at some examples of decomposable involutions in small degrees and characterise when an involution of the first kind on a biquaternion algebra is decomposable.

The link between the two concepts was pointed out in [5], and it was thoroughly studied there for orthogonal involutions in degree 8, where it is related to the phenomenon of triality. A crucial property of Pfister forms is that they are anisotropic or hyperbolic over any field extension. The same holds for totally decomposable involutions. In this lecture we will see the main steps of the proof of this statement in the case of orthogonal involutions in characteristic different from 2, which was given in [7].

A main ingredient is the analysis of the situation when an involution becomes hyperbolic over a generic splitting field of the underlying algebra. For orthogonal involutions, Karpenko proved in [14] that this cannot happen unless the involution is already hyperbolic. For algebras of index two this result is more elementary and was obtained by I. Dejaiffe [12] and by Parimala-Shridharan-Suresh [24]. This special case is crucial in the proof that totally decomposable algebras with orthogonal involution in the split case are adjoint to Pfister forms.

This statement on totally decomposable algebras with involution is related to a (meanwhile proven) conjecture by D. Shapiro, which characterises Pfister forms by the existence of a composition formula with another quadratic form of a particular dimension. See [27].

1.10. **The real theory of forms and involutions (T. Unger).** In this lecture we will give an introduction to ordered and (formally) real fields, and to quadratic forms over such fields we associate a natural invariant: the *signature* (see [18], for example). We will discuss how signatures can be defined for involutions on central simple algebras, via trace forms [19, 25], and for hermitian forms over algebras with involution, via Morita equivalence, [4], [1], [2]. We will also discuss the characterization of *weakly hyperbolic* forms and involutions via signatures (Pfister's local-global principle) [20, 8], and, time permitting, some other applications.

1.11. **Exceptional isomorphisms and triality (A. Quéguiner-Mathieu).** Algebras with involution of small degree play a specific role in the theory. Indeed, their automorphism groups are algebraic groups of small rank, which have specific properties. Two examples will be studied in this talk. *Exceptional isomorphims* come from the fact that some low degree Dynkin diagrams of different types coincide. *Triality* is related to the specific symmetries of the Dynkin diagram of type D_4. In both cases, we may derive from this that the Clifford algebra of a an algebra with orthogonal involution is a classifying invariant in some low degree cases. Some consequences will be studied, such as decomposability theorems for algebras with involution. We will also prove, using triality, that there exists non isomorphic involutions that become isomorphic after generic splitting of the underlying central simple algebra.

1.12. **Serre's Conjecture II (R. Parimala).** We define the notion of cohomological dimension of a field and state Conjecture II due to Serre ([29]) concerning the triviality of principal homogeneous spaces under semisimple simply connected linear algebraic groups over perfect fields of cohomological dimension at most 2. We explain an approach taken in [3] to proving this conjecture for classical groups via classification of hermitian forms by invariants in Galois cohomology and then dealing with the principal homogeneous spaces arising from the center through certain norm surjectivity results. Of fundamental importance to this approach is the result of Merkurjev–Suslin solving the conjecture for $SL(A)$, where A is a central simple algebra over the given field. Triviality of principal homogeneous spaces in this case is equivalent to reduced norm surjectivity. Conjecture II is open in general for groups of types trialitarian D_4, E_6, E_7 and E_8.

1.13. **Valuation theoretic methods (J.-P. Tignol).** Valuation theory is an important tool in the analysis of quadratic forms, which is often indispensable to prove anisotropy. We see in this lecture how the scope of this tool can be expanded to apply to noncommutative structures, in particular to division algebras and anisotropic hermitian spaces, yielding residue algebras and forms defined over the residue field. The leading motivation is a recent construction in [26] of a central simple algebra of degree 16 with an orthogonal involution that is not totally decomposable, even though it is totally decomposable after scalar extension to a generic splitting field of the underlying algebra. This example defeats the hope to characterize totally decomposable orthogonal involutions of degree 16 by the vanishing of a cohomological invariant of degree 3.

2. Special lectures

2.1. **Totally decomposable quadratic pairs (A. Dolphin).** Given its intimate connection to the theory of quadratic forms, it is not surprising that the theory of algebras with involution is somewhat unusual over fields of characteristic 2. Involutions on split algebras correspond to symmetric bilinear forms, and these are not equivalent to quadratic forms over such fields. Therefore involutions are not the correct object to use if one wishes to extend the theory of quadratic forms to the setting of central simple algebras in order to, for instance, study twisted orthogonal groups in characteristic 2.

Instead, quadratic pairs, introduced in the Book of Involutions, must be used. These pairs correspond to quadratic forms in an analogous manner to the correspondence between involutions and symmetric bilinear forms. That is, a quadratic pair on a split central simple algebra is adjoint to some quadratic form, unique up to similarity.

In this talk, we give an introduction to quadratic pairs. In particular we will see what the correct notion of a totally decomposable quadratic pair is, and see how they correspond to quadratic Pfister forms. We shall also discuss the interaction between totally decomposable quadratic pairs and involutions of both orthogonal and symplectic type.

2.2. **Mixed Witt ring of an algebra with involution (N. Garrel).** The lack of ring structure on the Witt group of hermitian forms over an algebra with involution makes it difficult to adapt many methods from the theory of quadratic forms, where the Witt ring is a central tool. We present a construction of a "mixed" Witt ring, in which the product of two hermitian forms is a quadratic form, involving trace forms in a crucial way. Furthermore, as is the case for quadratic forms, we can define lambda-operations on a related mixed Grothendieck-Witt ring. We will examine how these new algebraic structures allow to define new invariants, and interpret previously known ones in a more natural setting.

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