ALGEBRA AND NUMBER THEORY (ALGAR) 2018 – SUMS OF SQUARES IN FIELDS

Contents

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1. Sums of squares and quadratic forms (Becher)

1.1. **The Pythagoras number and Pfister forms.** In this session, the *Pythagoras number* and the *level* of a commutative ring are introduced and basic properties for fields are discussed. To study these invariants for fields, we introduce and recall basic facts from the theory of quadratic forms over fields, in particular covering so-called Pfister forms.

While composition formulas for sums of $2, 4$ and 8 squares imply that the elements in a commutative ring which are sums of 2^n squares are closed under multiplication for $n \leqslant 3$, the same fails in general for $n > 3$. With the theory of Pfister forms one easily sees that this fact still holds for all positive integers n in fields; there the nonzero sums of 2^n form a group. This yields the fact that the level of an arbitrary field is either infinite or a power of 2. The Pythagoras number is mainly of interest in the case of a real field, that is where the level is infinite.

Later lectures at this summer school will illustrate how difficult it is to determine the value for the Pythagoras number of a field and how diverse can be the methods and arguments involved.

In the first lecture we will further introduce the orthogonal group of a quadratic form and its transitive action on vectors of a given length, which also yields results on sums of squares.

1.2. **Pfister's bound for function fields over the real numbers.** The problem to give upper bounds for the Pythagoras number can be approached in a more systematic way than the problem for lower bounds. This relies again on some meanwhile classical results on Pfister forms concerning factorisation properties and their behaviour under quadratic field extensions.

We will first show that in a function field of a curve over $\mathbb{R}$ every totally positive element is a sum of 2 squares; note that for $\mathbb{R}(X)$ this is an easy exercise. We will then look at the situation of an arbitrary extension F of $\mathbb{R}$ of finite transcendence degree n and give Pfister's proof showing that the Pythagoras number of F is bounded by 2^n. One main ingredient is a bound on the dimensions of anisotropic quadratic forms over $F(\sqrt{-1})$ which can be shown by Tsen-Lang theory, but which we will not explain in detail.

2. Sums of squares in rational function fields (Leep)

2.1. **A survey on Pythagoras numbers of fields and related techniques.** I will talk about the Pythagoras numbers of number fields and rational function fields over a number field. In some of these cases local-global principles lead to upper bounds on the Pythagoras number.

I will further discuss results on Pythagoras numbers of function fields over the real numbers. The Cassels-Pfister theorem will be formulated and applied to obtain some basic results on lower bounds on Pythagoras numbers. I will explain a method due to Cassels-Elison-Pfister to showing that certain polynomials of degree 4 in one variable over a field k (such as the Motzkin polynomial seen as a polynomial in X over $k = \mathbb{R}(Y)$) is equivalent to showing that a certain elliptic curve defined over k has no k-rational point. Verifying this condition on the elliptic curve for the case of the Motzkin polynomial is the topic of one of the other series of lectures.

2.2. **The Cassels-Pfister theorem and bounds on the Pythagoras number.** In this talk I will sketch a proof of the Cassels-Pfister theorem and discuss various extensions of this theorem and their applications to problems on sums of squares. I will further make some remarks on the behaviour of the Pythagoras number under field extensions.

2.3. **Sums of squares of linear forms.** I will talk about sums of squares of linear forms over a field and introduce the so-called *Mordell function* $g_F : \mathbb{N} \to \mathbb{N}$; roughly $g_F(n)$ is the smallest integer g such that any quadratic form in n variables over F which represents only sums of squares is a sum of g squares of linear forms. We discuss a related field invariant, the *length*. These notions and results are motivated by the open problem of expressing the Pythagoras number of a rational function field in terms of the Pythagoras number of the base field. I will also look at special cases where one can improve the bounds on Pfister's bound of Pythagoras numbers of function fields over the real numbers.

2.4. **Rational function fields over complete discretely valued fields.** I will compare the Pythagoras numbers of finite field extensions of a field k with the Pythagoras number of the rational function field $k(X)$. This leads us to an open question whether $p(k(\!(t)\!)(X)) = p(k(X))$. At least in the case where k is itself a field of formal power series I can show that the answer is positive. I also discuss some related results and problems.

3. Sums of four squares in function fields of surfaces (Grimm)

3.1. **The elliptic curve method.** We discuss the main geometric ideas from Cassels-Ellison-Pfister [CEP71] which applies to show that under some particular conditions a positive definite polynomials in $\mathbb{R}[X, Y]$ cannot be a sum of three squares in $\mathbb{R}(X, Y)$, while being certainly a sum of four squares.

The Cassels-Pfister Theorem for quadratic forms over rational function fields in one variable helps to reduce the problem to studying points on a certain related elliptic curve defined over $\mathbb{R}(X)$. Similarly to the situation of elliptic curves defined over the integers of a number field, the algebraic group structure on elliptic curves helps immensely with this task. In particular, the Mordell-Weil theorem ('The group of rational points is finitely generated') and the Nagel-Lutz theorem ('torsion points are integral' with certain divisibility properties with the discriminant) have a verbatim analogue for the situation of an elliptic curve defined over a polynomial ring in one variable over a field. We will see how these structure theorems permit to determine the torsion part, and to some extent, the rank of the elliptic curve. We will also place some of the seemingly ad-hoc methods and calculations used in [CEP71] in the context of Galois cohomology, where the method appears very naturally and conceptually.

3.2. **The Motzkin polynomial.** Cassels-Ellison-Pfister [CEP71] showed that the so-called *Motzkin polynomial*

$$1 + (X^2Y^2)(X^2 + Y^2 - 3)$$

cannot be written as a sum of three squares in $\mathbb{R}(X, Y)$. This is done by studying the rational points over $\mathbb{R}(X)$ of the elliptic curve given by the Motzkin polynomial. We discuss the number theoretic methods involved in showing that the rank of that elliptic curve is zero, i.e. that the previously determined torsion part is already the full group of rational points. This implies the desired result that the Motzkin polynomial is not a sum of three squares.

In this approach one has the advantage of the fact that the coefficients of the Motzkin-polynomial are rational numbers, whereby the corresponding elliptic curve is defined over $\mathbb{Q}[X]$. This yields a natural action of the absolute Galois group of the rational numbers on the points of the curve. This is used to determine the rank to be equal to 1 and thus to find a non-divisible point of infinite order in the group of points over $\mathbb{C}(X)$. This ultimately shows the non-existence of such a point over $\mathbb{R}(X)$.

Together with the long-known upper bound $p(\mathbb{R}(X, Y)) \leqslant 4$, due to Hilbert and Siegel, the fact that Motzkin polynomial is not a sum of three squares in $\mathbb{R}(X, Y)$ shows that the Pythagoras number of this field is exactly 4.

At the end of the talk, we sketch a method due to Bogomolov, which applies readily to show for any real finitely generated extension $F/\mathbb{R}$ of transcendence degree 2 that $p(F) = 4$.

4. Sums of even powers in function fields of real curves (Becker)

4.1. **Continuous functions on $\mathbb{S}^1$ and applications to sums of even powers.** The elements of the rational function field $\mathbb{R}(X)$ admit a natural interpretation as continuous functions on the real projective line $\mathbb{P}^1$. In this way one can identify $\mathbb{R}(X)$ with a subset of $C(\mathbb{P}^1, \mathbb{P}^1)$ or, more convenient, of $C(\mathbb{S}^1, \mathbb{S}^1)$. This subset is shown to be dense by invoking homotopy theory. The density of this representation has striking consequences. Firstly, one identifies a large set of sums of squares which are, surprisingly, sums of $2n$th powers for every integer n. E.g. this is true for the rational function $(1 + X^2)/(2 + X^2)$. Secondly, using this set, one can produce a presentation of non-negative rational functions as sums of some higher powers for even exponents.

4.2. **The real holomorphy ring and Mason's theorem.** The concept of the real holomorphy ring of a function field F of a real curve will be used to rephrase and summarize some of the fundamental results of E. Witt presented in his seminal work of 1934 on sums of squares in function fields over $\mathbb{R}$. In addition, the image of the map $F \to C(M, \mathbb{P}^1)$, M the space of real places, is shown to be dense. The proof will be sketched. The special cases of $\mathbb{R}(X)$ and of function fields of elliptic curves will be presented in detail.
The real holomorphy ring is the subring of all elements of F, understood as functions, with no poles at the real points of the real curve in question. It is a Dedekind ring with finite class number, the totally positive units are sums of $2n$th powers for every n. One can derive a kind of 'unique factorization' for sums of squares in such function fields.
Quantitative studies of sums of higher powers are making use of the so called Mason's theorem, a consequence of the Riemann-Hurwitz genus formula. Mason's theorem will be applied to the rational function field.

Prerequisites: In this lecture the theory of discrete valuations, resp. discrete valuation rings, and their extensions to finite field extensions will be applied. Also, basic facts on Dedekind domains and affine algebraic curves will be used.

4.3. **Hilbert identities and higher Pythagoras numbers.** Higher Pythagoras numbers p_n refer to the length of sums of nth powers completely analogous to the case of sums of squares; p_2 is nothing but the ordinary Pythagoras number. The main result in this lecture states that in any formally real field all higher Pythagoras numbers are finite provided the ordinary Pythagoras number is finite, and there are bounds for p_{2n} in terms of p_2. The proofs rely on the structure of the real holomorphy ring in general formally real fields and the famous identities of Hilbert used in his solution of the Waring problem (1909). In the case of function fields over $\mathbb{R}$ recent results of Kucharz et al. on the topology of real algebraic varieties will be applied.

Prerequisites: The concept of affine varieties over $\mathbb{R}$ will be freely used along with basic facts.

5. Sums of squares in function fields (Hu)

5.1. **Function fields of real surfaces.** In this lecture, we consider positive polynomials in $\mathbb{R}[X, Y]$. These are known to be sums of four squares in the rational function field $\mathbb{R}(X, Y)$ (Hilbert and Landau). We shall relate the problem of writing a positive polynomial as a sum of three squares in $\mathbb{R}(X, Y)$ to a Leftschetz type problem in complex algebraic geometry. With this approach, Colliot-Thélène proved that for any even degree $d \geq 6$, there exist positive polynomials that are not sums of three squares of rational functions. We will explain his proof. Time permitting, we will mention without proof some more recent progress on this topic, mostly due to Olivier Benoist.

Prerequistes:

- Basic knowledge of algebraic geometry, including a good understanding of notions such as *birational equivalence, smooth projective varieties and divisors*, etc.; see [Sha94, Chapter I, Chapter II §§1-4, Chapter III, §1], or any other standard textbook on algebraic geometry.
- Basic knowledge of *Galois cohomology*, especially the *cohomological description of Brauer group*. See in particular [EKM08, §99], [Ser79, Chapters VII and X], [GS06, Chapters 3–4].

Sources and references:[CT93]

5.2. **Finitely generated fields.** This lecture will focus on fields having finite transcendence degree over $\mathbb{Q}$. In the case of number fields, the well known Hasse–Minkowski theorem is a fundamental tool to study various problems concerning quadratic forms over these fields. For a rational function field over a number field, this theorem can also help to determine the precise value of the Pythgoras number. In more general cases (a non-rational function field, or a function field in more variables), the best known result seems to be an upper bound given by a 2-power that only depends on the transcendence degree. The method is to use a local-global principle in Galois cohomology, first conjectured by Kazuya Kato in 1986 and complete proved by Jannsen in 2009.

I will first explain the statement of Kato's conjecture. Then I will prove that Kato's conjecture combined with Milnor's conjecture implies an explicit upper for the Pythagoras number of finitely generated fields. If time permits, some other applications of Kato's conjecture will be discussed.

Prerequistes: Basic knowledge of local and global fields. See e.g. [O'M73, Chapters 1–3], or any other standard textbook on algebraic number theory. Furthermore, basic knowledge of *Galois cohomology*, especially the *cohomological description of Brauer group*. See above.

Sources and references: [CTJ91], [CT86], [Kat86].

5.3. **Sums of squares in Laurent series fields.** The study of Pythagoras number for Laurent series fields in more than one variable was initiated by Choi, Dai, Lam and Reznick in 1982. They obtained the equality $p(\mathbb{R}((x, y))) = 2$, and proved more generally that if $p(k(t))$ is bounded by a 2-power, then $p(k((x, y)))$ is bounded by the same 2-power. In this lecture we will explain a proof of a similar result for Laurent series in 3 variables. We will use some valuative argument that works in a more general situation and some special techniques concerning Laurent series in several variables. The case of three variables is solved by relating the problem to certain local-global principles over two-dimensional complete local rings.

Prerequistes: Some commutative algebra towards a good understanding of basic facts about *complete local rings*; see [Eis95, Chapter 7], or any other standard textbook on commutative algebra.

Sources and references: [CDLR82], [BGVG14], [Hu15], [Hu17].

6. Special talks

6.1. **On sums of squares representations of symmetric forms (C. Goel).** The relationship between the cone of positive semidefinite (psd) real forms and its subcone of sum of squares (sos) of forms is of fundamental importance in real algebraic geometry and optimization. The study of this relationship goes back to the Hilbert's seminal paper [Hil88] , where he gave a complete characterisation of the pairs $(n, 2d)$ for which a psd n-ary form of degree $2d$ can be written as a sos. He proved that a n-ary $2d$-ic psd form is sos if and only if either $n = 2$, or $d = 1$, or $(n, 2d) = (3, 4)$. In all other cases there exists psd not sos forms; the first explicit examples of psd not sos forms for the pairs $(3, 6)$ and $(4, 4)$ were given by Motzkin and Robinson in the late 1960s.

In this talk, I will show how this relationship changes under the additional assumptions of invariance. I will present our recent results from a joint work with Salma Kuhlmann and Bruce Reznick giving the analogues of Hilbert's characterisation for symmetric and even symmetric forms respectively.

6.2. **On some cancellation algorithms (M. Zakarczemny).** Given an injective mapping $g : \mathbb{N} \to \mathbb{N}$ the discriminator $D_g(n)$ is the smallest natural number m such that $g(1), g(2), \ldots, g(n)$ are distinct modulo m. The problem of determining or estimating discriminators was studied by various authors. There is also a slightly different definition of a discriminator in terms of cancellation algorithms. We define $b_f(n)$ to be the smallest positive integer m such that all the values $f(n_1, n_2, ..., n_m)$,where $n_1 + n_2 + ... + n_m \leqslant n$ are not divisible by m. For the given functions $f : \mathbb{N}^m \to \mathbb{N}$ we will find the sequence of the least non cancelled numbers $(b_f(n))_{n\in\mathbb{N}}$ or estimate elements of this sequence. Browkin and Cao have shown that, in the case of the function $f : \mathbb{N}^2 \to \mathbb{N}, (n_1, n_2) \mapsto n_1^2 + n_2^2$, the sequence $(bf(n))_{n\in\mathbb{N}}$ is the increasing sequence of all elements of the set of all square-free positive integers which are products of prime numbers congruent to 3 modulo 4. In this talk we will especially focus on the case where the function is the sum of squares.

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Books

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Articles

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Other notes