On the stability of recurrence relations for hypergeometric functions

Alfredo Deaño*, Javier Segura

Universidad Carlos III de Madrid (Spain), Universidad de Cantabria (Spain)

Kummer differential equation

$$x\frac{d^2y(x)}{dx^2} + (c-x)\frac{dy(x)}{dx} - ay(x) = 0$$

- This equation is obtained from Gauss hypergeometric equation, considering the change $x \to x/b$, followed by the limit $b \to \infty$.
- It has a regular singular point at $x = 0$ and an irregular singular point at $x = \infty$.
- **References:** Abramowitz and Stegun (1970), Luke (1969), Olver (1974), Temme (1996) ...

Confluent hypergeometric functions

If $a \neq 0, -1, -2, \ldots$ a pair of independent solutions is:

- First kind (Kummer function).

$$M(a;c;x) = {}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}, \qquad x \in \mathbb{R}$$

- Second kind (Tricomi function).

$$U(a;c;x) = C_1 M(a;c;x) + C_2\, x^{1-c} M(a+1-c;2-c;x)$$

- Here we will suppose that $a, c, x > 0$ unless otherwise stated.

Confluent hypergeometric functions

Special cases of CHF include:

- Elementary functions.
- Laguerre and Hermite polynomials.
- Incomplete gamma functions.
- Error functions.
- Parabolic cylinder functions.
- ...

Three term recurrence relations

We will consider the functions:

$$y_n(x) = \{M(a+kn; c+mn; x),\ \ U(a+kn; c+mn; x)\}$$

where $k, m = 0, \pm 1$ and $n \in \mathbb{Z}$, possibly large. In particular the cases:

- $(k, m) = (1, 1)$
- $(k, m) = (0, 1)$
- $(k, m) = (1, 0)$

With suitable normalizations, both functions satisfy the same three term recurrence relation (TTRR).

$$y_{n+1}(x) + a_n(x)y_n(x) + b_n(x)y_{n-1}(x) = 0$$

Minimal and dominant solutions I

Given a TTRR

$$y_{n+1}(x) + a_n(x) y_n(x) + b_n(x) y_{n-1}(x) = 0$$

we say that a solution $f_n(x)$ is minimal if

$$\lim_{n\to\infty} \frac{f_n(x)}{g_n(x)} = 0$$

for any solution $g_n(x)$ which is not a multiple of f_n.

We say that $g_n(x)$ is a dominant solution.

Minimal and dominant solutions II

If we use the recurrence in the forward direction then we obtain a solution:

$$y_n(x) = Af_n(x) + Bg_n(x)$$

where in general $B \neq 0$, which is dominant. The error is:

$$E_n = \left| \frac{y_n - f_n}{f_n} \right| \to \infty, \qquad \text{if } B \neq 0$$

Thus, **a minimal solution cannot be computed using the TTRR in the forward direction**.

- **References:** Gautschi (1967), Wimp (1983).

Continued fractions I

From the TTRR we construct the associated CF:

$$\frac{y_n(x)}{y_{n-1}(x)} = \frac{-b_n}{a_n +} \frac{-b_{n+1}}{a_{n+1} +} \frac{-b_{n+2}}{a_{n+2} +} \cdots$$

- Does this CF converge?
- Under what conditions does it converge and to what function?

Continued fractions II

Theorem (Pincherle).

If $b_n \neq 0$, $n = 1, 2, \ldots$ then the TTRR

$$y_{n+1}(x) + a_n(x)y_n(x) + b_n(x)y_{n-1}(x) = 0$$

has a minimal solution f_n if and only if the associated CF converges. In that case for $n = 1, 2, 3, \ldots$

$$\frac{f_n}{f_{n-1}} = \frac{-b_n}{a_n+}\ \frac{-b_{n+1}}{a_{n+1}+}\ \frac{-b_{n+2}}{a_{n+2}+} \cdots$$

General strategy

Given a three term recurrence relation:

- Identify the minimal solution (Perron+asymptotics).
- If we want to compute a dominant solution we use the TTRR in the forward direction.
- If we want to compute a minimal solution we can use the CF.

Is this information enough in order to compute Kummer functions safely?

Some remarks

- Minimal and dominant solutions are defined in the limit $n \to \infty$.
- What happens when n is not very large?
- We will show that in many cases the behaviour for moderate values of n can be opposite to what is given by asymptotic information.

Recurrence (1,1)

$$y_{n+1}(x) + a_n(x)y_n(x) + b_n(x)y_{n-1}(x) = 0 \qquad (1)$$

where

$$a_n = -\frac{(c+n)(1-c-n+x)}{(a+n)x}$$

$$b_n = -\frac{(c+n)(c+n-1)}{(a+n)x}$$

Solutions:

$$f_n(x) = M(a+n, c+n, x) \qquad \text{Minimal}$$

$$g_n(x) = (-1)^n \Gamma(c+n) U(a+n, c+n, x)$$

Two examples

- First $a = 0.3$, $c = 0.8$ and $x = 1.1$
- Then $a = 0.3$, $c = 0.8$ and $x = 31.1$

We will compute in Fortran using the recurrence for $n = 0, 1, 2, \ldots, 100$ and plot the relative error with respect to the result given by Maple (40 digits).

Computation of $U(a+n; c+n; x)$

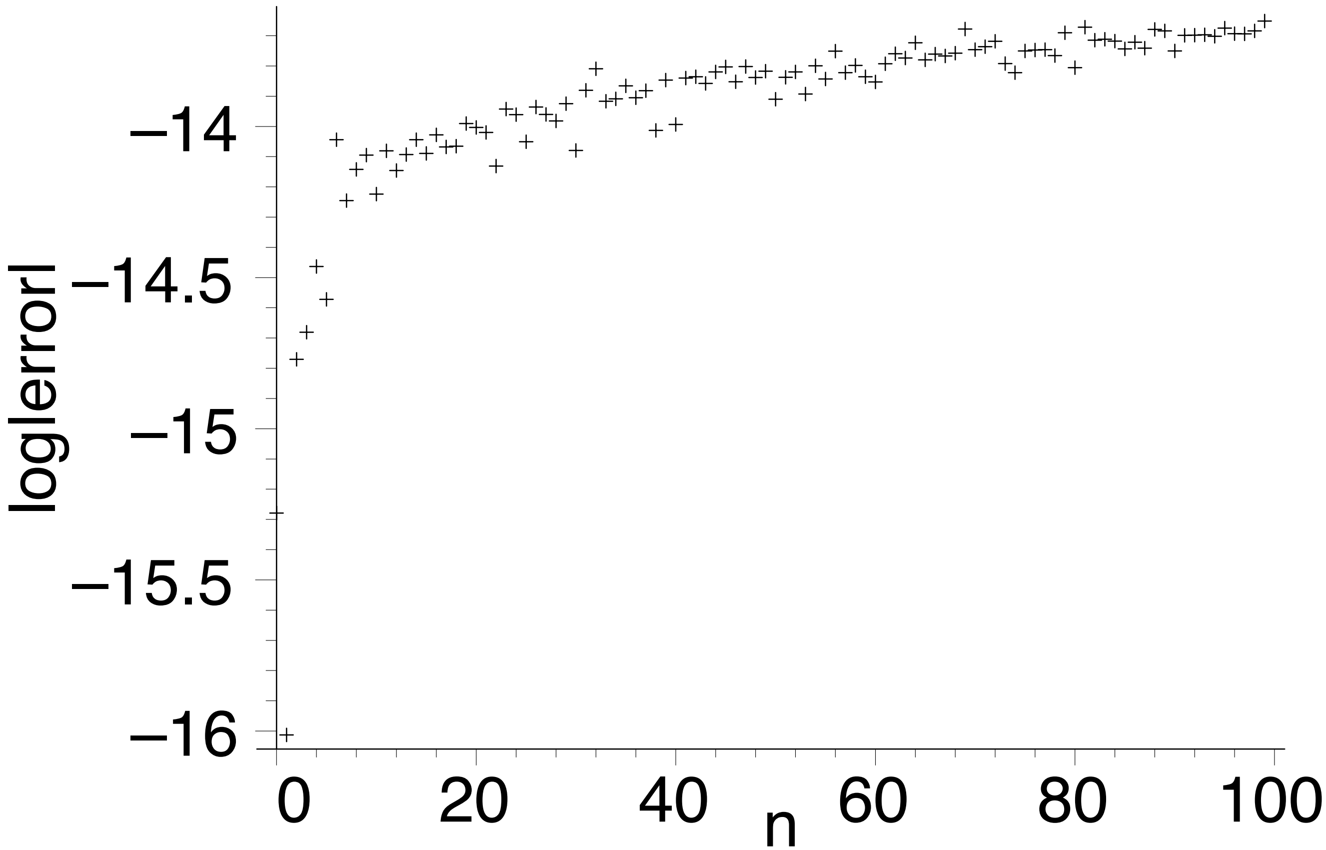


Figure 1: Dominant solution, $x = 1.1$

Computation of $M(a+n; c+n; x)$

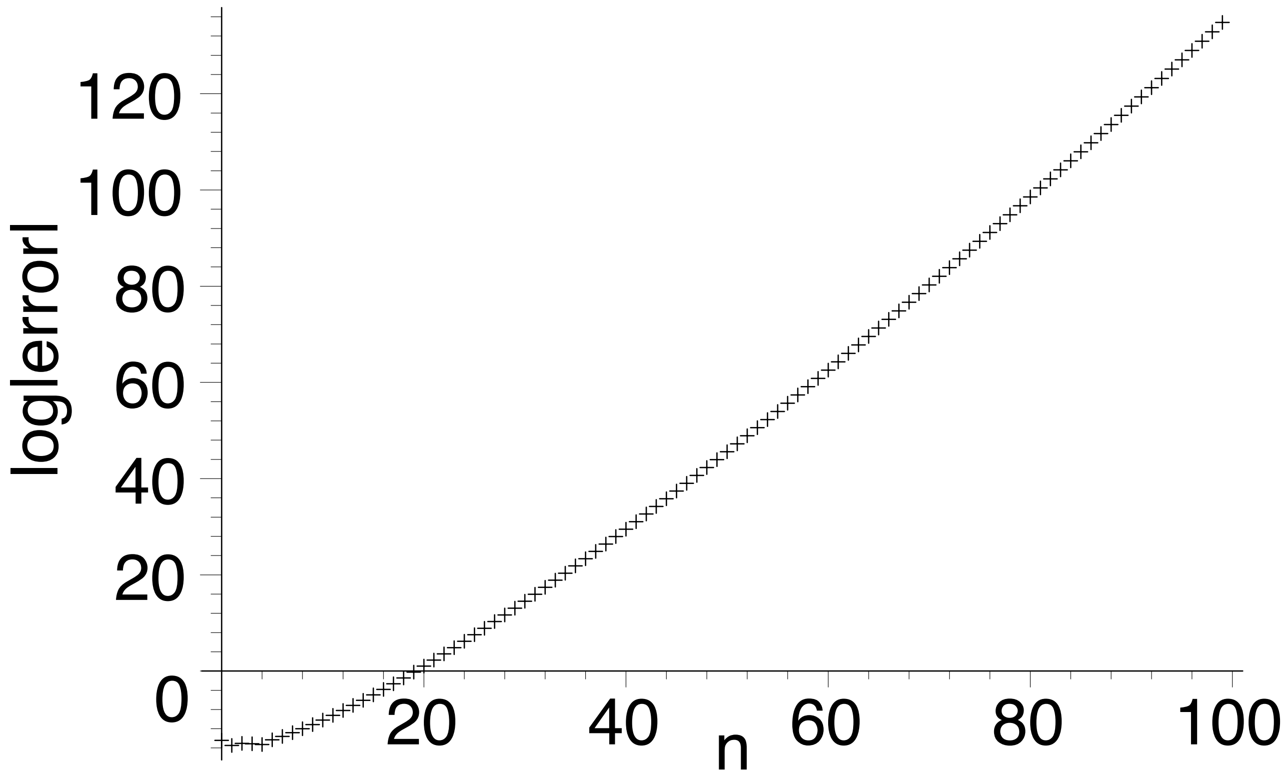


Figure 2: Minimal solution, $x = 1.1$

Computation of $U(a + n; c + n; x)$

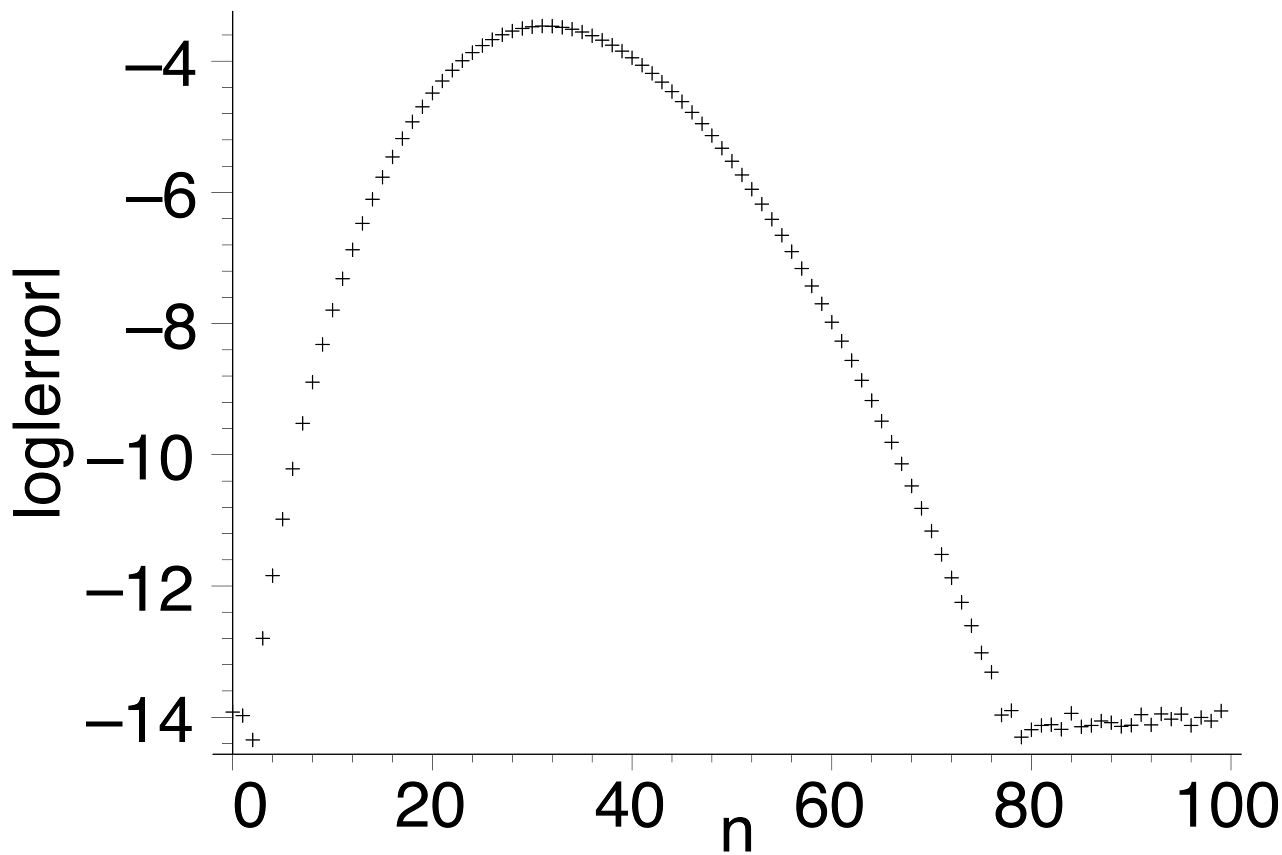


Figure 3: Dominant solution, $x = 31.1$

Computation of $M(a+n; c+n; x)$

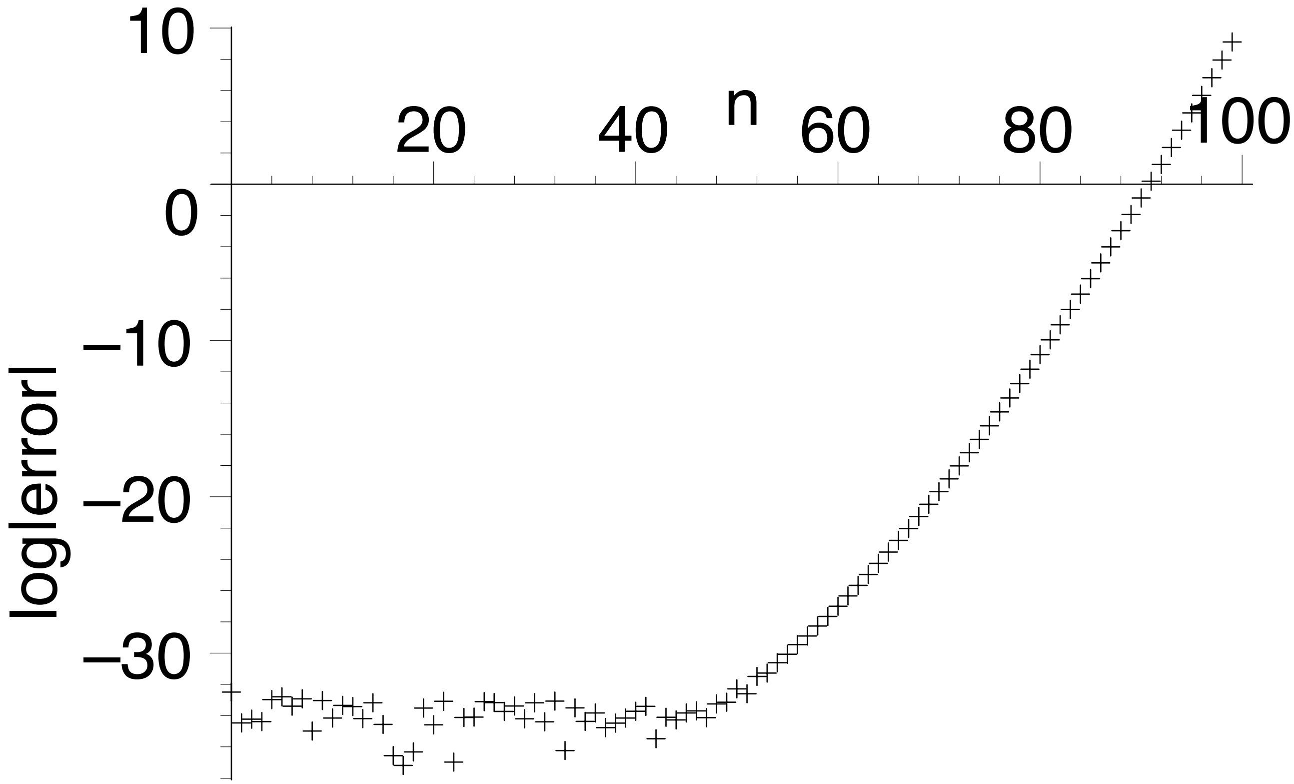


Figure 4: Minimal solution, $x = 31.1$

Analysis of the continued fraction I

$$\frac{f_n(x)}{f_{n-1}(x)} = \frac{c+n-1}{c+n-1-x+} \; \frac{(a+n)x}{c+n-x+} \; \frac{(a+n+1)x}{c+n+1-x+} \cdots$$

For $k \in \mathbb{N}$ we define the $k-th$ **approximant**:

$$F_1 := \frac{c+n-1}{c+n-1-x}$$

$$F_k := \frac{c+n-1}{c+n-1-x+} \; \frac{(a+n)x}{c+n-x+} \cdots \frac{(a+n+k-2)x}{c+n+k-2-x}$$

for $k = 2, 3, 4, \ldots$

Analysis of the continued fraction II

This is a **T-fraction**, that corresponds to:

$$\frac{f_n(x)}{f_{n-1}(x)} = \frac{M(a+n;c+n;x)}{M(a+n-1;c+n-1;x)}$$

at $x = 0$. We have convergence to this ratio for $x \in \mathbb{R}$.

The T-fraction corresponds to:

$$-\frac{c+n-1}{x}\frac{{}_2F_0(a+n, a+1-c; -1/x)}{{}_2F_0(a+n-1, a+1-c; -1/x)}$$

at $x = \infty$.

Analysis of the continued fraction III

If

$$g_n(x) = (-1)^n \Gamma(c+n) U(a+n; c+n; x)$$

then

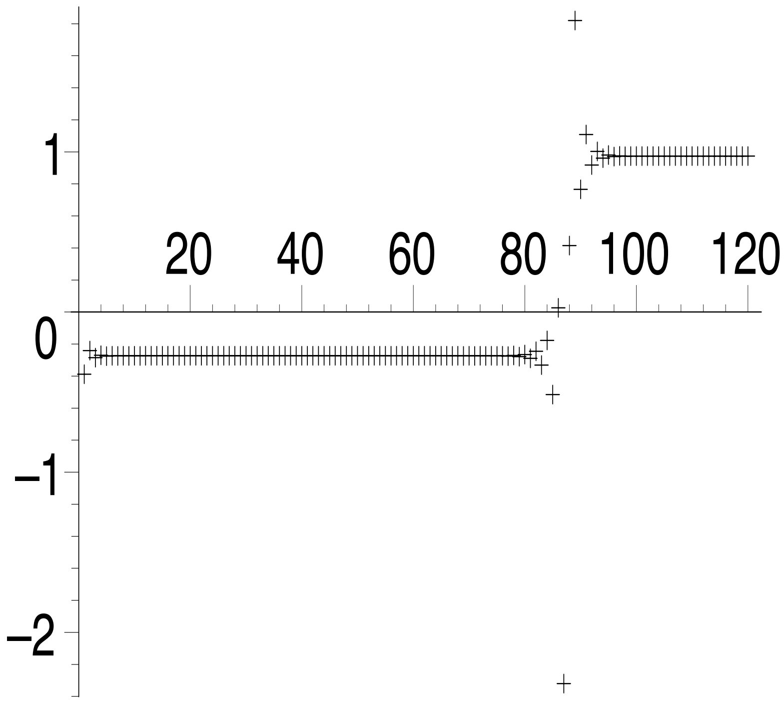
$$\frac{g_n(x)}{g_{n-1}(x)} \sim -\frac{c+n-1}{x} \frac{{}_2F_0(a+n, a+1-c; -1/x)}{{}_2F_0(a+n-1, a+1-c; -1/x)}$$

when $x \to \infty$.

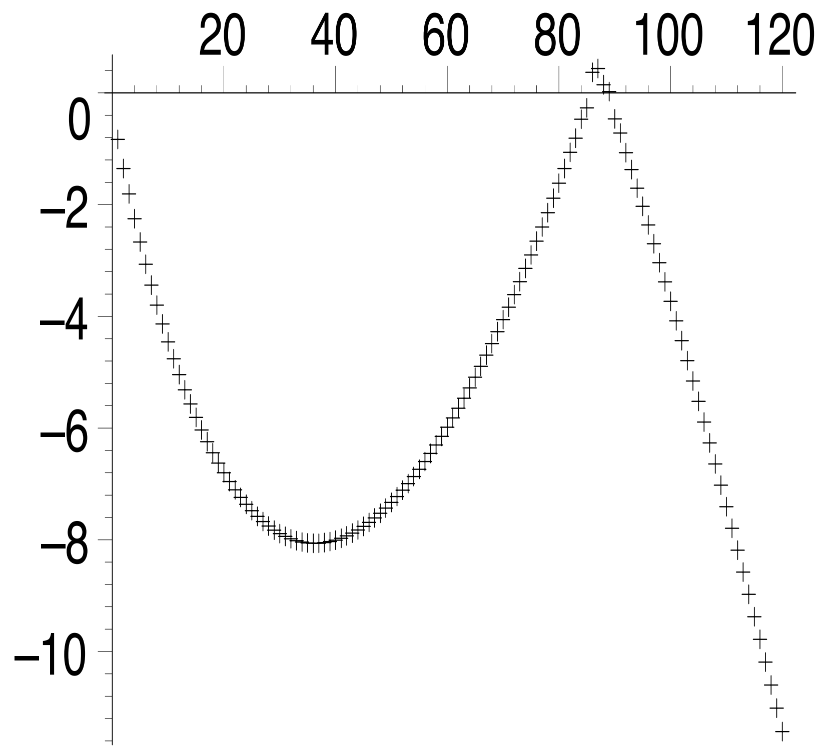
Note: The CF does **not** converge to this ratio, but numerically it can be accurately approximated.

Analysis of the continued fraction IV

We fix $a = 0.8$, $c = 0.3$, $n = 15$, $x = 51.1$:



(a) Plot of F_k



(b) Plot of $\log|F_{k+1} - F_k|$

A basic criterion I

Let us recall the continued fraction;

$$\frac{f_n}{f_{n-1}} = \frac{c+n-1}{c+n-1-x+}\ \frac{(a+n)x}{c+n-x+}\ \frac{(a+n+1)x}{c+n+1-x+}\cdots$$

If x is large enough then $c+n+k-2-x<0$ and the approximants interlace:

$$F_1 < F_3 < F_5 < F_7 < \ldots < F_8 < F_6 < F_4 < F_2 < 0$$

Remark: The ratio of minimal solutions is > 0, so this is wrong!!

A basic criterion II

If we reach K_0 odd such that $c + n + K_0 - x > 0$:

$$F_1 < F_3 < \ldots < F_{K_0} < F_{K_0+1} < F_{K_0-1} \ldots < F_4 < F_2 < 0$$

The next one is not right: $F_{K_0+2} < F_{K_0}$, and the behaviour changes.

If we reach K_0 even such that $c + n + K_0 - x > 0$:

$$F_1 < F_3 < \ldots < F_{K_0+1} < F_{K_0} < F_{K_0-2} \ldots < F_4 < F_2 < 0$$

The next one is not right: $F_{K_0+2} > F_{K_0}$, and the behaviour changes.

A basic criterion III

- If we add $[|c+n-x|]+1$ terms in the CF we obtain the best approximation to the "false" limit.
- The larger x is the better the approximation is.
- The absolute error is bounded by $|F_{K_0+1}-F_{K_0}|$.
- In the example, $a=0.8$, $c=0.3$, $n=15$, $x=51.1$, this bound is $\approx 8.645786\times 10^{-9}$.
- If $x=76.1$ then it is $\approx 2.857821\times 10^{-17}$.

Under construction

- This criterion is clearly related to the signs of the coefficients of the recurrence.
- We look for a criterion of pseudostability of the TTRR based on the signs of these coefficients.
- Other cases:
  - Recursion $(k, m) = (1, 0)$ when $c > 0$ is large.
  - Recursion $(k, m) = (0, -1)$ when $x > 0$ is large.
  - Recursions for Gauss hypergeometric functions.
- To be continued...