Continued Fractions for Special Functions in Maple

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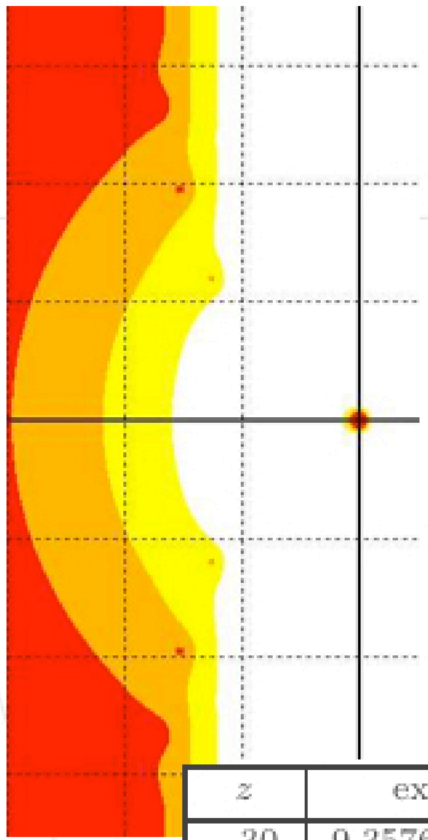

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \qquad (11.1.1)$$

$$= 1 + \frac{2z}{2 - z +} \frac{z^2/6}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty} \frac{a_m z^2}{1} \qquad (11.1.2)$$

$$= 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{c_m z}{1} \qquad (11.1.3)$$

$$= 1 + \frac{z}{1 - z +} \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{(m-1)z}{m - z} \qquad (11.1.4)$$

z	$\exp(z)$	(11.1.1)	(11.1.2)	(11.1.3)	(11.1.4)
−30	9.357623e−14	9.1e+23	1.4e+02	9.0e+09	2.7e+01
−5	6.737947e−03	1.1e−03	1.1e−32	6.8e−11	8.1e−08
1	2.718282e+00	7.5e−21	2.2e−61	1.1e−25	5.1e−20
15	3.269017e+06	8.3e−02	1.4e−12	8.7e−01	1.0e+00
43	4.727839e+18	1.0e+00	1.0e+00	1.0e+00	1.0e+00

In this talk we introduce a new Maple package for working with continued fractions.

Overview

1. Motivation

- [1964] *Handbook of Mathematical Functions (with Formulas, Graphs, and Mathematical Tables)* by Abramowitz and Stegun
  - continued fractions are not well covered
- [1999] NIST's *Digital Library of Mathematical Functions*
- [200X] *Handbook of Continued Fractions for Special Functions* by Annie Cuyt, Vigdis Petersen, Brigitte Verdonk, Haakon Waadeland, William B. Jones and Catherine Bonan-Hamada
  - includes numerical en graphical illustrations
  - includes worksheets
- only limited support in computer algebra systems
  - Mathematica : NumberTheory`ContinuedFractions`
  - Maple : numtheory[cfrac]
  - . . .

2. Maple's numtheory[cfrac]

```
> restart;
> with(numtheory):
```

Warning, the protected name order has been redefined and unprotected

```
> cfrac( erfc(z), 8, 'regular' );
```

$$1+\cfrac{z}{-\frac{1}{2}\sqrt{\pi}+\cfrac{z^2}{-\frac{6}{\sqrt{\pi}}+\cfrac{z^2}{5\sqrt{\pi}+\cfrac{z^2}{\frac{14}{39\sqrt{\pi}}+\ldots}}}} \tag{2.1}$$

```
> cfrac( % );
```

$$\frac{1}{3}\,\frac{210\sqrt{\pi}+180\,z^2\sqrt{\pi}+39\sqrt{\pi}\,z^4-420\,z-220\,z^3}{\sqrt{\pi}\left(70+60\,z^2+13\,z^4\right)} \tag{2.2}$$

```
> evalf( subs( z = 3.2, % ) );
```

$$0.2144280128 \tag{2.3}$$

```
> cfrac( erfc(z), z, 8, 'simregular' );
```

$$1-\cfrac{2\,z}{\sqrt{\pi}\left(1+\frac{1}{3}\cfrac{z^2}{1-\frac{1}{30}\cfrac{z^2}{1+\frac{39}{70}\cfrac{z^2}{1+\ldots}}}\right)} \tag{2.4}$$

```
> cfrac( % );
```

$$\frac{1}{3}\,\frac{210\sqrt{\pi}+210\sqrt{\pi}\ldots+180\,z^2\sqrt{\pi}+63\sqrt{\pi}\,z^2\ldots+39\sqrt{\pi}\,z^4-420\,z-420\,z\ldots-220\,z^3+14\,z^3\ldots}{\sqrt{\pi}\left(70+70\ldots+60\,z^2+21\,z^2\ldots+13\,z^4\right)} \tag{2.5}$$

> evalf(subs(z = 3.2, %));

$$\frac{0.1880631945\ (2334.59727 + 630.412737\ \ldots)}{2047.5488 + 285.04\ \ldots} \qquad (2.6$$

> cfrac(Ei(1,z), 'simregular');

Error, (in fincfrac) invalid series

3. Coverage

Families of continued fractions

- C-fraction : $b_0 + K_{m=1}^{\infty}\left(\frac{a_m z^{\alpha_m}}{1}\right)$ with $a_m \in \mathbb{C}\backslash\{0\},\ \alpha_m \in \mathbb{N}$
- regular C-fraction : if furthermore $\alpha_m = 1$ for $m \geq 1$
- S-fraction (Stieltjes) : $F(z) = K_{m=1}^{\infty}\left(\frac{a_m z}{1}\right)$ with $a_m > 0$
- modified S-fraction : $C(z)$ is a modified S-fraction if there exist transformations $C(z) \rightarrow B(C(z))$ and $z \rightarrow a(z)$ such that $B(C(a(z)))$ is a S-fraction
- J-fraction (Jacobi) : $\dfrac{\alpha_1}{\beta_1 + z + K_{m=2}^{\infty}\left(\frac{-\alpha_m}{\beta_m + z}\right)}$ with $a_m \in \mathbb{C}\backslash\{0\},\ \beta_m \in \mathbb{C}$
- general T-fraction (Thron) : $K_{m=1}^{\infty}\left(\frac{F_m z}{1 + G_m z}\right)$ with $F_m \in \mathbb{C}\backslash\{0\},\ G_m \in \mathbb{C}$
- T-fraction : if furthermore all $F_m = 1$
- M-fraction (Murphy) : if $F_1 z$ is replaced by F_1
- . . .

Available functions

- this is only a partial list of (families of) functions for which there are continued fraction representations available in the package *at this moment*
- series representations are not included in this list
- there are more formulas in the book

function	name	C	S	T
$atan(x),\ sinh(x), \ldots$	elementary functions	+	+	+
$J(z)$	Binet		+	
$\Gamma(a,z),\ \gamma(a,z)$ $\psi_k(z)$ $\psi_1(z)$ $\psi_2(z)$	(compl.) incompl. gamma polygamma trigamma tetragamma	+ + +	+ + + +	+
$erf(z), dawson(z)$ $erfc(z)$ $C(z),\ S(z)$	error, Dawson's integral complementary error Fresnel integral	+ +	 + 	+ +
$E_m(z)$ $Ei(z),\ li(z),\ Ci(z), Si(z)$	exponential integral exponential, logarithmic, (co)sine integral	+	+ +	+
${}_pF_q\left(a_1,\ldots,a_p; b_1,\ldots,b_q; z\right)$ $M(a;b;z),\ U(a;b;z)$	hypergeometric Kummer	+ +	+	+
$F_L(\eta,\rho)$ $D_\nu(z)$	Coulomb wave parabolic cylinder	 +		
$J_\nu(z), Y_\nu(z)$ $P_\nu^\mu(z),\ Q_\upsilon^\mu(z)$	Bessel associated Legendre	+ +	+	+
$f(x,\mu,\sigma)$ χ_2 $B(a,b)$	normal distribution chi-square distr. beta function	 +	+ +	
...	...			

4. The CFHB package

```
> restart;
> with(CFHB);
```

[*build*, *compare*, *create*, *default*, *evaluate*, *formula*, *nthapprox*, *nthdenom*, *nthelement*, *nthnumer*, *nthtail*, *query*, *tailestimate*, *transform*] (4.1

```
>
```

Creating a continued fraction

- form/notation : $b_0 + f\left(\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n +} K_{k=1,\, l=n+1+tk}^{\infty}\left(\frac{c_1(l)}{d_1(l) +} \cdots \frac{c_t(l+t-1)}{d_t(l+t-1)}\right)\right)$
  - front : b_0
  - factor : f
  - begin : $\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}$
  - general : $\frac{c_1(l)}{d_1(l)}, \ldots, \frac{c_t(l+t-1)}{d_t(l+t-1)}$
  - other items : even, odd, parameters, variable, index (and also lhs, category, function, comment, . . .)
- running example : the complementary error function $erfc(z)$

S-fractions. Based on (13.2.7) and the modified S-fraction representation for the complementary incomplete gamma function (12.5.17), we find

$$\operatorname{erfc}(z) = \frac{z}{\sqrt{\pi}} e^{-z^2} \left(\frac{a_1}{z^2} \,+\, \frac{a_2}{1} \,+\, \frac{a_3}{z^2} \,+\, \frac{a_4}{1} \,+\, \dots \right), \qquad \Re(z) > 0, \qquad (13.2.19a)$$

$$w(z) = -\frac{iz}{\sqrt{\pi}} \left(\frac{-a_1}{z^2} \,+\, \frac{-a_2}{1} \,+\, \frac{-a_3}{z^2} \,+\, \frac{-a_4}{1} \,+\, \dots \right), \qquad \Im(z) > 0, \qquad (13.2.19b)$$

where the coefficients are given by

$$a_1 = 1, \quad a_m = \frac{m-1}{2}, \quad m \geq 2. \qquad (13.2.19c)$$

The continued fraction (13.2.19b) is a modified S-fraction in $-1/z^2$.

```
> f := create( 'contfrac', factor = z/sqrt(Pi)*exp(-z^2), begin = [[ 1, z^2 ]],
      even = [ (m-1)/2, 1 ], odd = [ (m-1)/2, z^2 ] );
```

$$f := formula\left(\left[general = \left[\left[\frac{1}{2}m - \frac{1}{2}, 1 \right], \left[\frac{1}{2}m - \frac{1}{2}, z^2 \right] \right], variable = z, front = 0, type = contfrac, begin = \left[\left[1, z^2\right]\right], factor = \frac{z\, e^{(-z^2)}}{\sqrt{\pi}}, index = m \right]\right) \qquad (4.1.1)$$

```
> f := formula( "13.2.19a" );
```

$$f := formula(\text{"13.2.19a"}) \qquad (4.1.2)$$

```
> op(%);
```

$$\left[general = \left[\left[\frac{1}{2}m - \frac{1}{2}, 1 \right], \left[\frac{1}{2}m - \frac{1}{2}, z^2 \right] \right], variable = z, front = 0, type = contfrac, label = \text{"13.2.19a"}, function = \operatorname{erfc}, begin = \left[\left[1, z^2\right]\right], category = \text{"modified S-fraction"}, factor = \frac{z\, e^{(-z^2)}}{\sqrt{\pi}}, index = m, lhs = \operatorname{erfc}(z) \right] \qquad (4.1.3)$$

```
> query( function = erfc, category = "S-fraction" );
```

$$\{\text{"13.2.19a"}, \text{"13.3.5a"}\} \qquad (4.1.4)$$

Computing approximants

- symbolically :

```
> seq( nthelement( f, i ), i=0..5 );
```

$$0, \left[\frac{z\,e^{(-z^2)}}{\sqrt{\pi}}, z^2\right], \left[\frac{1}{2}, 1\right], \left[1, z^2\right], \left[\frac{3}{2}, 1\right], \left[2, z^2\right] \qquad (4.2.1)$$

```
> nthapprox( f, 5 );
```

$$\frac{2\,e^{(-z^2)}\left(2z^4+9z^2+4\right)}{z\sqrt{\pi}\left(4z^4+20z^2+15\right)} \qquad (4.2.2)$$

- numerically :

```
> nthapprox( f, 5, z = 6 );
```

$$\frac{2920}{17757}\,\frac{e^{(-36)}}{\sqrt{\pi}} \qquad (4.2.3)$$

```
> Digits := 40:
> r1 := evalf( %% );
```

$$r1 := 2.151973762067875570568349858491156421119\ 10^{-17} \qquad (4.2.4)$$

```
> r2 := evalf( erfc(6) );
```

$$r2 := 2.151973671249891311659335039918738463048\ 10^{-17} \qquad (4.2.5)$$

Tails, limits and modifications

- computing approximants means truncating the continued fraction, i.e. replacing a complete tail with a single value
- a tail does not need to converge to zero

```
> g := nthtail( f, 4 );
```

$$g := formula\left(\left[general = \left[\left[\frac{1}{2}m+\frac{3}{2}, z^2\right], \left[\frac{1}{2}m+\frac{3}{2}, 1\right]\right], variable = z, front = 0, type = contfrac, factor = 1, index = m\right]\right) \qquad (4.3.1)$$

- better approximations can be obtained when replacing the tail with a modification $w \neq 0$
- choice of modification :

- As an alternative to (7.7.6) we can choose

$$w_n := \frac{a_{n+1}}{1} + \frac{a_{n+1}}{1} + \frac{a_{n+1}}{1} \ldots = \frac{\sqrt{4a_{n+1}+1}-1}{2}$$

$$a_{n+1} \notin (-\infty, -\frac{1}{4}) \qquad \Re(a_{n+1} + \frac{1}{4}) > 0 \quad (7.7.8)$$

This modification, called the *square root modification*, gives an improvement of the order of [JJW87]

$$\left| \frac{f - S_n((\sqrt{4a_{n+1}+1}-1)/2)}{f - S_n(0)} \right| = O\left(\frac{1}{|a_{n+1}|}\right) . \qquad (7.7.9)$$

This choice may be of use when $a_n \to \infty$.

```
> tailestimate( f );
```

Warning, a modification could not be computed, returning 0 instead

$$0 \qquad (4.3.2)$$

```
> assume( z, real );
> about( z );
Originally z, renamed z~:
  is assumed to be: real
> tailestimate( f );
```

$$nthdenom(f, m) \left(\frac{1}{2} \sqrt{\frac{4\, nthnumer(f, m+1)}{nthdenom(f, m)\, nthdenom(f, m+1)} + 1} - \frac{1}{2} \right) \qquad (4.3.3)$$

```
> nthapprox( f, 5, 'modification' = tailestimate(f), z = 6 );
```

$$\frac{1}{3} \frac{e^{(-36)} (3068 + 154\sqrt{23}\sqrt{18})}{\sqrt{\pi}\left(6219 + \frac{1873}{6}\sqrt{23}\sqrt{18}\right)} \qquad (4.3.4)$$

```
> r3 := evalf(%);
```

$$r3 := 2.1519736703368421005239187163103276383\,82\ 10^{-17} \qquad (4.3.5)$$

```
> r2 := evalf( erfc(6) );
```

$$r2 := 2.151973671249891311659335039918738463048\ 10^{-17} \qquad (4.3.6)$$

- improvement :

```
> abs( (r1-r2)/r2 );
```

$$4.220218187249515667311748961742562936742\ 10^{-8} \qquad (4.3.7)$$

```
> abs( (r3-r2)/r2 );
```

$$4.242845641346098501406822782220897515434\ 10^{-10} \qquad (4.3.8)$$

• graphical illustration :

Contractions and equivalence transformations

- a contraction of a continued fraction is another continued fraction for which the approximants form a subset of the approximants of the latter
  - even_contraction, odd_contraction

```
> z := 'z':
> h := transform( f, even_contraction );
```

$$h := \mathrm{formula}\left(\left[\mathit{general} = \left[\left[-\left(m-\frac{3}{2}\right)(m-1), -\frac{3}{2}+2m+z^2\right]\right], \mathit{variable} = z, \mathit{front} = 0, \mathit{type} = \mathit{contfrac}, \mathit{begin} = \left[\left[\frac{z\,\mathrm{e}^{(-z^2)}}{\sqrt{\pi}}, \frac{1}{2}+z^2\right]\right], \mathit{index} = m\right]\right) \quad (4.4.1)$$

- an equivalence transformation has the property that the approximants are not changed
  - simregular : constructs a new continued fraction with all denominators 1
  - Euler : creates a continued fraction from a given series
  - approximants : creates a continued fraction with prescribed approximants

```
> k := transform( f, simregular );
```

$$k := \mathrm{formula}\left(\left[\mathit{general} = \left[\left[\frac{\frac{1}{2}m-\frac{1}{2}}{z^2}, 1\right]\right], \mathit{variable} = z, \mathit{front} = 0, \mathit{type} = \mathit{contfrac}, \mathit{begin} = \left[\left[\frac{1}{z^2}, 1\right]\right], \mathit{factor} = \frac{z\,\mathrm{e}^{(-z^2)}}{\sqrt{\pi}}, \mathit{index} = m\right]\right) \quad (4.4.2)$$