Old and new ways of computing the gamma function

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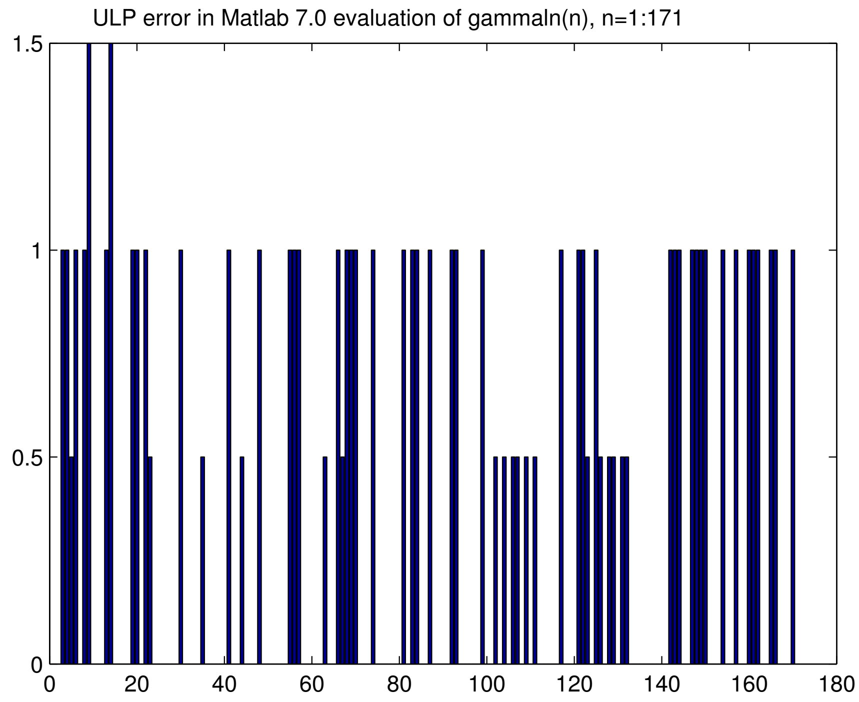
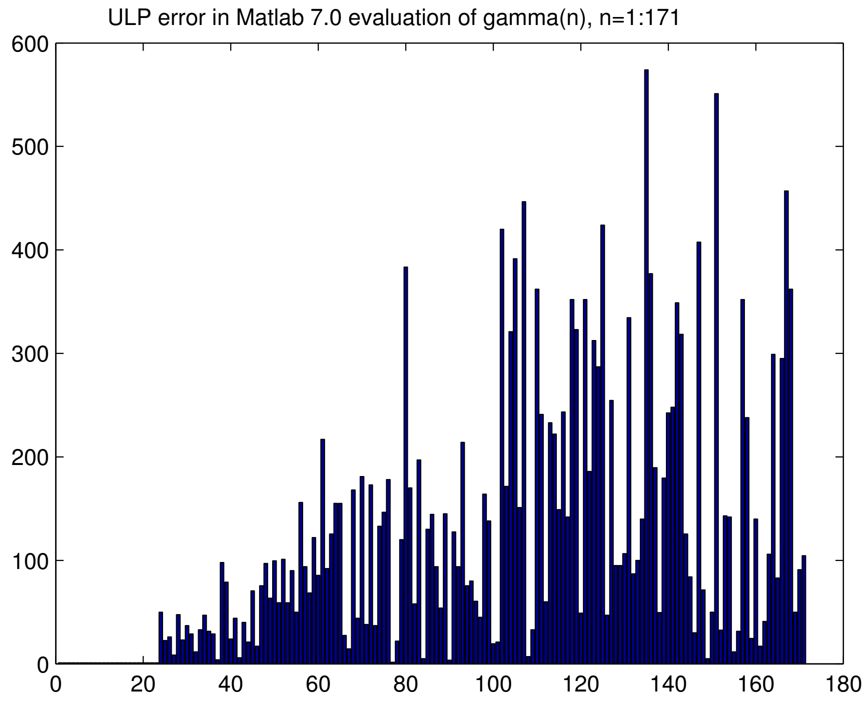
What got me on this topic

- Problem 5 of Trefethen's 100-Digit Challenge requires about 100 evaluations of $1/\Gamma(z)$ for z on the unit circle.

- Matlab and Octave do not have complex Γ, so I used a series for $1/\Gamma$ from the Handbook.

- Internet consensus is that a routine by Paul Godfrey implementing a Lanczos approximation is best available for general complex z.
  I didn't even know what a Lanczos approximation was.





You can't accurately evaluate $\Gamma(x)$, even when x is real, by the popular method (Matlab, Octave, etc) of accurately evaluating $\log \Gamma(x)$ and using the formula $e^{\log \Gamma(x)}$.

What really got me hooked

- When writing a book on the 100-Digit Challenge, we tried doing all the problems to 10 000 digits.
- The bottleneck in Problem 5 (using Pari/GP, Maple, or Mathematica) was in computing Γ, more precisely, in computing the *first* value of Γ.
- Those packages all use (or seem to use) Stirling's formula.

Has nothing useful happened in the last 270 years except decent error estimates (Hare[1997]) for Stirling's formula?

Time (Pari/GP 2.2.10) to compute Γ to d digits

	Time in seconds	
d	First value	Next value
500	0.12	< 0.01
1000	0.85	0.06
2000	6.45	0.33
4000	50.37	1.75
8000	392.11	9.94
c	$1.2\ 10^{-9}$	$2.7\ 10^{-9}$
p	2.95	2.45

Time taken is approximately cd^p. This behaviour is typical of most multiprecision packages: first evaluation is much more expensive than later evaluations.

Software for mathematical functions: two games and their rules

IEEE double precision

- Hardware arithmetic
- Precision 53 bits, approximately 16 digits
- 64 bits available for intermediate results
- Constants required by algorithm are precomputed; cost of this irrelevant
- Cost measured as average number of flops

Arbitrary precision

- Software arithmetic
- Precision typically $32n$ bits, approximately $d = 9.6n$ digits
- n can be increased for intermediate results
- Constants must be recomputed whenever precision is increased
- Cost measured as approximately cd^p, for some constants c, d

Outline of this talk

- How Stirling did it
- Some other intriguing ideas (overview)
- The Lanczos approximation and others like it (detailed)
- Representations of rational functions with known poles
- Unified classification of approximations to $z!$

An old question

$$\begin{array}{c|c|r}
0 & 1 & 1 \\
\frac{1}{2} & ?? & ? \\
1 & 1 & 1 \\
2 & 1 \times 2 & 2 \\
3 & 1 \times 2 \times 3 & 6 \\
4 & 1 \times 2 \times 3 \times 4 & 24 \\
5 & 1 \times 2 \times 3 \times 4 \times 5 & 120 \\
6 & 1 \times 2 \times 3 \times 4 \times 5 \times 6 & 720
\end{array}$$

What is the numerical value?

What is the analytical definition??

Solution by polynomial interpolation: Stirling[1730]

- The differences of $n!$ form a divergent progression $\Rightarrow$ polynomial interpolation of $n!$ will not work.

- Therefore interpolate $\log_{10} n!$ instead, whose differences can form a rapidly convergent progression.

- But the differences of the initial terms are slowly convergent.

- Therefore calculate $\log_{10}(10.5!)$ instead, and assume that the recursion formula $x! = x(x-1)!$ holds for non-integer x.

Stirling's interpolation of $\log_{10} 10.5!$, **using** $\log_{10} n!,\ n = 5, 6, \ldots, 16.$

	Logarithms	2nd diff.	4th diff.	6th diff.	8th	10th
	2.0791812460					
	2.8573324964	669467896				
	3.7024305364	579919470	21154180			
	4.6055205234	511525224	14443928	2568588		
	5.5597630328	457574906	10302264	1446210	541511	
A	6.5597630328	413926852	7606810	865343	259252	156590
B	7.6011557180	377885608	5776699	543728	133583	65082
	8.6803369640	347621063	4490316	355696	72996	
	9.7942803163	321846834	3559629	240660		
	10.9404083520	299632234	2869602			
	12.1164996111	280287236				
	13.3206195938					

$\frac{1}{2}(A_0 + B_0) + \sum_{k=1}^{K} (-16)^{-k} \binom{2k-1}{k-1} (A_{2k} + B_{2k})$ "Bessel's interpolation formula"

$$\log_{10} 10.5! \approx 7.08045937540 - 494882787 + 1568380 - 34401 + 2098 - 266$$

Stirling's conclusion

$$\begin{aligned}\log_{10} 10.5! &\approx 7.07552590569\\ 10.5! &\approx 11899423.08\\ 0.5! &\approx \frac{11899423.08}{10.5 \times 9.5 \times 8.5 \times 7.5 \times 6.5 \times 5.5 \times 4.5 \times 3.5 \times 2.5 \times 1.5}\end{aligned}$$

"From this it is established that the term between 1 and 1 is .8862269251, whose square is .7853 . . .etc., namely the area of a circle whose diameter is one. And twice that term, 1.7724538502, [. . .] is equal to the square root of the number 3.1415926 . . .etc., which denotes the circumference of a circle whose diameter is one." *(English translation by Tweddle[2003])*

Stirling is claiming, on the basis of numerical evidence alone, that $\frac{1}{2}! = \frac{1}{2}\sqrt{\pi}$.

What is now called "experimental mathematics" was invented in 1730!

Stirling's formula for logarithmic sums [1730]

"But it will be shown in what follows how series of this type can be interpolated without logarithms."

$$\log(x+h)+\log(x+3h)+\log(x+5h)+\cdots+\log(z-h)=S(z,h)-S(x,h)$$

$$S(x,h)=\frac{x\log(x)}{2h}-\frac{x}{2h}-\frac{h}{12x}+\frac{7h^3}{360x^3}-\frac{31h^5}{1260x^5}+\frac{127h^7}{1680x^7}-\frac{511h^9}{1188x^9}+\cdots$$

The coefficients $A=-\frac{1}{12}$, $B=+\frac{7}{360}$, $C=-\frac{31}{1260}$ are given by:

$-\frac{1}{3.4}=A$	1 1
$-\frac{1}{5.8}=A+3B$	1 3 3 1
$-\frac{1}{7.12}=A+10B+5C$	1 5 10 10 5 1
$-\frac{1}{9.16}=A+21B+35C+7D$	1 7 21 35 35 21 7 1
etc.	Odd-numbered binomial coefficients

Stirling's formula for $\log n!$

$$\log\left(z-\tfrac{1}{2}\right)! = S\left(z,\tfrac{1}{2}\right) + \tfrac{1}{2}\log(2\pi)$$

- Stirling [1730] only guessed the value of the constant, and did not recognize the relationship between his coefficients $A, B, C, \ldots$ and the Bernoulli numbers [1713]: these refinements are due to De Moivre [1730].
- Stirling's formula for sums of logarithms is a special case of Maclaurin's version (midpoint rule) of the Euler-Maclaurin summation formula.
- What is nowadays called "Stirling's formula" is a special case of Euler's version (trapezoidal rule) of the Euler-Maclaurin summation formula, and is also due to De Moivre (after he saw Stirling's version).
- Stirling's formula answers the numerical but not the analytical question.
- The series is not convergent, only asymptotic.

From $\log\left(z-\frac{1}{2}\right)!$ to $z!$

It is very convenient that the series for $S(z,h)$ contains only odd powers. Therefore one will in practice always use Stirling's formula as $\left(z-\frac{1}{2}\right)! = e^{\log(z-1/2)!}$, accepting either precision loss or the use of extended precision when z is large. In theory the precision loss can be avoided by replacing z by $z+\frac{1}{2}$ and taking antilogarithms.

$$z! = \sqrt{2\pi}\left(z+\tfrac{1}{2}\right)^{z+1/2} e^{-z-1/2}\, f_{1/2}(z),$$

where $f_{1/2}$ can be expanded in the form

$$f_{1/2}(z) = 1 - \frac{1}{24(z+1/2)} + \frac{1}{1152(z+1/2)^2} + \frac{1003}{414720(z+1/2)^3} - \cdots$$

The coefficients are complicated, but in principle this is the special case $a=\frac{1}{2}$ of a factorization

$$z! = F_a(z) f_a(z), \quad F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a},$$

where f_a satisfies $f_a(z) = 1 + O((z+a)^{-1})$ and has an asymptotic expansion in negative powers of $z+a$.

A property of the factors F_a and f_a

If $a + b > 0$, $\quad \dfrac{F_a(b)}{F_b(a)} = (a+b)^{a-b} = \dfrac{f_b(a)}{f_a(b)}$

Stirling's formula with backward recursion

Since high precision can only be obtained from Stirling's formula when z is large, we pick a large enough m and evaluate $(z+m)!$ by the formula. Then the recursion formula gives

$$\frac{\sqrt{2\pi}\left(z+m+\frac{1}{2}\right)^{z+m+1/2} e^{-z-m-1/2} f_{1/2}(z+m)}{(z+1)(z+2)\cdots(z+m)}.$$

Note that some of the poles of the approximation obtained by truncation the series for $f_{1/2}$ are the same as the first m poles of $z!$.

Take-home message #1

A slightly generalized form of Stirling's formula, namely:

$$z! = F_a(z) f_a(z), \quad F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a},$$

where f_a satisfies $f_a(z) = 1 + O\left((z+a)^{-1}\right)$, is in principle at the heart of almost all competitive methods for evaluating the gamma function.

Approximations based on $z! = F_a(z) f_a(z), \quad F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a}$

- The original Stirling's formula is
$$z! = F_{1/2}(z)\left(1 - \frac{1}{24(z+1/2)} + O\left(z^{-2}\right)\right)$$
- De Moivre's version, now ubiquitous, of Stirling's formula is
$$z! = F_0(z)\left(1 + \frac{1}{12z} + O\left(z^{-2}\right)\right)$$
- The two-term Lanczos approximation is
$$z! = F_2(z)\left(0.999779 + \frac{1.084635}{z+1}\right)(1+\eta), \quad |\eta| < 0.00024, \ \operatorname{Re} z \geqslant 0$$
- The two-term Spouge approximation is (note $e/\sqrt{2\pi} \doteq 1.08443755$)
$$z! = F_2(z)\left(1 + \frac{e}{\sqrt{2\pi}(z+1)}\right)(1+\eta(z)), \quad |\eta(z)| < \frac{0.0143}{\operatorname{Re}(z+2)}, \ \operatorname{Re} z \geqslant 0$$

Integral formulas (old)

It is sufficient to evaluate $z!$ for $\mathrm{Re}(z) \geqslant 0$, since $\quad (-z)! = \dfrac{\pi z}{z! \sin(\pi z)}$

- Euler[1729] $\quad z! = \displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^z \, dx.$

  Reformulated by Legendre[1809] as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$

  The normalization of the gamma function to $\Gamma(n+1)$ *instead of* $\Gamma(n)$ *is void of any rationality.* – Lanczos[1964]; like Lanczos, we use $z!$ most of the time.

- Laplace[1812], Hankel[1864] $\quad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C s^{-z} e^t \, dt$

  Contour in general: anticlockwise, such that the negative real axis is inside.

  Specific contours: Trefethen[2001]; also lecture by Weideman, later today.

Integral formulas (modern)

- Lanczos[1964] $\quad z! = (z+a)^{z+1} e^{-z-a} \int_0^e (v(1-\log v))^z v^{a-1}\,dv$
  The constant $a \geqslant 1$ is to be chosen conveniently.
- Spouge[1994] $\quad z! = \frac{F_a(z)}{2\pi i} \int_C \frac{u!}{(u-z)F_a(u)}\,du$
  F_a as before; constant $a \geqslant 0$ to be chosen conveniently; contour has negative integers outside but z inside.
- Temme[1996] $\quad \dfrac{1}{\Gamma(z)} = \dfrac{e^z z^{1-z}}{2\pi} \displaystyle\int_{-\pi}^{\pi} e^{-z(1-\theta/\tan\theta+\log(\theta/\sin\theta))}\,d\theta$

Each formula is to be used with its own quadrature method, different for all.

Note significant difference from old formulas: the factor $z^z e^{-z}$ of Stirling's formula appears explicitly outside the integral.

Plausible methods for computing $\Gamma(z)$ for $\operatorname{Re} z \geqslant 1$

- Stirling's formula with backward recursion
- Luke's method [1970]: Use Padé approximations for the meromorphic factors of the two incomplete gamma functions $\gamma(a, z+1) = \int_0^a t^z e^{-t}\,dt$ and $\Gamma(a, z+1) = \int_a^\infty t^z e^{-t}\,dt$, then add them together. (Thanks to Annie Cuyt for telling me this!)
- Numerical evaluation of integral representations
- Special-purpose quadratures for certain integral representations (including rational approximations of $e^{-t},\ t \in [0, \infty]$ — Trefethen, Weideman et al.)

Another look at Stirling's formula with backward recursion

Let $f_{1/2}^{[n]}$ be the truncation of the asymptotic series of $f_{1/2}(z)$ after the term in $\left(z+\frac{1}{2}\right)^{-n}$. Then Stirling's formula with backward recursion is

$$
\begin{aligned}
z! &\approx \frac{F_{1/2}(z+m)f_{1/2}^{[n]}(z+m)}{(z+1)(z+2)\cdots(z+m)} \\
&= \frac{F_{m+1/2}(z)\left(z+m+\frac{1}{2}\right)^{m} f_{1/2}^{[n]}(z+m)}{(z+1)(z+2)\cdots(z+m)}
\end{aligned}
$$

Since the last term in $f_{1/2}^{[n]}(z+m)$ is a multiple of $\left(z+m+\frac{1}{2}\right)^{-n}$, we are in effect approximating $f_{m+1/2}$ by a rational function which has m correct poles at $-1,-2,\ldots,-m$ and a spurious pole of multiplicity $(n-m)$ at $(-m-\frac{1}{2})$. This is a compelling reason to stop when $m=n$ even though the smallest term has not yet been reached. We will call the (m,m) rational function thus obtained the *balanced* form of Stirling's approximation.

Lanczos series

Based on Lanczos integral formula $z! = (z+a)^{z+1} e^{-z-a} \int_0^e (v(1-\log v))^z v^{a-1}\,dv$; substitute $v(1-\log v) = \cos^2\theta$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$; expand even part of v^{a-1} as a cosine series (very clever, this). Integrate series term-by-term, the factor $z!/(z+\frac{1}{2})!$ appears, so we have a formula for $(z+\frac{1}{2})!$; replace $z+\frac{1}{2}$ by z, and voilà!
(All this takes six pages of wizardry in Lanczos's paper.)

- Lanczos series $\quad z! = F_a(z) f_a(z)$, $a \geqslant \frac{1}{2}$ to be chosen conveniently, where
$$f_a(z) = \rho_0 + \rho_1 \frac{z}{z+1} + \rho_2 \frac{z(z-1)}{(z+1)(z+2)} + \cdots$$ and the ρ_k depend on a.
This series converges for all $\operatorname{Re} z \geqslant 0$.

Lanczos suggests: *Decide how many terms $m = m(a)$ will be retained after the constant (a good choice is an integer near a); then express the finite series in terms of partial fractions*.

Lanczos approximation

$z! = F_a(z)\tilde{f}_a(z) + \tilde{\eta}_a(z)$, where $F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2}e^{-z-a}$, and for certain constants depending on a,

$$\tilde{f}_a(z) = \tilde{c}_0 + \frac{\tilde{c}_1}{z+1} + \frac{\tilde{c}_2}{z+2} + \cdots + \frac{\tilde{c}_m}{z+m}.$$

- To follow the paper is hard; to compute the coefficients the way Lanczos says is rather complicated: "Lanczos' formulas for the c_k take about a page to write out" — Spouge[1994]; but the final approximation is very easy to use.
- The way that this approximation is derived — truncate an exact Fourier series — suggests that the approximation will be near-best among approximations having this form.
- Like the balanced Stirling's formula, the Lanczos approximation has correct poles only.

Pari/GP code for the Lanczos approximation

```
{ faux(z,a) = (z+a)^z*sqrt(z+a)*exp(-z) }
{ rho(m,a)=local(c,c0,c1,s0,s1); s1=1;
  c1=c=vector(m,k,  s0=s1; s1=s1*(k-1/2); s0/faux(k-1,a) );
  forstep(k=m,2,-1, c[k]=2*c[k]-c[k-1]);
  for(j=3,m, c0=c1; c1=c;
    forstep(k=m,j,-1, c[k]=4*c[k]-2*c[k-1]-c0[k-2] )
    );
  c }
{ fac(z,a,m)=local(r,s,t);
  r=rho(m,a); s=r[1]/2; t=1;
  for(k=1,m-1, t=t*(z-k+1)/(z+k); s=s+t*r[k+1] );
  2.*s*faux(z,a) }
```

Calculation of coefficients involves m evaluations of F_a at integer arguments and about m^2 binary-shifted additions. Evaluation of approximand takes one evaluation of F_a and about $3m$ multiplications.

The Spouge approximation

Based on Spouge's integral formula $z! = \frac{F_a(z)}{2\pi i}\int_C \frac{u!}{(u-z)F_a(u)}\,du$ which is of course just a special case of Cauchy's integral formula. Spouge's idea is very simple: let C_m be a contour that includes z and the first m poles $-1, -2, \ldots, -m$ of $z!$. Using residue calculus, we get:

$$z! = F_a(z)\left(1 + \sum_{k=1}^{m} \frac{(-1)^{k-1}}{(k-1)!(z+k)F_a(-k)} + \frac{1}{2\pi i}\int_{C_m} \frac{u!}{(u-z)F_a(u)}\,du\right)$$

This gives in one shot Spouge's approximation $z! = F_a(z)\hat{f}_a(z) + \hat{\eta}_a(z)$, with

$$\hat{f}_a(z) = 1 + \frac{\hat{c}_1}{z+1} + \frac{\hat{c}_2}{z+2} + \cdots + \frac{\hat{c}_m}{z+m}, \qquad \hat{c}_k = \frac{(-1)^{k-1}}{(k-1)!(z+k)F_a(-k)},$$

and an integral representation for the error $\hat{\eta}_a(z)$. (There is of course a lot of analysis involved in getting a practically useful estimate for $|\hat{\eta}_a(z)|$.)

Since $F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2}e^{-z-a}$, the process is valid only for $a > m$, and the largest legal value of m, which is also the best one, is $m = \lceil a \rceil - 1$.

Take-home message #2

The Lanczos and Spouge approximations have the same form,

$$z! \approx \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a} \left(c_0 + \frac{c_1}{z+1} + \frac{c_2}{z+2} + \cdots + \frac{c_m}{z+m} \right),$$

which respects the poles of $z!$, but not quite the same coefficients.

A derivation of the Lanczos approximation in the spirit of Stirling

Take another look at the Lanczos series

$$f_a(z) = \rho_0 + \rho_1 \frac{z}{z+1} + \rho_2 \frac{z(z-1)}{(z+1)(z+2)} + \cdots .$$

Note that this series terminates when z is an integer. Since the Lanczos approximation

$$\tilde{f}_a(z) = \tilde{c}_0 + \frac{\tilde{c}_1}{z+1} + \frac{\tilde{c}_2}{z+2} + \cdots + \frac{\tilde{c}_m}{z+m}$$

is obtained by truncating the series at ρ_m, the Lanczos approximation is exact for $z = 0, 1, 2, \ldots, m$. This implies that the approximant $\tilde{f}_a$ can be found by solving the polynomial interpolation problem

$$p(z) = q(z) f_a(z), \quad z = 0, 1, 2, \ldots, m,$$
$$q(z) = (z+1)(z+2)\cdots(z+m),$$

and putting $\tilde{f}_a = p/q$.

Lanczos vs Spouge approximations: an example

The derivation of the Lanczos and Spouge approximations is very different, and the coefficients are not the same, but very close. For example, for Godfrey's recommendation $a = 671/128 = 5.2421875$, $m(a) = 14$, we get

k	$\tilde{c}_k$ (Lanczos)	$\hat{c}_k$ (Spouge)
0	1.0000000000	1.0000000000
1	57.1562356659	57.1562356659
2	−59.5979603555	−59.5979603555
3	14.1360979747	14.1360979744
4	−0.4919138161	−0.4919138308
5	0.0000339946	0.0000358560
6	0.0000465236	0.0000000000
7	−0.0000983745	0.0000000000

The remaining seven $\tilde{c}_k$ values are all of the same order of magnitude as the last three shown. Note that there are two fairly large coefficients of opposite sign. This gets worse, quickly, as we increase a.

Numerical properties of partial fraction approximations to $z!$

We now look only at approximations of the same form as the Lanczos and Spouge approximations, i.e. $z! \approx F_a(z)\phi_a(z)$, where

$$\phi_a(z) = c_0 + \frac{c_1}{z+1} + \frac{c_2}{z+2} + \cdots + \frac{c_m}{z+m}.$$

Since the Spouge approximation exactly matches the residues at $-1, -2, \ldots, -m$, it is to be expected that any good approximation of this form will have the following properties (as in fact the Lanczos approximation has):

- The first $\lceil a \rceil - 1$ coefficients are close to the Spouge coefficients.
- Taking more than $\lceil a \rceil - 1$ coefficients is not as effective as increasing a.

Unfortunately, as a increases, roundoff becomes non-neglible. For example, when $a = 50$ and $z = 10$, the ratio between the largest term and the sum is about 10^{12}, i.e. twelve significant digits are lost to smearing.

Take-home message #3

All good partial fraction approximations to $z!$ with poles at $-1, -2, \ldots, -m$, are increasingly prone to roundoff as m increases.

Rational functions with known poles

The space of (m, m) rational functions with poles $z_k, k = 1, 2, \ldots, m$, is a linear space, and the obvious way to represent such a function is via its coefficients relative to some basis $\{f_k,\ k = 0, 1, \ldots, m\}$ for that space. We define $e_k(z) = (z - z_k)^{-1}$.

- $f_k = e_k$, i.e. the partial fraction decomposition. Obtainable instantly when the residues at the poles are known. The interpolation problem is ill conditioned.
- $f_k = p_k/q$, where $q = \prod_{k=1}^{m} e_k$, and $\{p_k,\ k = 0, 1, \ldots, m\}$ form a basis for the space of polynomials of degree $\leqslant m$. The interpolation problem reduces to polynomial interpolation.
- $f_k = \prod_{j=1}^{k} e_j$, i.e. the inverse factorial series. The interpolation problem is as ill conditioned as for the partial fraction decomposition.
- For our case with $z_k = -k$, one can try $f_k(z) = \prod_{j=1}^{k}(z - j + 1)(z + j)^{-1}$, as in the Lanczos series. The interpolation problem is as well conditioned as one can hope for, having a triangular matrix.

The "best" basis

It turns out the coefficients are all positive if one uses the inverse factorial series for the Spouge approximation, also for the Lanczos approximation if the same value of m is used. That is:

$$\phi_a(z) = \sum_{k=0}^{\lceil a \rceil - 1} b_k \prod_{j=1}^{k} \frac{1}{z+j}.$$

For the same case as before, we get:

k	$\tilde{b}_k$ (Lanczos)	$\hat{b}_k$ (Spouge)
0	0.9999999980	1.0000000000
1	11.2025045504	11.2024953100
2	32.8013256287	32.8013624751
3	25.3212507533	25.3211432361
4	2.9504205951	2.9506224402
5	0.0010400114	0.0008605449

The interpolation problem involving this basis is ill-conditioned and extra precision may be required.

Continued fraction interpolation

Aitken's continued fraction interpolation can be applied to calculate the Lanczos approximation, as follows:

- In IEEE arithmetic it is OK to have infinite function values. But this can be avoided if necessary by working with $1/f_a$, which is finite.
- Zero divided by zero occurs if two consecutive function values are equal, so we arrange the z values in the order $0, -1, 1, -2, 2, \ldots, -m, m$. The calculation can be simplified in this case (work cut by a factor of four).

Continued fraction interpolation of f_a is equivalent to the original Lanczos series

$$\rho_0 + \rho_1 \frac{z}{z+1} + \rho_2 \frac{z(z-1)}{(z+1)(z+2)} + \cdots$$

in the sense that the coefficients calculated by Aitken's algorithm are just the ρ_k's.

Classification of approximations to the factorial function as interpolants

$$z! = F_a(z) f_a(z) \approx F_a(z)\phi_a(z); \quad F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a}$$

where ϕ_a is an (m, m) rational function interpolating f_a at $2m+1$ points. The Stirling and De Moivre interpolants refer to the balanced form.

Original source	a	Interpolation points
Stirling[1730]	$m+\frac{1}{2}$	$-m, -m+1, -m+2, \ldots, -1, \infty^{m+1}$
De Moivre[1730]	m	$-m, -m+1, -m+2, \ldots, -1, \infty^{m+1}$
Lanczos[1964]	$\geqslant m$	$-m, -m+1, -m+2, \ldots, m-1, m$
Spouge[1994]	$\geqslant m$	$(-m)^2, (-m+1)^2, \ldots, (-1)^2, \infty$

Take-home message #4

All the main approximations to $z!$ can be derived as rational interpolants, but when it comes to evaluating them, the stablest way is via an inverse factorial series.

Take-home messages

1. A slightly generalized form of Stirling's formula, namely:
$$z! = F_a(z) f_a(z), \quad F_a(z) = \sqrt{2\pi}\,(z+a)^{z+1/2} e^{-z-a},$$
where f_a satisfies $f_a(z) = 1 + O(z^{-1})$, is at the heart of almost all competitive methods for evaluating the gamma function.

2. The Lanczos and Spouge approximations have the same form,
$$z! \approx F_a(z) \left(c_0 + \frac{c_1}{z+1} + \frac{c_2}{z+2} + \cdots + \frac{c_m}{z+m} \right),$$
which respects the poles of $z!$, but not quite the same coefficients.

3. All good partial fraction approximations to $z!$ with poles at $-1, -2, \ldots, -m$, are increasingly prone to roundoff as m increases.

4. All the main approximations to $z!$ can be derived as rational interpolants, but when it comes to evaluating them, the stablest way is via an inverse factorial series.

5. For fixed precision, the Lanczos approximation is better than Stirling's formula; in arbitrary precision, Stirling's formula is still going strong despite being around for 270 years.

Remarks on implementation

- $z!$ in IEEE double precision
- $z!$ in arbitrary precision

IEEE: Why $\Gamma(z) = e^{\log \Gamma(z)}$ does not work perfectly

Here is an example. Note that the evaluation of $\log \Gamma(z)$ gives the closest machine number.

```
> f=prod(1:152); lf=log(f); ULPs_lost=(1-exp(lf)/f)/eps
ULPs_lost = 255
> error_in_lgamma = lgamma(153)-lf
error_in_lgamma = 0
```

When x is a very large (or a very small) floating-point number, the relative error in $e^{\log x}$ is approximately equal to the absolute error in $\log x$, namely $\mu |\log x|$, where μ is a number at machine roundoff level. That is, the relative roundoff error has been amplified by a factor $\log |x|$.

The only way round this difficulty is to have $\log x$ available to enough extra precision (extended precision in IEEE arithmetic will do): Clark and Cody[1969].

IEEE: Why $z! = F_a(z) f_a(z)$ requires great care

$(z+a)^{z+1/2} e^{-z-a}$ must not be evaluated as it stands: $(z+a)^{z+1/2}$ can overflow even when the result is in bounds. It must also not be evaluated as $e^{(z+1/2)\log(z+a)-(z+a)}$, for the same reason as the case $e^{\log\Gamma(z)}$.

A good solution (Godfrey[2001]) is $F_a(z) = (z+a)^{z/2+1/4} e^{-z-a} (z+a)^{z/2+1/4}$, which cannot overflow for reasonable values of a unless the result is out of bounds. Of course, the underlying software for a^b should be careful, not just barge in with $e^{b \log a}$.

Multiprecision: Methods based on Stirling's formula

Evaluation: calculate $(z + a)!$ with $a = O(d)$, to get enough accuracy, then use then recursion formula to get back. Thus a typical evaluation of Γ requires $O(d)$ multiplications. Each multiplication requires about $O(d^{1.6})$ time: total complexity, about $O(d^{2.6})$.

Initialization: Requires Bernoulli numbers up to B_{2n}, where $n = O(d)$. To get B_{2k} given all the previous numbers requires k multiplications, so it looks like $O(d^2)$ multiplications. But there is a way to organize this process so that one of the factors in each multiplication is a smallish integer. So each multiplication is of time $O(d)$: total complexity, $O(d^3)$.

The natural form of the series is already the numerically stable form.

Bernoulli numbers, possible useful elsewhere, are generated.

Node at ∞ with high multiplicity concentrates too much accuracy in one area.

Multiprecision: the Lanczos and Spouge approximations

In both cases, we need $m = \lceil a \rceil - 1 = O(d)$ terms.

Evaluation: $O(d)$ multiplications, complexity about $O(d^{2.6})$.

Initialization: We need $O(d)$ values of F_a; elementary functions take about $O(d^2)$ time, so we can do this in $O(d^3)$ time. The Lanczos algorithm given above takes $O(d^2)$ additions, also $O(d^3)$ time.

The advantageously placed nodes give near-uniform accuracy in the Lanczos case; Spouge not bad either.

The natural form of the Lanczos approximation is the continued fraction.

The natural form of the Spouge approximation is the partial fration.

Take-home message #5

For fixed precision, the Lanczos approximation is better than Stirling's formula; in arbitrary precision, Stirling's formula is still going strong despite being around for 270 years.

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