ALGEBRA AND ARITHMETIC (ALGAR) 2017 – LOCAL-GLOBAL PRINCIPLES FOR QUADRATIC FORMS

Contents

Date: July 14, 2017.

1. Quadratic forms over fields (Becher)

1.1. Quadratic forms. We revisit basic notions for quadratic forms over fields and related algebraic objects such as the discriminant and the orthogonal group. We formulate Witt's fundamental theorems and describe the Witt ring of a field. We further introduce residue forms.

1.2. Quaternion algebras and detecting isotropy by invariants. We introduce quaternion algebras and recall their relations to quadratic forms. We show how isotropy of quadratic forms in small dimensions can be determined by basic invariants. Some special properties of norm forms of quaternion algebras extend to Pfister forms. The way how a quaternion algebra can be associated to a quadratic form of dimension 2 or 3 gives rise to the definition of the Clifford algebra of a quadratic form.

1.3. Arithmetic of fields and isotropy criteria. We have two general approaches to the problem of showing that certain quadratic forms are isotropic, on the one hand local-global arguments, on the other hand, systems of quadratic forms. Any of these approaches requires the field to have special arithmetic properties. On the example of the Hasse-Minkowski Theorem, it will be explained how the analysis of isotropy of quadratic forms (in particular of dimension 4) can be reduced by weak approximation to analyse quaternion algebras. We further discuss the u-invariant of a field and how to bound it by means of Tsen-Lang Theory. Some other criteria for bounding the u-invariant will be indicated.

2. Local-global principles in number theory and geometry (Auel)

2.1. Galois cohomology and quadratic forms. We will introduce Galois cohomology with an eye toward the theory of quadratic form and the Brauer group. Along the way, we will encounter Hilbert's theorem 90, Kummer theory, Galois descent for tensors on a vector space, and the Milnor and Bloch–Kato conjectures.

2.2. Local-global principles for global fields. We study the two original examples of local-global principles, namely the Albert–Hasse–Brauer–Noether exact sequence and the Hasse–Minkowski theorem, describing the structure of the Brauer group and of quadratic forms over a global field, respectively. This will involve the input of other kinds of local-global phenomena, such as the Grunwald–Wang theorem and the local and global Artin reciprocity theorems from class field theory.

2.3. Geometric view on local-global principles. We will explain how considerations from classical algebraic geometry can give rise to explanations for why local-global principles fail to hold for function fields of (complex) algebraic varieties, especially in dimensions one and two. We will cover counterexamples to local-global principles for isotropy of quadratic forms arising from unramified covers of curves and the unramified Brauer groups of surfaces.

3. Field patching and local-global principles (Parimala)

3.1. **Patching problems.** We discuss patching problems and local-global principles for vector spaces with tensors in the abstract setting of factorisation an inverse system of fields, which we call a system of patching fields. We illustrate this by giving a patching system of fields associated to $k[\![t]\!](X)$.

3.2. **Function fields of arithmetic surfaces and patching.** We study regular proper models of function fields of curves over $k[\![t]\!]$, associated patching fields, the intersection property for the patches, and factorisation for the general linear group GL_n. We then describe how these results generalise. On the one hand these properties hold more generally for suitable linear algebraic groups and with patching fields associated to function fields of curves over complete discretely valued fields. This leads to a local-global principle for isotropy of quadratic forms. Furthermore, one obtains period-index relations for central simple algebras.

3.3. **Local-global principles with respect to discrete valuations.** From the local-global principle for isotropy of quadratic forms with respect to field patches we derive local-global principles with respect to divisorial discrete valuations. We obtain analogous results for the index of central simple algebras. This leads us to a discussion of u-invariant and Brauer dimension of such fields.

4. Splitting ramification and residue characteristic two (Suresh)

4.1. **Splitting the ramification of central simple algebras.** Let X be a regular two dimension excellent scheme which is proper over an affine scheme and let F be its function field. In this talk we explain a method of Saltman [Salt08] on splitting the ramification of elements in the Brauer group of F.

4.2. **The u-invariant of a function field over a complete field.** Let K be a complete discretely valued field and F the function field of a curve over X. Let k be the residue field of K and $p = char(k)$. Suppose that $p \neq char(K)$ and $[k : k^p] = p^n$. Using the filtration of Kato on the Brauer group of a complete discretely valued field and the work of Hartmann-Harbater-Krashen on patching, we give a bound on the indices of p-torison elements in the Brauer group of F which depends on n. As a consequence we also show that the u-invariant of F is finite. [PS15]

4.3. **Local-global principle for quadratic forms over $K(\!(X, Y)\!)$.** Let K be a field of characteristic not equal to 2 and $F = K(\!(X, Y)\!)$. Following the work of Hartmann-Harbater-Krashen on refinements to patching, we explain the proof of the fact that a quadratic form q over F of rank at least 3 is isotropic over F if and only if it is isotropic over the completions of F at all discrete valuations of F. [HHK15]

5. Special talks

5.1. **Locally isotropic forms over a rational function field (D. Leep).** Let K be a field with characteristic different from 2 and assume that the u-invariant of every finite extension of K is 2. We discuss what can be said about the u-invariant of the rational function field $K(x)$ and connections to the local-global question of isotropy over $K(x)$.

5.2. **Locally isotropic forms over $\mathbb{Q}(X)$ (P. Gupta).** In 1940-42, Lind-Reichardt gave a locally isotropic pair of quadratic forms over $\mathbb{Q}$ which is anisotropic. Using an extended version of their example I show that certain four-dimensional quadratic forms over the rational function field $\mathbb{Q}(X)$, do not satisfy local-global principle for isotropy with respect to the set of all valuation on $\mathbb{Q}(X)$.

5.3. **Systems of quadratic forms over a complete field (D. Leep).** Let K be a complete discretely valued field with residue field k. We discuss the proof of the following theorem. Suppose there is a positive number A such that every system of r quadratic forms over k in more than Ar variables is isotropic over k. Then every system of r quadratic forms over K in more than $2Ar$ variables is isotropic over K. For the special case when K is the field of p-adic numbers, we improve a result of Heath-Brown. As a corollary, we discuss the u-invariant of $K(x_1, ..., x_m)$ for arbitrary m.

References

Books

[CF67] J.W.S. Cassels, A. Fröhlich, *Algebraic Number Theory*, Academic Press; Thompson Book Co., 1967.

[EKM08] R. Elman, N. Karpenko, A. Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.

[GS06] P.Gille, T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Vol. 101. Cambridge University Press, 2006.

[Lam05] T.Y. Lam, *Introduction to Quadratic Forms over Fields*, Vol. 67. American Mathematical Soc., 2005.

[Kahn08] B. Kahn, *Formes quadratiques sur un corps*, Société mathématique de France, 2008. `https://webusers.imj-prg.fr/~bruno.kahn/preprints/fqbook3.pdf`

[Roq06] P. Roquette, *The Brauer-Hasse-Noether theorem in historical perspective*, Vol. 15. Springer Science & Business Media, 2006. `https://www.mathi.uni-heidelberg.de/~roquette/brhano.pdf`

[Scha12] W. Scharlau, *Quadratic and Hermitian Forms*, Springer Science & Business Media, 2012.

[Ser73] J.-P. Serre, *A Course in Arithmetic*, Springer New York, Graduate Texts in Mathematics, Volume 7, 1973.

[Ser96] J.-P. Serre, *Galois Cohomology*, Springer Monogr. Math., 1996.

[Sha16] S.S. Shatz, *Profinite Groups, Arithmetic, and Geometry*, Vol. 67. Princeton university press, 2016.

Articles

[APS15] A. Auel, R. Parimala, V. Suresh, *Quadratic surface bundles over surfaces*, Doc. Math., Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday (2015), pp. 31–70.

[Auel13] A. Auel, *Failure of the local-global principle for isotropy of quadratic forms over surfaces*, Quadratic Forms and Linear Algebraic Groups, Oberwolfach Reports 10 (2013), issue 2, report 31, pp. 1823–1825.

[HHK09] D. Harbater, J. Hartmann, D. Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. 178 (2009), 231-263.

[HHK11] D. Harbater, J. Hartmann, D. Krashen, *Local-global principles for torsors over arithmetic curves*, arXiv: 1108.3323v2, 2011.

[HHK15] D. Harbater, J. Hartmann, D. Krashen, *Refinements to Patching and Applications to Field Invariants*, Int. Math. Res. Notices 2015; 2015 (20): 10399-10450. doi: 10.1093/imrn/rnu278

[OP03] M. Ojanguren, R. Parimala, *Quadratic forms over complete local rings*, Adv. in algebra and geometry (Hyderabad, 2001), Hindustan Book Agency, New Delhi, 2003, pp. 53–55.

[PS15] R. Parimala, V. Suresh, *On the u-invariant of function fields of curves over complete discretely valued fields*, Advances in Mathematics 280 (2015), 729-742.

[Salt08] D.J. Saltman, *Division algebras over surfaces*, Journal of Algebra, Volume 320, Issue 4, 15 August 2008, Pages 1543-1585.

[Srid00] R. Sridharan, *On the theorem of Hasse-Minkowski*, in Cyclotomic Fields and Related Topics (Pune, 1999), Bhaskaracharya Pratishthana, Pune, 2000, pp. 127–134. `https://www.bprim.org/sites/default/files/sri.pdf`

Other notes

[Conr] Keith Conrad, *Ostrowski's theorem over* $\mathbb{Q}$
`http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/ostrowskiQ.pdf`

[Conr2] Keith Conrad, *Ostrowski's theorem over* $F(T)$
`http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/ostrowskiF(T).pdf`

[Open] Open Problem Session in the Workshop *Local-Global Principles and Their Obstructions*. October 1-3, 2015, University of Pennsylvania, Philadelphia, Pennsylvania. Organizers: D. Harbater, J. Hartmann, D. Krashen, R. Parimala, V. Suresh.
`https://www.math.upenn.edu/~hartmann/sha/problems.pdf`