

*Production of Commodities by Means of  
Commodities* with Non-Uniform Rates of Profits,  
Credit and Debt \*

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**Abstract**

Sraffa's book *Production of Commodities by Means of Commodities* (PCMC) begins with the following:

*Let us consider an extremely simple society which produces just enough to maintain itself. Commodities are produced by separate industries and are exchanged for one another at a market held after the harvest (Sraffa, 1960a, p.3).*

One of Sraffa's objectives is to determine the

*... set of exchange-values which if adopted by the market restores the original distribution of the products and makes it possible for the process to be repeated; such values spring directly from the methods of production (Sraffa, 1960a, p.3).*

so that the system is in a *self-replacing state*. In this paper we study the *self-replacing* properties without assuming *uniform rate of profits*

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## Introduction

Sraffa's book *Production of Commodities by Means of Commodities* (PCMC) begins with the following:

*Let us consider an extremely simple society which produces just enough to maintain itself. Commodities are produced by separate industries and are exchanged for one another at a market held after the harvest (Sraffa, 1960a, p.3).*

The subtitle of the book indicates that Sraffa's contribution is a *A Prelude to a Critique of Economic Theory*. Sraffa's one of main aims seems to have been to examine the tenability of the (*neoclassical*) postulate that relative commodity prices are a monotonic function of the profit rate (or of the wage rate). According to *neoclassical* requirements, as the profit rate increases and the wage rate decreases the price of a commodity of an industry having a high proportion of the value of the means of production to labour should, *ceteris paribus*, increase relative to the price of commodities where the proportion is low. This is well explained in Sraffa (1960a, Ch. 3, pp. 14-5).

His investigation led to conclude that the notion of capital intensive techniques as opposed to labour intensive ones would be most of the times meaningless.

*... as the wages fall the price of the product of a low proportion ... may rise or it may fall, or it may even alternate in rising and falling, relative to its means of production; while the price of the product of a high-proportion ... industry may fall, or it may rise or it may alternate. What neither of such products can do ... is to remain stable in price relative to its means of production throughout any range, long or short, of the wage-variation.*

These results were at the core of the 60s' famous capital controversy debate between the two Cambridges. If the prices of the individual products vary as the wage-rate varies, also the value of the means of production will vary as the wage-rate varies. This could lead to situation where as the profit rate increases the capital intensity could as well increase. Contradicting a fundamental neoclassical postulate<sup>1</sup>.

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<sup>1</sup>Ferguson, a neoclassical economist, conveys this very clearly:

*[Neoclassical Postulate] ... the lower the rate of interest, the greater the capital intensity of production. All other neoclassical properties follow immediately from this simple relation (Ferguson, 1969, p.252)*

. A discussion of this postulate in light of the Cambridges debate and an investigation of its empirical validity is to be found in Zambelli (2017).

A conclusion of that debate was that the neoclassical aggregate production function may not always be well behaved because the value of aggregate capital may not be a monotonic function of the wage rate or of the profit rate.

The problem was acknowledged from a logical point of view by some neoclassical authors (Samuelson (1966) and Ferguson (1969)), who anyway considered it to be an empirically irrelevant problem (Sato, 1974). To the point that today there is very little debate on the empirical relevance of the aggregate neoclassical function as if aggregation was unproblematic. Recently Zambelli (2017), following a Sraffian methodology, has retraced the 60s' debate and has computed the neoclassical aggregate production function using empirical observations. The conclusion is that the aggregate production, as defined by Samuelson (1962), Arrow et al. (1961), does not have any empirical foundation, hence it does not exist. The logical problems exposed during the Cambridges Controversy cannot be confined to be just a *curiosum* because they seem to be the rule.

An explicit assumption in Sraffa's PCMC is that of a uniform rate of profits for all industries.

For the purposes of the *Prelude to a Critique of Economic Theory* the assumption of a uniform rate of profits may be sufficient. To go through the complications of non-uniform rate of profits is not necessary for the critique. In fact the Austrian School would maintain that the inverse of the profit rate - which they assume to be uniform - would be correlated to the average period of production and this could be related with the *round-aboutness* of capital determining a relation with capital intensive methods<sup>2</sup>. Furthermore, the complications of non-uniform rate of profits are not necessary for the foundations of the critique to the incorporated labour theory of value embedded in Ch. 6 (*Reduction to Dated Quantities of Labour*) of PCMC. The reason being that the critique holds also for the general case. In other words, if the critique holds for the case of *uniform rate of profits* it would hold even more strongly for *non-uniform rate of profits*.

Sraffa himself considered the uniform rate of profits to be an assumption.

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<sup>2</sup>On this point Sraffa is crystal clear:

(The reduction to dated labour terms has some bearing on the attempts that have been made to find in the 'period of 'production' an independent measure of the quantity of capital which could be used, without arguing in a circle, for the determination of prices and of the shares in distribution. But the case just considered seems to be conclusive in showing the impossibility of aggregating the 'periods' belonging to the several quantities of labour into a single magnitude which could be regarded as representing the the quantity of capital. The reversal in the direction of the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with any notion of capital as a measurable quantity independent of distribution and prices) (Sraffa, 1960a, p.38)

In the text, at a very early stage, he writes that the profit rates “*must be uniform (Sraffa, 1960a, p.6, emphasis added)*”, but in the index he writes under the entry relative to the Rate of profits “*assumed*” to be uniform for all industries Sraffa (1960a, p.98, emphasis added)”<sup>3</sup>. To say that the rate of profit is assumed to be uniform is very different from saying that it must. Hence it is not very clear what Sraffa really meant.

Much research has taken place having the *uniform rate of profits* as an more or less *implicit* assumption. Several authors claim that this is not really an assumption, but a property of the system a) determined by the methods of production and distribution (Sinha, 2010, 2011), or b) describing the long term convergent properties for the case of free competition (Kurz and Salvadori, 1995), or c) determining the *natural prices* around which the market prices gravitate (Garegnani, 1976, 1997).

Another important implicit assumption is the absence, in Sraffa’s market after the *harvest*, of any mechanism that might allow for trade to take place thanks to the use of deferred means of payments (i.e., endogenous debt and credit relations). In the presence of these means of payments it is still possible that the “[production] process be repeated”, i.e. that the system repeats itself even when the prices are not *production prices*.

Following the work presented in Zambelli (2016) here we study the relation between these means of payments and the existing *methods of production* and study the conditions in which the system could be in a *self-replacing* state for the case of single-product industries and circulating capital<sup>4</sup>.

## 1 The self-replacing prices

*The significance of the equations is simply this: that if a man fell from the moon on the earth, and noted the amount of things consumed in each factory and the amount produced by each factory during a year, he would deduce at which values the commodities must be sold, if the rate of interest must be uniform and the process of production repeated. In short, the equations show that the conditions of exchange are entirely determined by the conditions of production. Sraffa(1927 or 1928, D3/12/7, emphasis added)*<sup>5</sup>

In the sequel we will compute these prices for the general case in which the

<sup>3</sup>Also the Italian edition of *Production of Commodities by Means of Commodities* refers to, in the Indice Analitico: *Saggio del profitto, supposto uniforme in tutte le industrie* (Sraffa, 1960b, p.127)

<sup>4</sup>We believe that in doing so we follow Sraffa’s declared methods, which is the study of prices without changing the quantities.

<sup>5</sup>Catalogue of Sraffa Papers, Wren Library, Trinity College, Cambridge, edited by Jonathan Smith. This quotation is from the archives of the Wren Library. It is dated between 1927 and 1928. See Gilibert (2006, p.28) and Gilibert (2003)

rates of interest (interpreted here as being the same as the rate of profits) may not be uniform. Sraffa thought about the equations that came to be core of his 1960's book probably since his works Sraffa (1925, 1926). From the Sraffa archives we see that he had written already in the twenties several equations trying to link the quantities used in production, distribution of the physical surplus generated by the economic system with the theory of value. On the relation between the equations written and shown to Keynes in 1928 and PCMC see Gilibert (2003) and Gilibert (2006).

The quotation above is illuminating, especially when put in relation with the opening sentences of the opening section of PCMC. Sraffa principal scope in PCMC is to determine the

*... set of exchange-values which if adopted by the market restores the original distribution of the products and makes it possible for the process to be repeated; such values spring directly from the methods of production (Sraffa, 1960a, p.3).*

so that the system is in a *self-replacing state*.

The focus is on the determination of the prices and wages that, “*if adopted*”, will “*restore the original distribution of the product*”.

At the end of the harvest and before the market day the  $n$  producers have produced  $n$  commodities. In this market producers will sell their produced goods and the workers their labour.

## 1.1 Physical Quantities, Surplus and Distribution

### 1.1.1 Physical Quantities to be Exchanged

Without going into the details of the exchanges what is that we can say about the possibility of reproducing production and consumption as it did occur the previous year? We have some *objective* observations. Our privileged observer, *the man from the moon*, at the beginning of the market day observes the produced  $b$ -quantities *brought* to the market. In the sequel we will focus on the prices that will restore the *original distribution of the product*.

$$[b_1, b_2, \dots, b_i, \dots, b_n]^T \tag{1.1}$$

$i = 1, 2, \dots, n$ . He also knows that there are, at the beginning of the market day, workers that are willing to work. Sraffa, in his thought experiment, gives to the *man from the moon* also information of the quantities that were used during the previous production period. He calls this information *the methods of production*. If the system has to reproduce itself the exchanges during the market day would have to be such that at the end of the market

day the producers would have the means of production necessary to replicate the production process of the previous year.

$$b_i \xrightarrow{\text{exchange}} a_i^1, a_i^2, \dots, a_i^j, \dots, a_i^{n-1}, a_i^n, \ell_i \xrightarrow{\text{production}} b_i \quad (1.2)$$

where  $a_i^j$  is the mean of production  $j$  used in the production of good  $i$ , the quantity  $b_i$ . For the whole system this circularity is summarized with the following standard notation

$$\begin{array}{ccccccc} b_1 & \xrightarrow{\text{exchange}} & a_1^1 & \dots & a_1^j & \dots & a_1^n & \ell_1 & \xrightarrow{\text{production}} & b_1 \\ b_2 & \xrightarrow{\text{exchange}} & a_2^1 & \dots & a_2^j & \dots & a_2^n & \ell_2 & \xrightarrow{\text{production}} & b_2 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \xrightarrow{\text{production}} & \vdots \\ b_i & \xrightarrow{\text{exchange}} & a_i^1 & \dots & a_i^j & \dots & a_i^n & \ell_i & \xrightarrow{\text{production}} & b_i \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \xrightarrow{\text{production}} & \vdots \\ b_n & \xrightarrow{\text{exchange}} & a_n^1 & \dots & a_n^j & \dots & a_n^n & \ell_n & \xrightarrow{\text{production}} & b_n \end{array} \quad (1.3)$$

These quantities may be written in compact matrix notation in the following way:

$$\mathbf{b} \xrightarrow{\text{exchange}} \mathbf{A}, \mathbf{L} \xrightarrow{\text{production}} \mathbf{b} \quad (1.4)$$

where:  $\mathbf{A}$  is an  $n \times n$  matrix whose components are the used means of production  $\{a_i^j\}$ ;  $\mathbf{L}$  is a  $n \times 1$  vector whose elements  $\{\ell_i\}$  is the labour used in production;  $\mathbf{b}$  is  $n \times 1$  vector whose element  $\{b_i\}$  is the harvest of good  $i$ <sup>6</sup>.

### 1.1.2 Surplus available for distribution or consumption

Note that once we have the observed and hence known quantities of eq. 1.3 or eq. 1.4 we know also the Physical Surplus that was produced during the production period. That is

$$\begin{array}{rcl} s_1 & = & b_1 - \sum_{i=1}^n a_i^1 \\ s_2 & = & b_2 - \sum_{i=1}^n a_i^2 \\ \vdots & = & \vdots - \vdots \\ s_j & = & b_j - \sum_{i=1}^n a_i^j \\ \vdots & = & \vdots - \vdots \\ s_n & = & b_n - \sum_{i=1}^n a_i^n \end{array} \quad (1.5)$$

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<sup>6</sup>In the sequel we will indicate with bold lowercase letters the vectors, with the exception of the labour vector  $\mathbf{L}$  and the physical surplus vector  $\mathbf{S}$ , and uppercase letter matrices. To simplify notation the the row  $i$  of a matrix, let us take as example matrix  $\mathbf{A}$ , would be written in bold in the following way,  $\mathbf{a}_i$ , while the column  $j$  will be written as  $\mathbf{a}^j$ .

where  $s_i$  is the surplus of commodity  $i$  available for distribution after the quantities  $\{a_i^j\}$  have been put aside for the next year production or, alternatively, is the quantity produced in the previous period which is left once the inputs used in production have been removed. In compact matrix notation we have:

$$\mathbf{S} = (\mathbf{B} - \mathbf{A})^T \mathbf{e} \quad (1.6)$$

where:  $\mathbf{e}$  is an  $n \times 1$  unit or summation vector (each element is 1);  $T$  is the transpose operator;  $\mathbf{S}$  is the  $n \times 1$  Physical Surplus vector or Physical *Net National Product*;  $\mathbf{B}$  is an  $n \times n$  diagonal matrix having as elements in the diagonal the elements of gross production  $\mathbf{b}$  and the other elements are 0s.

## 2 The *Man from the Moon* and the computation of *Self-Replacing Prices*

The *man from the moon* has, before the opening of the market, to compute the prices that will allow the system to replicate for next year (or this year) what happened the current year (or last year). After the harvest and before the market day begins we have only the produced quantities  $\mathbf{b}$ . These quantities could be transformed through trade, at the end of the market day, into a different allocation of resources.

That is, we have that exchange may reverse the process. The whole circular process could become

$$\mathbf{A}, \mathbf{L} \xrightarrow{\text{production}} \mathbf{b} \xrightarrow{\text{exchange}} \mathbf{A}, \mathbf{L}, \mathbf{S} \quad (2.1)$$

provided there are uniform prices that restore the quantities and hence the distribution of means of production and of the surplus that were produced at the beginning of the production or transformation period.

### 2.1 Self-Replacing Conditions: the General Case.

Our *man from the moon* has the task of finding the vector of prices and wage rate that would allow self-reproduction of the system as in 2.1 and would allow the same distribution the surplus  $\mathbf{S}$ , i.e., matrix  $\mathbf{H}$ . This is a matrix with  $n \times 1$  rows and  $n$  columns. The rows from 1 to  $n$  are the aggregate physical consumption of the owners of the industries, i.e. each industry has final consumption  $\mathbf{h}_i$  with  $i = 1, \dots, n$ . The  $n \times 1$  row is the aggregate consumption of the workers,  $\mathbf{h}_{n+1}$ <sup>7</sup>.

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<sup>7</sup>For this paper we mean by consumption both final consumption and eventual *net investment*. If the thought experiment concerns the computation of the prices that would allow the system to replicate, net-investment (and dis-investment) would have to be zero. The possibility of net investment to be positive (or negative) is considered in another paper.

In synthesis the prices would have to be such that the exchange process during the market day would lead to the following distribution of the gross product available (and employment):  $\mathbf{b} \xrightarrow{\text{exchange}} \mathbf{A}, \mathbf{L}, \mathbf{S}$ . The quantity  $\mathbf{S}$ , distributed as in  $\mathbf{C}$ , is consumed while the means of production  $\mathbf{A}$  and the  $\mathbf{L}$  is transformed into the output ( $\mathbf{A}, \mathbf{L} \xrightarrow{\text{production}} \mathbf{b}$ ).

Given a vector of prices and a wage rate we would have the following:

$$\begin{array}{rcl}
a_1^1 p_1 + \dots + a_1^j p_j + \dots + a_1^n p_n + \ell_1 w & \leq & b_1 p_1 \\
a_2^1 p_1 + \dots + a_2^j p_j + \dots + a_2^n p_n + \ell_2 w & \geq & b_2 p_2 \\
\vdots + \vdots + \vdots + \vdots + \vdots + \vdots & \geq & \vdots \\
a_i^1 p_1 + \dots + a_i^j p_j + \dots + a_i^n p_n + \ell_i w & \leq & b_i p_i \\
\vdots + \vdots + \vdots + \vdots + \vdots + \vdots & \geq & \vdots \\
a_n^1 p_1 + \dots + a_n^j p_j + \dots + a_n^n p_n + \ell_n w & \leq & b_n p_n
\end{array} \tag{2.2}$$

which in matrix notation could be written as:

$$\mathbf{A}\mathbf{p} + \mathbf{L}w \begin{array}{l} \leq \\ \geq \end{array} \mathbf{B}\mathbf{p} \tag{2.3}$$

The left hand sides of eqs. 2.2 and 2.1 would be the individual industries production expenditures (or costs of production) while the right hand side would be the industries production revenues.

A general definition of profit rates is the one for which the following equation holds:

$$(\mathbf{I} + \mathbf{R})\mathbf{A}\mathbf{p} + \mathbf{L}w = \mathbf{B}\mathbf{p} \tag{2.4}$$

where  $\mathbf{R} = \text{diag}(\mathbf{r})$ , with  $\mathbf{r} = [r_1, r_2, \dots, r_i, \dots, r_n]^T$ <sup>8</sup>.

There are two cases of importance.

**First case or no financial balances: non negative rates of profits.**

This is the case where prices and wages are such that the revenues of the industries are greater than the production costs. In this case all industries are in the self-replacing state because they would sell more than enough to buy the means of production. That is:  $\mathbf{A}\mathbf{p} + \mathbf{L}w \leq \mathbf{B}\mathbf{p}$ . Zambelli (2016) studies this case. A special case of this case, which is well studied in the literature and the one investigated by Sraffa (1960a) himself, is the one in which the profit rates are uniform. That is  $r = r_1 = r_2 = \dots = r_i = \dots = r_n$ .

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<sup>8</sup>The *accounting* implicit in eq. 2.4 ?? ?? is consistent with the choice made in PCMC. We think that the alternative choice of computing the profit rates as including also labour costs would be simpler and more appropriate. Nevertheless the qualitative conclusions would not change. The difference is that, for example, eq. 2.4 would have to be written as  $(\mathbf{I} + \mathbf{R})(\mathbf{A}\mathbf{p} + \mathbf{L}w) = \mathbf{B}\mathbf{p}$ . The proposition made in the sequel of the article may be appropriately modified to consider this different accounting. We leave an analysis of the consequences of this alternative to another article.

### Second case or *with financial balances*: negative rates of profits. .

Clearly if there is at least one sector, as in the general case of eqs. 2.2 let us say sector  $j$ , for which we have  $\mathbf{a}_j \mathbf{p} - \ell_j w > b_j p_j$  the system would not be able to reproduce simply because the industry  $j$  would not have the necessary purchasing power. But industry  $j$  may be able to buy means of production if the owners of these means of production are willing to exchange against a future promise to pay. In this case this is possible  $\mathbf{A} \mathbf{p} + \mathbf{L} w \stackrel{\leq}{\cong} \mathbf{B} \mathbf{p}$ .

## 2.2 Surplus, distribution and future promises to pay (Financial Balances).

Associated to each industry  $i$  and the workers the revenues are:

$$\begin{aligned} \text{Revenues}(i) &= b_i p_i \quad i = 1, \dots, n \\ \text{Revenues}(n+1) &= \mathbf{e}^T \mathbf{L} w \end{aligned} \quad (2.5)$$

While the expenditures are:

$$\begin{aligned} \text{Expenditures}(i) &= \mathbf{a}_i \mathbf{p} + \ell_i w + \mathbf{h}_i \mathbf{p} \quad i = 1, \dots, n \\ \text{Expenditures}(n+1) &= \mathbf{h}_{(n+1)} \mathbf{p} \end{aligned} \quad (2.6)$$

Clearly those industries that are in the condition for which the expenditures would be higher than the revenues would not have the purchasing power to buy the means of production necessary to replicate the production of the previous period. These industries are in a condition of potential financial deficit. They would be able to purchase the necessary means of production only by agreeing to a deferred payment to take place during the years to follow in favor of the industries in potential surplus. Recall that the current exercise is to study the conditions that would allow the economic system to be in a self-replacing state as described above and summarized in the relation 2.1.

### 2.2.1 A *virtual* bank as clearing house.

If the quantities  $\mathbf{A}, \mathbf{L}, \mathbf{S}$  and  $\mathbf{H}$  have to be restored independently from any given prices we must have that in general the inequalities of eq. 2.2 may be “*eliminated*” if we allow for deferred payments which might take the form of *I Owe You*s (*IOUs*) or debt and credit relations (i.e. selling of real quantities or labour now by an agent with the promise of another agent of paying back at a future point of time). Therefore a producer may sell a part of the physical quantities produced in return of other physical quantities (*Cash Revenues*)<sup>9</sup> or in exchange of a future promise to pay ( $\Delta \text{Credit}(\cdot)$ ).

<sup>9</sup>Whether these exchanges take place triangularly or by use of a common accepted mean of payments is here not relevant. What is relevant is that during the *market day*

In the case of the workers the same would apply, in the sense that they could sell a part of labour in exchange of liquid means of payments and another part in exchange of future promises to pay by the employer.

Whether this takes place bilaterally or through a clearing house is here not important. For the simplicity of the argument we might presume here that there is a bank or clearing house: a central bank which takes care of the good functioning of the paying system.

For each industry and for the workers we have the following:

$$Revenues(\cdot) = Cash\ Revenues(\cdot) + \Delta\ Credit(\cdot) \quad (2.7)$$

The same would apply for the expenditures where for each industry and the workers we have the following:

For each industry and for the workers we have the following:

$$Expenditures(\cdot) = Cash\ Expenditures(\cdot) + \Delta\ Debt(\cdot) \quad (2.8)$$

### 3 The economic system as whole

An implicit interpretation of eqs. 2.5 (or 2.7) is that a quantity of goods (or of labour) is sold against ‘cash’ and a quantity against *credit*. Symmetrically an implicit interpretation of eqs. 2.6 (or 2.8) is that a quantity of goods (and of labour) are bought against ‘cash’ and a quantity against *credit*.

Therefore by taking into account this and by substituting 2.7 and 2.8 into 2.5 and 2.6 we obtain the following:

$$\begin{aligned} Cash\ Revenues(i) + \Delta\ Credit(i) &= b_i^{Cash} p_i + b_i^{Credit} p_i \\ &\quad i = 1, \dots, n \\ Cash\ Revenues(n+1) + \Delta\ Credit(n+1) &= \mathbf{e}^T \mathbf{L}^{Cash} w + \mathbf{e}^T \mathbf{L}^{Credit} w \end{aligned} \quad (3.1)$$

$$\begin{aligned} Cash\ Expenditures(\cdot) + \Delta\ Debt(\cdot) &= \mathbf{a}_i^{Cash} \mathbf{p} + \ell_i^{Cash} w + \mathbf{h}_i^{Cash} \mathbf{p} + \\ &+ \mathbf{a}_i^{Credit} \mathbf{p} + \ell_i^{Credit} w + \mathbf{h}_i^{Credit} \mathbf{p} \\ &\quad i = 1, \dots, n \\ Cash\ Expenditures(n+1) + \Delta\ Debt(n+1) &= \mathbf{h}_{(n+1)}^{Cash} \mathbf{p} + \mathbf{h}_{(n+1)}^{Credit} \mathbf{p} \end{aligned} \quad (3.2)$$

where:

$$b_i = b_i^{Cash} + b_i^{Credit} \text{ and hence } \mathbf{B} = \mathbf{B}^{Cash} + \mathbf{B}^{Credit};$$

$$\ell_i = \ell_i^{Cash} + \ell_i^{Credit} \text{ and hence } \mathbf{L} = \mathbf{L}^{Cash} + \mathbf{L}^{Credit};$$

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the quantity sold with the use of means of payments is used to buy produced goods by others.

$$\mathbf{a}_i = \mathbf{a}_i^{Cash} + \mathbf{a}_i^{Credit} \text{ and hence } \mathbf{A} = \mathbf{A}^{Cash} + \mathbf{A}^{Credit}$$

By rearranging in matrix notation we have:

$$\begin{aligned} \mathbf{CashRevenues}_{n \times 1} + \Delta \mathbf{Credit}_{n \times 1} &= \mathbf{B}^{Cash} \mathbf{p} + \mathbf{B}^{Credit} \mathbf{p} \\ \mathbf{Cash Revenues}(n+1) + \Delta \mathbf{Credit}(n+1) &= \mathbf{e}^T \mathbf{L}^{Cash} w + \mathbf{e}^T \mathbf{L}^{Credit} w \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \mathbf{CashExpenditures}_{n \times 1} + \Delta \mathbf{Debt}_{n \times 1} &= \mathbf{A}^{Cash} \mathbf{p} + \mathbf{L}^{Cash} w + \mathbf{H}_{n \times n}^{Cash} \mathbf{p} + \\ &+ \mathbf{A}^{Credit} \mathbf{p} + \mathbf{L}^{Credit} w + \mathbf{H}_{n \times n}^{Credit} \mathbf{p} \\ \mathbf{Cash Expenditures}(n+1) + \Delta \mathbf{Debt}(n+1) &= \mathbf{h}_{(n+1)}^{Cash} \mathbf{p} + \mathbf{h}_{(n+1)}^{Credit} \mathbf{p} \end{aligned} \quad (3.4)$$

By rearranging the above we have that the following must hold:

$$\mathbf{B}^{Cash} \mathbf{p} + \Delta \mathbf{CreditDebt}_{n \times 1} = \mathbf{A}^{Cash} \mathbf{p} + \mathbf{L}^{Cash} w + \mathbf{H}_{n \times n}^{Cash} \mathbf{p} \quad (3.5)$$

where:

$$\Delta \mathbf{CreditDebt}_{n \times 1} = \mathbf{B}^{Credit} \mathbf{p} - \mathbf{A}^{Credit} \mathbf{p} - \mathbf{L}^{Credit} w - \mathbf{H}_{n \times n}^{Credit} \mathbf{p} \quad (3.6)$$

$\Delta \mathbf{CreditDebt}_{n \times 1}$  is a vector composed of the first  $n$  elements of the  $n+1$  rows of the column vector  $\Delta \mathbf{CreditDebt}$ . Where the first  $n$  elements are relative to the net financial condition of the  $n$  industries and the element  $n+1$  is the net financial condition of the workers.

When prices and wages are such that each industry does not have to rely on credit in order to buy the necessary means of production we would have that  $\Delta \mathbf{CreditDebt} = \mathbf{0}$ , which it does imply that  $\mathbf{B} = \mathbf{B}^{Cash}$ ,  $\mathbf{L} = \mathbf{L}^{Cash}$  and  $\mathbf{A} = \mathbf{A}^{Cash}$ ,  $\mathbf{H} = \mathbf{H}^{Cash}$ .

### 3.1 The Surplus as *numéraire*

It is convenient to measure prices,  $\mathbf{p}$  in terms of the purchasing power of the Physical Surplus or Physical Net National Product. Once the *numéraire* is picked to be the Surplus  $\mathbf{S}$  we have by definition that the following relation should hold:

$$\mathbf{S}^T \mathbf{p} = \mathbf{e}^T (\mathbf{B} - \mathbf{A}) \mathbf{p} = 1 \quad (3.7)$$

This simplifies the analysis without changing the substance of the argument. Relative price ratios do not change as the *numéraire* changes. Therefore one

can shift from one *numéraire* to another without having to change the accounting relation or other things<sup>10</sup>. But most importantly with this particular choice for the *numéraire* we have that the wage rate  $w$ , under certain conditions, could also be interpreted as the share of the physical surplus that goes to workers. Furthermore, as it will be shown in the next section because the value of the surplus would be 1, we also have that the Share of the Surplus has the same numerical value as the measured quantities. Last, the financial magnitudes are too in units of the Net National Product and this simplifies further the analysis by making apparent what could otherwise remain hidden.

### 3.2 Prices and profit rates once the distribution (C or d) and the lending and borrowing balances are given

We are now in the position of studying the prices, profit rates and financial conditions that would allow the system to replicate.

We have the “*objective*” observations of the quantities or “*things*”, produced and available at the beginning of the market day: the gross product  $\mathbf{b}$ . Given the knowledge of used past methods,  $\mathbf{A}$  and  $\mathbf{L}$ , the surplus  $\mathbf{S}$  we can compute the prices that would allow replication for next year.

Each producer  $i$  can buy the consumption vector  $\bar{\mathbf{h}}_i$  if he has the purchasing power. The purchasing power necessary to buy a given composite vector a given share of the total Physical Surplus is  $\mathbf{dS}^T \mathbf{p} = \mathbf{Hp}$ .

The agents would have this purchasing power if the prices, the wage rate and *Lending and Borrowing* are such that:

$$\mathbf{dS}^T \mathbf{p} = \overbrace{\begin{bmatrix} d_1 \mathbf{S}^T \mathbf{p} \\ d_2 \mathbf{S}^T \mathbf{p} \\ \vdots \\ d_n \mathbf{S}^T \mathbf{p} \\ d_w \mathbf{S}^T \mathbf{p} \end{bmatrix}}^{\text{Value of Effective Demands}} = \overbrace{\begin{bmatrix} b_1 p_1 - \mathbf{a}_1 \mathbf{p} - \ell_1 w - \Delta \text{CreditDebt}(1) \\ b_2 p_2 - \mathbf{a}_2 \mathbf{p} - \ell_2 w - \Delta \text{CreditDebt}(2) \\ \vdots \\ b_n p_n - \mathbf{a}_n \mathbf{p} - \ell_n w - \Delta \text{CreditDebt}(n) \\ \mathbf{e}^T \mathbf{L} w - \Delta \text{CreditDebt}(n+1) \end{bmatrix}}^{\text{Purchasing Capacity}} \quad (3.8)$$

Written in a more compact notational form we have:

$$\mathbf{dS}^T \mathbf{p} = \left[ \begin{array}{c} (\mathbf{B} - \mathbf{A}) * \mathbf{p} - \mathbf{L} \\ \mathbf{e}^T \mathbf{L} w \end{array} \right] - \Delta \text{CreditDebt} \quad (3.9)$$

<sup>10</sup>Another obvious choice for the *numéraire* could be the Standard Commodity. But while the standard commodity was important for the purpose of PCMC, here it is not essential. The reason being that we are searching for the vectors of prices that would allow the economic system to self-reproduce. For this purpose the choice of the *numéraire* does not really matter. Naturally it is also Sraffa himself that did give to the Net National Product or surplus great importance (Sraffa, 1960a, Sec.12, p.11).

If the prices and the wage rate are expressed in terms of the surplus we have that  $\mathbf{S}^T \mathbf{p} = 1$ , see eq.3.7. If we had a different *numéraire* by dividing left and right sides of the above equation we would obtain the equation below, where the above accounting identity eq. 3.8, may be rewritten as:

$$\begin{array}{c}
 \textit{Physical} \\
 \textit{and} \\
 \textit{Value Distribution} \\
 \\
 \mathbf{d} = \overbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \\ d_w \end{bmatrix}} \\
 \\
 = \overbrace{\begin{bmatrix} (\mathbf{B} - \mathbf{A}) & -\mathbf{L} \\ \mathbf{0}_{1 \times n} & \mathbf{e}^T \mathbf{L} \end{bmatrix}}^{\textit{Purchasing Capacity}} \underbrace{\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}}_{\textit{Income}} - \underbrace{\begin{bmatrix} \Delta \textit{CreditDebt}_{n \times 1} \\ \Delta \textit{CreditDebt}(n+1) \end{bmatrix}}_{\textit{Lending and Borrowing}}
 \end{array} \tag{3.10}$$

The distributional vector  $\mathbf{d}$  is the distribution of the produced surplus  $\mathbf{S}$  among producers and workers both in value terms and as fraction of the physical surplus  $\mathbf{S}$ .

### 3.3 An algorithm for the determination of self replacing conditions with credit and debt

Eq. 3.10 is the core of the analysis. Once the prices and wage rates are given the system is in self-replacing state if the vector  $\Delta \textit{CreditDebt} = \mathbf{0}$  and the vector  $\mathbf{d}$  has each element greater or equal 0 and its sum equal to 1. The domain of all possible vectors  $[\mathbf{p}, w]^T$  for which the system is in a self-replacing condition can be found by trying all possible combinations of  $\mathbf{d}$  (Zambelli, 2016). The knowledge of the distribution vector and of the lending and borrowing among the producers and among the workers are enough information for the computation of prices, wage rates and profit rates that would allow the system to be in the self-replacing state.

Here the problem is reversed: once the prices and the wage rate are given the problem is to find a vector  $\Delta \textit{CreditDebt}$  so that  $\mathbf{d} \geq \mathbf{0}$  and  $\mathbf{e}^T \mathbf{d} = 1$ .

HERE DESCRIPTION OF THE ALGORITHM - With coherence with the accounting framework described in section 3.

$$\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} (\mathbf{B} - \mathbf{A}) & -\mathbf{L} \\ \mathbf{0}_{1 \times n} & \mathbf{e}^T \mathbf{L} \end{bmatrix}^{-1} (\mathbf{d} + \Delta \textit{CreditDebt}) \tag{3.11}$$

Once the inverted matrix is expanded we have the following:

$$\begin{bmatrix} \mathbf{p}_d \\ w_d \end{bmatrix} = \begin{bmatrix} (\mathbf{B} - \mathbf{A})^{-1} & -\frac{\mathbf{L}}{\mathbf{e}^T \mathbf{L}} \\ \mathbf{0}_{1 \times n} & \frac{1}{\mathbf{e}^T \mathbf{L}} \end{bmatrix} (\mathbf{d} + \Delta \text{CreditDebt}) \quad (3.12)$$

Which can be written explicitly in terms of the distribution and financial balances of the industries and of the workers:

$$\mathbf{p}_d = (\mathbf{B} - \mathbf{A})^{-1} (\mathbf{d}_{n \times 1} + \Delta \text{CreditDebt}_{n \times 1}) + \frac{\mathbf{L}}{\mathbf{e}^T \mathbf{L}} (d_w + \Delta \text{CreditDebt}(n+1)) \quad (3.13)$$

where:  $\mathbf{p}_d$  is the vector of prices consistent or determined by the distribution  $\mathbf{d}$  of the surplus  $\mathbf{S}$ ;  $\mathbf{d}_{n \times 1}$  is the distribution among the  $n$  industries;  $d_w$  is the distribution of the surplus to the workers – given our choice of

$$w_d = \frac{1}{\mathbf{e}^T \mathbf{L}} (d_w + \Delta \text{CreditDebt}(n+1)) \quad (3.14)$$

#### 4 Example – Sraffa’s Standard System - Self-Replacing without and with borrowing and lending

If we exclude the possibility of borrowing and lending we would be in the conventional and traditional world of the Sraffian Schemes. The only difference being, with respect to what normally found in the literature, the absence of the imposition of a uniform rate of profits.

Here we study the situation where prices and wages are given and try to determine the credit and debt conditions that would allow the system to be in a self replacing state.

Let us take an example using the numbers defining a Standard System in Sraffa (1960a, p.19). The economic production system is given by the following:

$$\mathbf{A} = \begin{bmatrix} 120 & 160 & 80 \\ 40 & 100 & 120 \\ 40 & 40 & 200 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} \frac{4}{16} \\ \frac{4}{16} \\ \frac{8}{16} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 240 \\ 360 \\ 480 \end{bmatrix} \quad (4.1)$$

The first row would be the iron industry (industry and commodity 1), the second the coal industry (industry and commodity 2) and the third the wheat industry (industry and commodity 3). The columns of  $\mathbf{A}$  would indicate the means of production used. The first column would be iron, the second coal and the third wheat. These are the quantities we observe. At the end of the production period (the harvest) producers have produced quantities  $\mathbf{b} = [240, 360, 480]^T$  which have to be exchanged to organize production for the next year.

Following Sraffa's method, we do not know what would happen during the market day, but if production has to be repeated the gross production  $\mathbf{b}$  have to be exchanged in such a way that producers can buy the means of production  $\mathbf{A}$  and labour  $L$ .

The total Surplus to be distributed after exchanges have taken place and  $\mathbf{A}$  has been put aside for production is  $\mathbf{S} = [40, 60, 80]^T$ .

Clearly the system would be in a self replacing state for any distribution vector  $\mathbf{d}$  that would not imply any debt and credit relations.

#### 4.1 After the First Harvest and Before the First Market Day

We as the *man from the moon*, before the market day begins, are asked to say something about the self replacing conditions of the system. We are also told that *IOUs* could be issued. We are also told that t the moment there are not old *IOUs*. We would use the equations above and our algorithm to compute the domains for self replacing states.

Figures 4.1 report the domain (blue triangles) of the values of prices that would allow the system to be in the self replacing state without resorting to financial balances (with the value of the wage rate going from 0 to 1). There is a unique quadruple of prices  $[\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{w}]$ . The domains of the figures have to be interpreted as two dimension projection of a 4 dimension vectors. These projections (the blue triangles) have to be seen as necessary conditions but not as sufficient ones. That is, it is not enough to pick values belonging to the blue triangles to ensure self reproduction without credit and debt. But once three values have been picked that belong to the blue triangles, say  $\bar{p}_1, \bar{p}_2, \bar{p}_3$  the wage rate would be uniquely determined and would have a value belonging to the interval  $[0, 1]$ .

The domain of self replacing profit rates associated to all possible combinations of self replacing prices are given by figures 4.2.

Any point falling on the red area (both in the price domains of figure 4.1 and in the profit rates domain of figure 4.2 would imply credit and debt to allow for self reproduction.

Fig. 4.3 gives the relation between the distribution of the surplus to workers  $d_w \in [0, 1]$  and the wage rate  $w$ . When the wage rate is greater than  $d_w$  it implies that the workers have "lended" labour force to some producers.

Fig. 4.4 show the debt and credit associated to the set of prices in relation to the distribution of the surplus to workers  $d_w$ .

#### 4.2 After the First Harvest and After the First Market Day

When the market is closed we are given the prices and wage rate in which the exchanges took place.

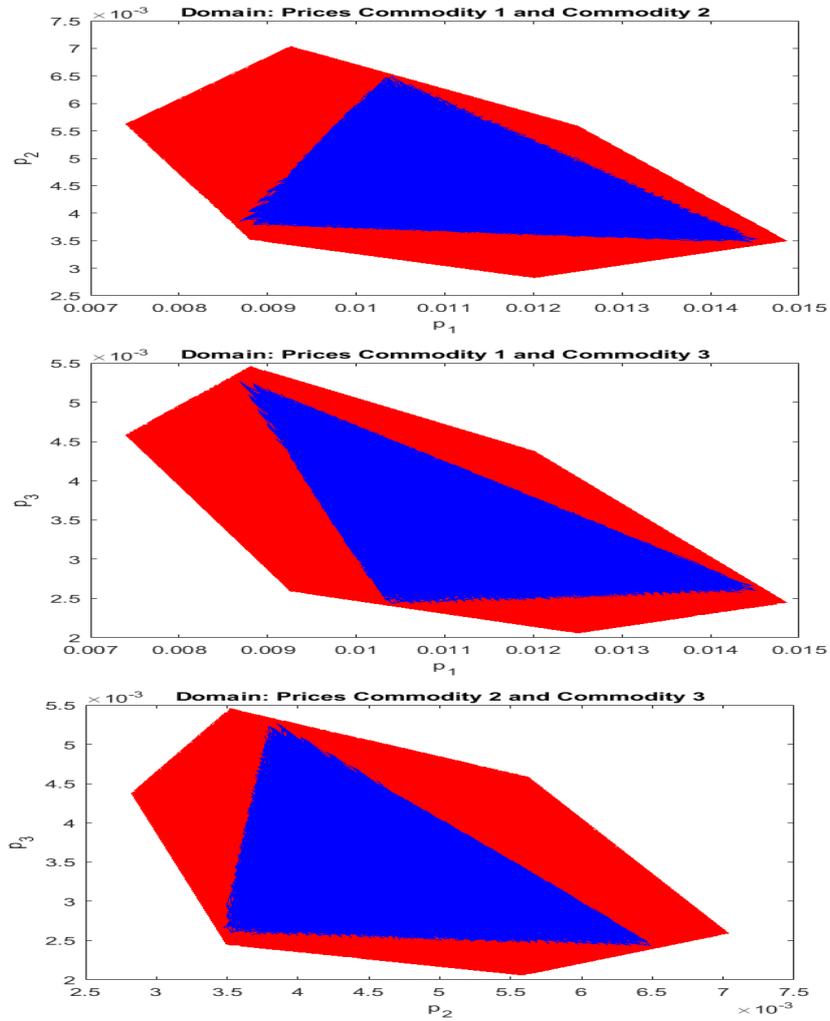


Fig. 4.1: The blue triangles are the domains of prices that would allow self reproduction. The wage rate  $w$  can take any value from 0 to 1..

The triple  $\mathbf{A}, \mathbf{L}, \mathbf{b}$  used for this figure is reported above in eq. 4.1 (Sraffa, 1960a, p.19).

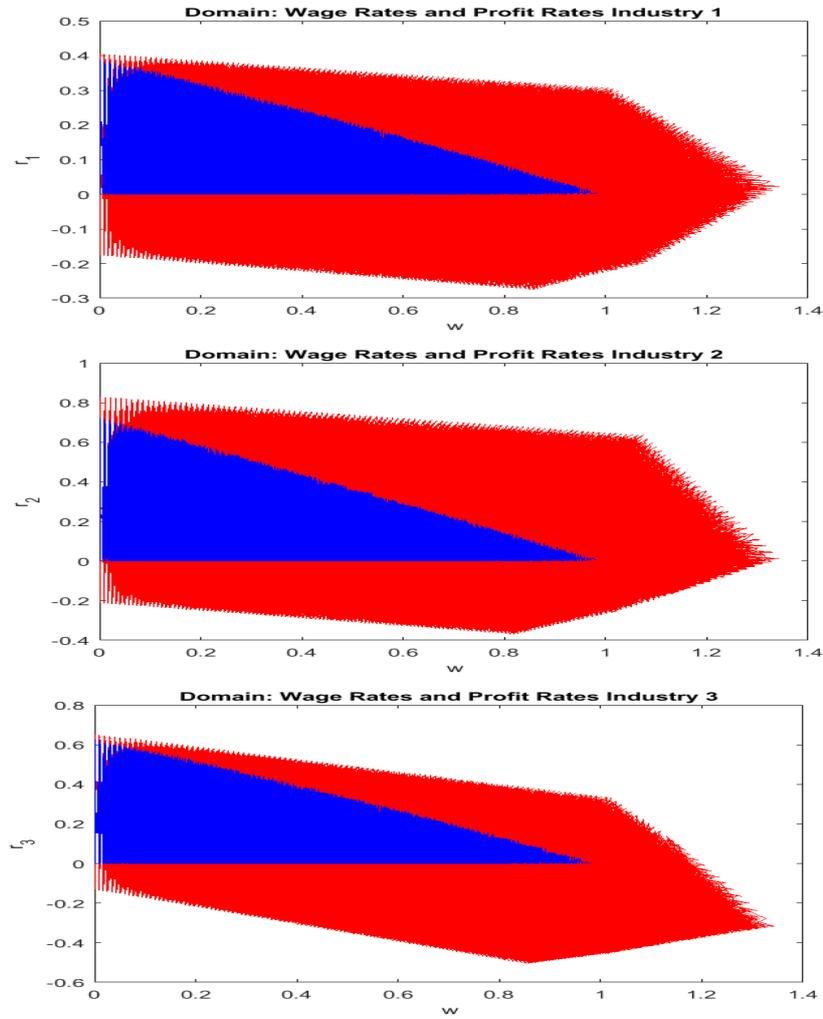


Fig. 4.2: The blue triangles are the domains of profit rates that would allow self reproduction. The wage rate  $w$  can take any value from 0 to 1.

The triple  $\mathbf{A}$ ,  $\mathbf{L}$ ,  $\mathbf{b}$  used for this figure is reported above in eq. 4.1 (Sraffa, 1960a, p.19).

Table 4.1: *Prices and wage rate observed after the first market day*

	Prices	Prof.Rates	$\Delta$ CreditDebt	Shares ( $\mathbf{d}$ )
1. Iron	9.73* ( $p_1$ )	-10 % ( $r_1$ )	-0.25	0 % ( $\mathbf{d}_1$ )
2. Coal	6.82* ( $p_2$ )	70 % ( $r_2$ )	0.35	61 % ( $\mathbf{d}_2$ )
3. Wheat	2.51* ( $p_2$ )	- 17 % ( $r_3$ )	- 0.20	0 % ( $\mathbf{d}_3$ )
4. Workers	0.49 ( $w$ )	-	0.10	39 % ( $\mathbf{d}_w$ )

(\*) Multiplied by  $\times 10^{-3}$

These prices are the first column of table 4.1. Based on this information we are asked whether there was self replacing (or whether it was a possible outcome). We compute the associated profit rates, the credit and debt values and the associated shares. These values are reported in table 4.1.

### 4.3 After the Second Harvest and Before the Second Market Day

After the second harvest we have the same situation as before, where the quantity  $\mathbf{b}$  has been produced and has to be exchanged so as to organize production for next year. A substantial difference is that there is a debt that has to be paid back.

From table 4.1, we see that the iron industry owes (-) 0.25 to the coal industry and to the workers and that the wheat industry owes (-) 0.20 to the coal industry and to the workers.

We can now search for the domain of prices that would allow the economic system to be in the self replacing state and at the same time pay the debt.

In figures 4.5 and 4.6 we report the new price domains. The broader yellow triangles are domains where the debt is repaid without payment of interest. The colored in cyan areas are the domains where 20% of interest rates were paid.

As it was probably to be expected the possibility to repay the debt requires limits to the prices and to distribution.

## 5 Conclusions

This is a **work in progress**. We have investigated the conditions that would allow the economic system to be in a self-replacing state also when profit rates are negative.

TO BE COMPLETED ...

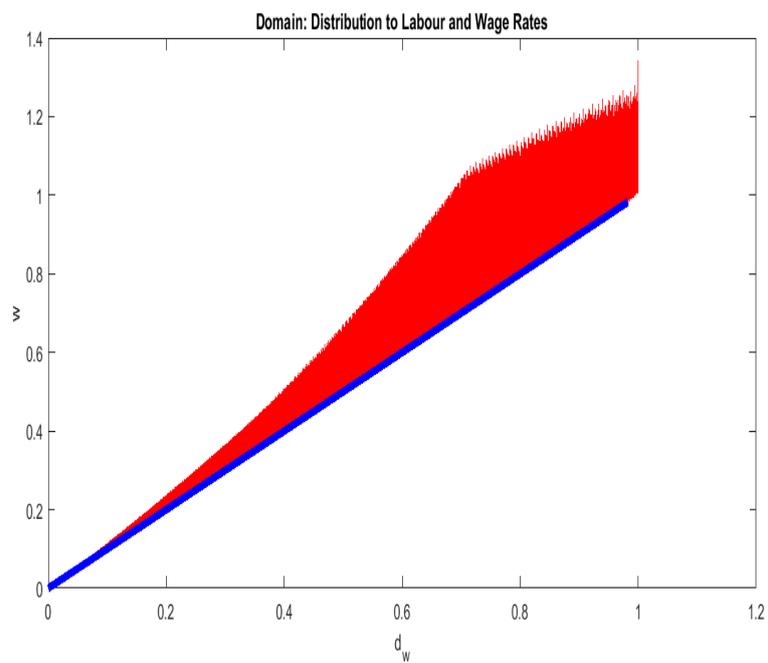


Fig. 4.3: Distribution to labour and wage rates. The blue line is the the identity line where  $w = d_w$ .

The triple  $\mathbf{A}$ ,  $\mathbf{L}$ ,  $\mathbf{b}$  used for this figure is reported above in eq. 4.1 (Sraffa, 1960a, p.19).

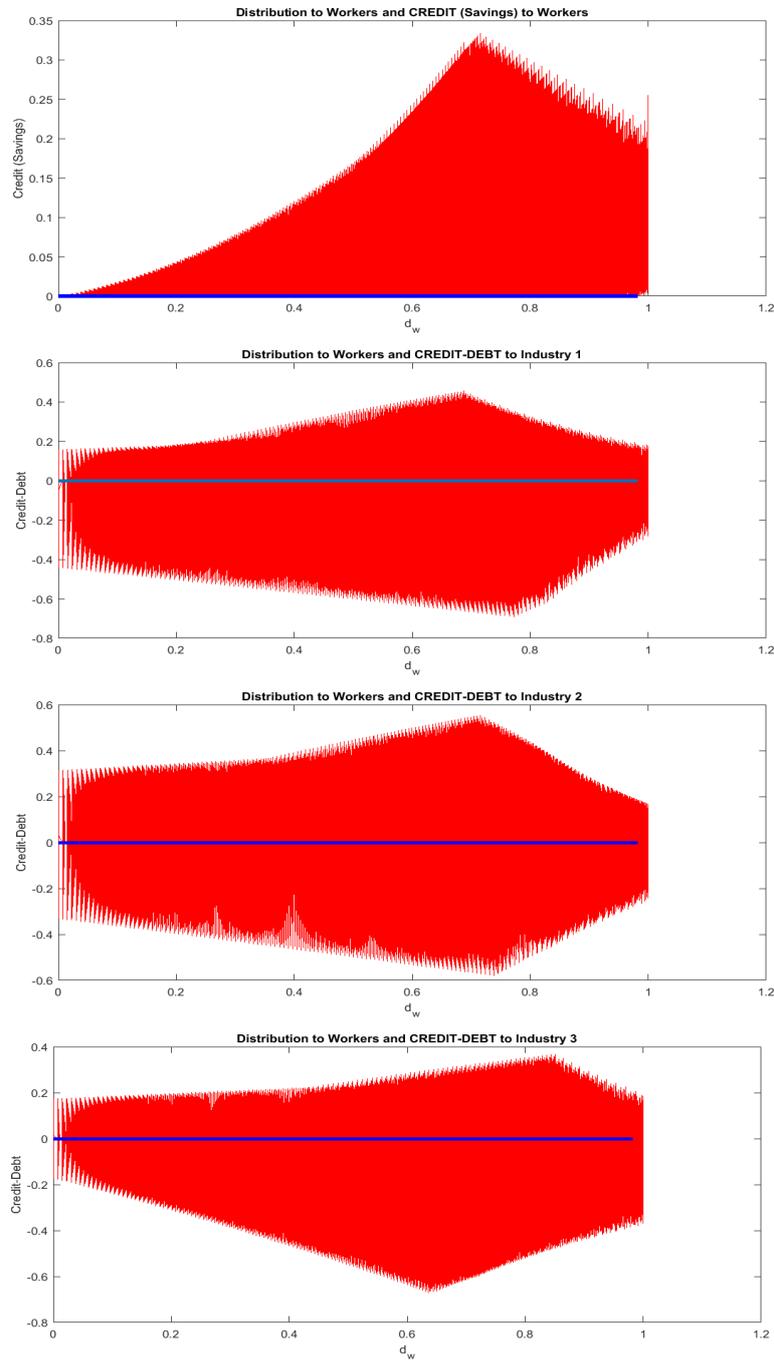


Fig. 4.4: Credit and Debt associated to the distribution to the workers  $d_w$ . The blue line corresponds to zero credit and debt relations. The red area the region of credit or debt values.

The triple  $\mathbf{A}$ ,  $\mathbf{L}$ ,  $\mathbf{b}$  used for this figure is reported above in table 4.1 (Sraffa, 1960a, p.19).

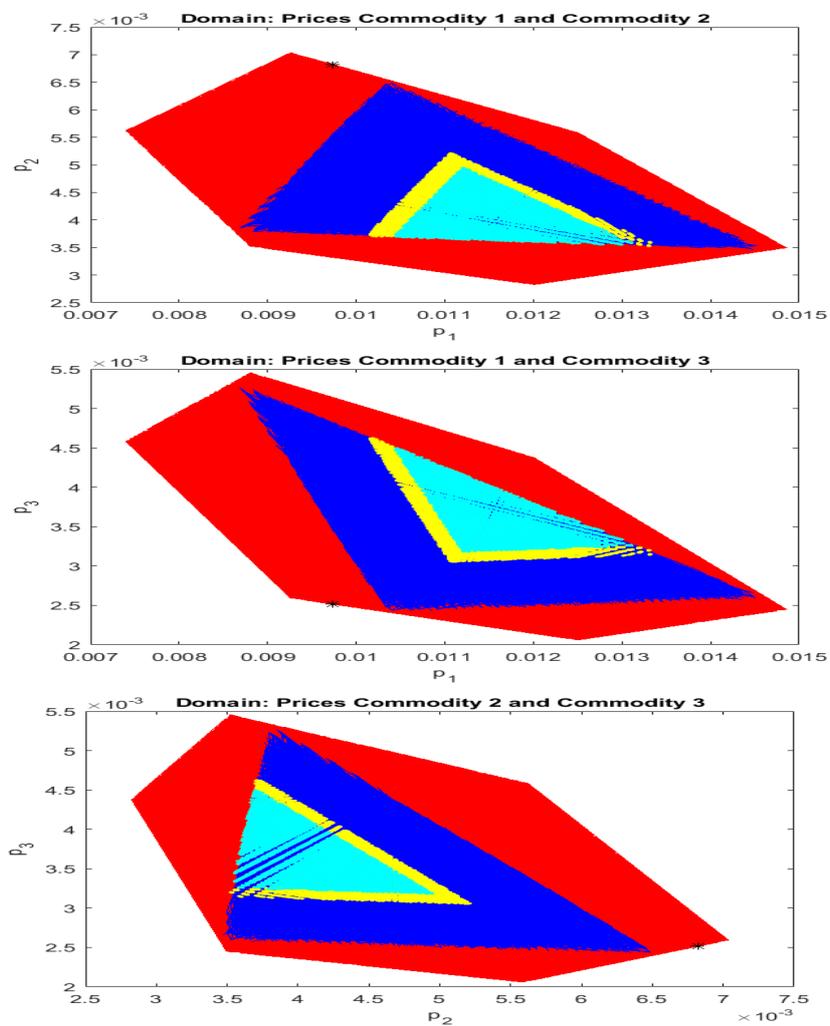


Fig. 4.5: The yellow triangles are the domains of prices that would allow self reproduction and the writing off of the debts. The star \* represents the conditions occurred after the first market day. See table 4.1 The cyan triangles are relative to a payment of 20% interest rates. .

The triple  $\mathbf{A}$ ,  $\mathbf{L}$ ,  $\mathbf{b}$  used for this figure is reported above in eq. 4.1 (Sraffa, 1960a, p.19).

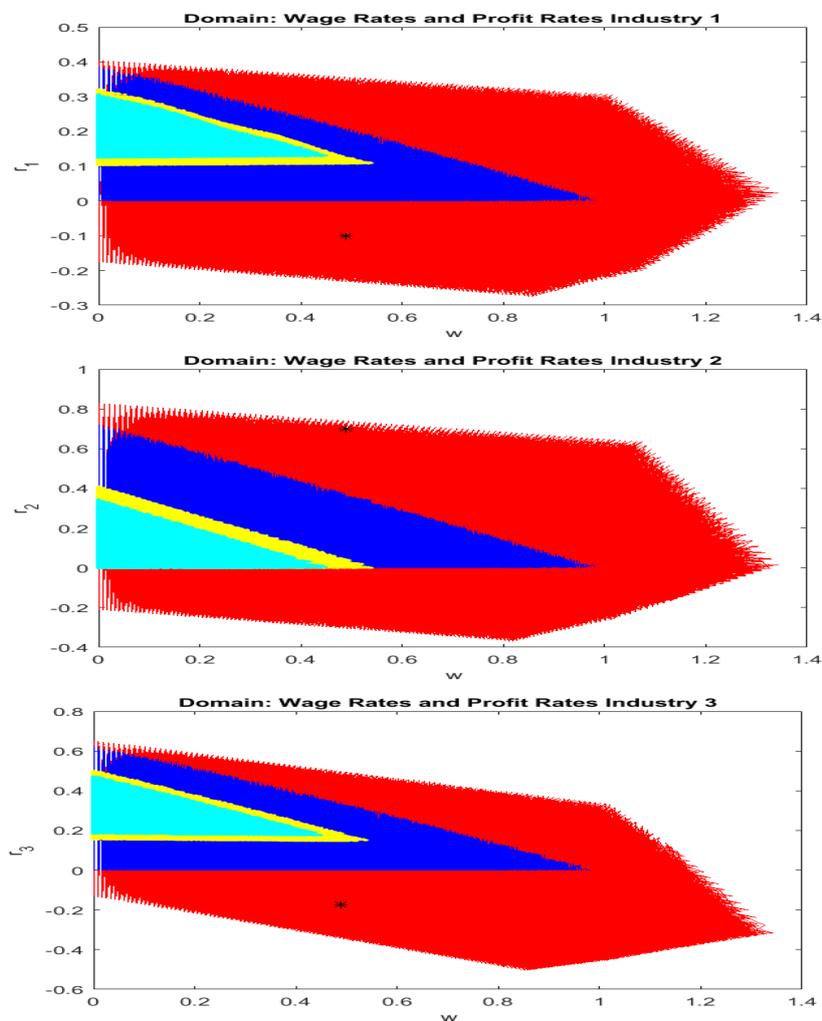


Fig. 4.6: The yellow triangles are the domains of prices that would allow self reproduction and the writing off of the debts. The star \* represents the conditions occurred after the first market day. See table 4.1. The cyan triangles are relative to a payment of 20% interest rates.

The triple  $\mathbf{A}, \mathbf{L}, \mathbf{b}$  used for this figure is reported above in eq. 4.1 (Sraffa, 1960a, p.19).

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