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Radial orthogonality, Symbolic-numeric integration, Padé approximation and Lebesgue constants

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Orthogonal polynomials

$$V_m(z) = \sum_{i=0}^{m} b_{m-i} z^i, \qquad \int_{-1}^{1} w(z)\, dz > 0,$$

$$\int_{-1}^{1} z^i V_m(z) w(z)\, dz = 0, \qquad i = 0, \ldots, m-1$$

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$$\int_{-1}^{1} z^i V_m(z) w(z)\,dz = 0, \qquad i = 0, \ldots, m-1$$

$$c_i := \int_{-1}^{1} z^i w(z)\,dz,$$

$$\sum_{j=0}^{m} c_{i+j} b_{m-j} = 0, \qquad i = 0, \ldots, m-1$$

Orthogonal polynomials

$$H_m := \begin{vmatrix} c_0 & \dots & c_{m-1} \\ \vdots & \iddots & \vdots \\ c_{m-1} & \dots & c_{2m-2} \end{vmatrix}, \qquad H_0 := 1$$

if $\gamma(z^i) = c_i$ positive definite, meaning $H_m > 0$ for $m \geq 0$,

then $V_m(z) = b_0 \prod_{i=1}^{m} (z - z_i^{(m)})$ with $z_i^{(m)}$ real and simple

Example

$w(z) = 1$: $V_m(z) = L_m(z)$, Legendre polynomial

$$L_3(z) = \frac{5}{2} z \left(z - \sqrt{\frac{3}{5}} \right) \left(z + \sqrt{\frac{3}{5}} \right)$$

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$w(z) = 1/\sqrt{1 - z^2}$: $V_m(z) = T_m(z)$, Chebyshev first kind

$$T_3(z) = 4z \left(z - \cos \frac{\pi}{6} \right) \left(z - \cos \frac{5\pi}{6} \right)$$

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$w(z) = (1-z^2)^{(\nu-\frac{1}{2})} : V_m(z) = C_m^{(\nu)}(z)$, Gegenbauer

$$C_3^{(\nu)}(z) = \nu(\nu+1) z \left(\frac{4}{3}(\nu+2) z^2 - 2 \right)$$

Quadrature and Padé approximation

$$\ell_i(z) = \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{z - z_j^{(m)}}{z_i^{(m)} - z_j^{(m)}}$$

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$$\int_{-1}^{1} g(z) w(z)\, dz \approx \sum_{i=1}^{m} A_i^{(m)} g\left(z_i^{(m)}\right)$$

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Quadrature and Padé approximation

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$$A_i^{(m)} = \int_{-1}^{1} \ell_i(z) w(z) \, dz$$

$$\int_{-1}^{1} g(z) w(z) \, dz \approx \sum_{i=1}^{m} A_i^{(m)} g\left(z_i^{(m)}\right)$$

exact for $g(z) \in \mathbb{C}_{2m-1}[z]$

$$\int_{-1}^{1} p_{2m-1}(z) w(z) \, dz = \sum_{i=1}^{m} A_i^{(m)} p_{2m-1}\left(z_i^{(m)}\right)$$

Quadrature and Padé approximation

$$f(z) = \int_{-1}^{1} \frac{w(t)}{1 - tz} \, dt$$

$$[m - 1/m]^f(z) = \sum_{i=1}^{m} A_i^{(m)} / \left(1 - z_i^{(m)} z\right)$$

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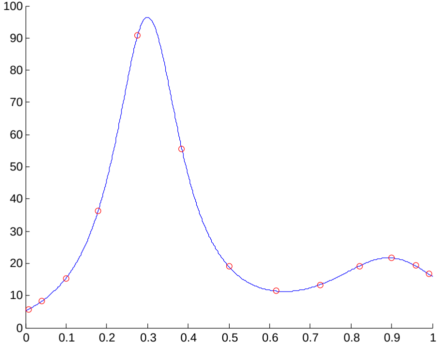
Example

$$\int_0^1 \left(\frac{1}{(x-0.3)^2+0.01} + \frac{1}{(x-0.9)^2+0.04} \right) dx = \int_{-1}^{1} g(z)\, dz \approx \sum_{i=1}^{13} A_i^{(13)} g\left(z_i^{(13)}\right)$$



Lebesgue constants

$$V_m(z) \text{ orthogonal}, \quad V_m\left(z_i^{(m)}\right) = 0$$

polynomial interpolation:

$$p_m(z) = \sum_{i=0}^{m} f_i \ell_i(z), \qquad \ell_i(z) = \prod_{\substack{j=1 \\ j\neq i}}^{m+1} \frac{z - z_j^{(m+1)}}{z_i^{(m+1)} - z_j^{(m+1)}}$$

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error:

$$\max_{z \in [-1,1]} |f(z) - p_m(z)| \le (1 + \Lambda_m) \max_{z \in [-1,1]} |f(z) - p_m^*(z)|,$$

$$\Lambda_m(z_1^{(m+1)}, \ldots, z_{m+1}^{(m+1)}) = \max_{z \in [-1,1]} \sum_{i=0}^{m} |\ell_i(z)|$$

Lebesgue constants

optimize:

$$\min_{\{x_0,\ldots,x_m\}\subset[-1,1]} \Lambda_m(x_0,\ldots,x_m)$$

Lebesgue constants

optimize:

$$\min_{\{x_0,\ldots,x_m\}\subset[-1,1]} \Lambda_m(x_0,\ldots,x_m)$$

near-optimal solution:

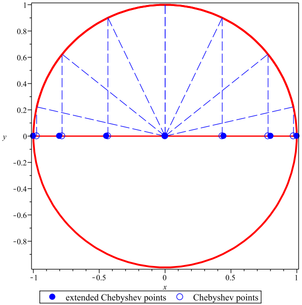
$$\Lambda_m(x_0,\ldots,x_m) < \frac{2}{\pi}\log(m+1) + 0.5829\ldots$$

extended Chebyshev nodes:

$$T_{m+1}\left(\cos\frac{(2i+1)\pi}{2(m+1)}\right) = 0, \qquad x_i = \frac{\cos\frac{(2i+1)\pi}{2(m+1)}}{\cos\frac{\pi}{2(m+1)}}, \qquad i = 0,\ldots,m$$

Example

extended Chebyshev nodes



$$m + 1 = 7$$

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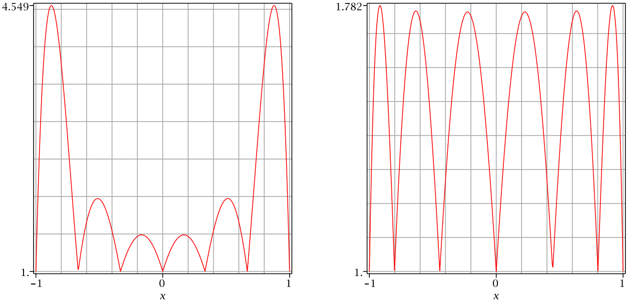
Example

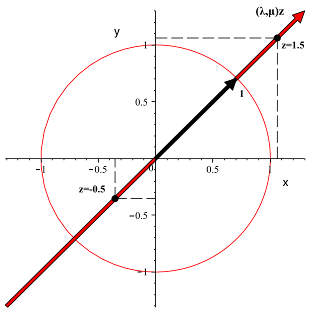
Lebesgue function for equidistant and for extended Chebyshev nodes



$$\mathfrak{m} + 1 = 7$$

Orthogonal polynomials

$$(x, y) = (\lambda, \mu)z, \qquad \|(\lambda, \mu)\|_p = 1$$



$$V_m(x, y) = \mathcal{V}_m(\lambda, \mu; z) = \sum_{i=0}^{m} b_{m-i}(\lambda, \mu) z^i,$$

$$b_{m-i}(\lambda, \mu) \in \mathbb{C}(\lambda, \mu)$$

$$V_m(x, y) \in \mathbb{C}(\lambda, \mu)[z] \notin \mathbb{C}[x, y]$$

$$\mathcal{V}_m(\lambda, \mu; \lambda x + \mu y) \in \mathbb{C}[x, y]$$

Orthogonal polynomials

$$\iint_{\|(x,y)\|_p \leq 1} (\lambda x + \mu y)^i \mathcal{V}_m(\lambda, \mu; \lambda x + \mu y) w(z) \, dx \, dy = 0,$$

$$i = 0, \ldots, m-1$$

Orthogonal polynomials

$$\iint_{\|(x,y)\|_p \le 1} (\lambda x + \mu y)^i \mathcal{V}_m(\lambda, \mu; \lambda x + \mu y) w(z)\, dx\, dy = 0, \qquad i = 0, \dots, m-1$$

$$\begin{aligned} c_i(\lambda, \mu) &= \iint_{\|(x,y)\|_p \le 1} (\lambda x + \mu y)^i w(z)\, dx\, dy \\ &= \sum_{j=0}^{i} \left(\binom{i}{j} \iint_{\|(x,y)\|_p \le 1} x^j y^{i-j} w(z)\, dx\, dy \right) \lambda^j \mu^{i-j} \end{aligned}$$

$$\sum_{j=0}^{m} c_{i+j}(\lambda, \mu) b_{m-j}(\lambda, \mu) \equiv 0, \qquad i = 0, \dots, m-1$$

Example

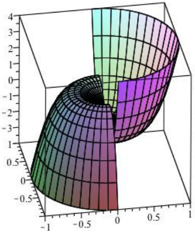
Euclidean unit disk: $\|(x, y)\|_2 \leq 1$

$w(z) = 1$

$$\mathcal{V}_m(\lambda, \mu; z) = \mathcal{L}_m(\lambda, \mu; z), \text{spherical Legendre}$$

$$\mathcal{L}_0(\lambda, \mu; z) = 1$$
$$\mathcal{L}_1(\lambda, \mu; z) = 2z$$
$$\mathcal{L}_2(\lambda, \mu; z) = 4z^2 - 1$$
$$\mathcal{L}_3(\lambda, \mu; z) = 8z^3 - 4z$$

Example



$\mathcal{L}_3(\lambda, \mu; z)$

bivariate:

$$\begin{aligned}\mathcal{L}_{m+1}(\lambda, \mu; z) &= 2z\mathcal{L}_m(\lambda, \mu; z) - \mathcal{L}_{m-1}(\lambda, \mu; z)\\ &= C_{m+1}^{(1)}(z)\end{aligned}$$

Example

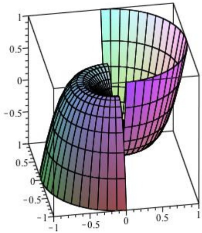
Euclidean unit disk: $\|(x, y)\|_2 \le 1$

$w(z) = 1/\sqrt{1 - z^2}$

$$\mathcal{V}_m(\lambda, \mu; z) = \mathcal{T}_m(\lambda, \mu; z), \text{spherical Chebyshev}$$

$$\mathcal{T}_0(\lambda, \mu; z) = 1$$
$$\mathcal{T}_1(\lambda, \mu; z) = z$$
$$\mathcal{T}_2(\lambda, \mu; z) = \frac{3}{2}z^2 - \frac{1}{2}$$
$$\mathcal{T}_3(\lambda, \mu; z) = \frac{5}{2}z^3 - \frac{3}{2}z$$

Example



$\mathcal{T}_3(\lambda, \mu; z)$

bivariate:

$$\mathcal{T}_{m+1}(\lambda, \mu; z) = \frac{2m+1}{m+1} z \mathcal{T}_m(\lambda, \mu; z) - \frac{m}{m+1} \mathcal{T}_{m-1}(\lambda, \mu; z)$$
$$= C_{m+1}^{(1/2)}(z)$$

Cubature and Padé approximation

$$H_m(\lambda,\mu) := \begin{vmatrix} c_0(\lambda,\mu) & \dots & c_{m-1}(\lambda,\mu) \\ \vdots & \iddots & \vdots \\ c_{m-1}(\lambda,\mu) & \dots & c_{2m-2}(\lambda,\mu) \end{vmatrix}, \qquad H_0(\lambda,\mu) := 1$$

$\Gamma(z^i) = c_i(\lambda,\mu)$ positive definite if $H_m(\lambda,\mu) > 0$ for $m \geq 0$

$$\mathcal{V}_m\left(\lambda,\mu; z_i^{(m)}(\lambda,\mu)\right) = 0$$

Cubature and Padé approximation

$$H_m(\lambda,\mu) := \begin{vmatrix} c_0(\lambda,\mu) & \dots & c_{m-1}(\lambda,\mu) \\ \vdots & \iddots & \vdots \\ c_{m-1}(\lambda,\mu) & \dots & c_{2m-2}(\lambda,\mu) \end{vmatrix}, \qquad H_0(\lambda,\mu) := 1$$

$\Gamma(z^i) = c_i(\lambda,\mu)$ positive definite if $H_m(\lambda,\mu) > 0$ for $m \geq 0$

$$\mathcal{V}_m\left(\lambda,\mu; z_i^{(m)}(\lambda,\mu)\right) = 0$$

$$A_i^{(m)}(\lambda,\mu) = \iint_{\|(x,y)\|_p \leq 1} \ell_i(z) w(z)\,dx\,dy$$

Cubature and Padé approximation

$$\iint_{\|(x,y)\|_p \le 1} g(x,y) w(z)\, dx\, dy \approx \sum_{i=1}^{m} A_i^{(m)}(\lambda,\mu) g\left(\lambda,\mu; z_i^{(m)}\right)$$

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Cubature and Padé approximation

$$\iint_{\|(x,y)\|_p \le 1} g(x,y) w(z)\, dx\, dy \approx \sum_{i=1}^{m} A_i^{(m)}(\lambda,\mu) g\left(\lambda,\mu; z_i^{(m)}\right)$$

exact for $g(x,y) \in \mathbb{C}_{2m-1}[x,y]$

$$\iint_{\|(x,y)\|_p \le 1} p_{2m-1}(x,y) w(z)\, dx\, dy = \sum_{i=1}^{m} A_i^{(m)}(\lambda,\mu) p_{2m-1}\left(\lambda,\mu; z_i^{(m)}\right)$$

Cubature and Padé approximation

$$\begin{aligned} f(x,y) &= \iint_{\|(r,s)\|_p \le 1} \frac{w(t)}{1-(xr+ys)}\,dr\,ds, \qquad (r,s)=(\alpha,\beta)t \\ &= \iint_{\|(r,s)\|_p \le 1} \frac{w(t)}{1-(\lambda r+\mu s)z}\,dr\,ds \end{aligned}$$

$$[m-1/m]_H^f(\lambda z,\mu z) = \sum_{i=1}^{m} A_i^{(m)}(\lambda,\mu) \Big/ \left(1 - z_i^{(m)}(\lambda,\mu) z\right)$$

Illustration

disk: $\|(x, y)\|_2 \leq 1$, $w(z) = 1$, $V_4(x, y) = \mathcal{L}_4(\lambda, \mu; z)$

$$z_1^{(4)}(\lambda, \mu) = -\frac{\sqrt{3 + \sqrt{5}}}{2\sqrt{2}}$$

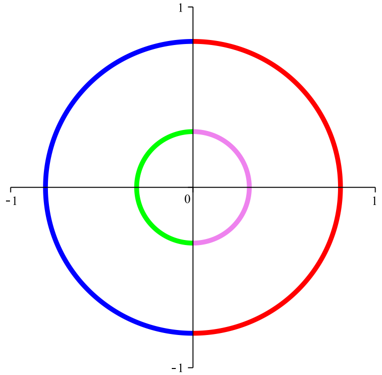
$$z_2^{(3)}(\lambda, \mu) = -\frac{\sqrt{3 - \sqrt{5}}}{2\sqrt{2}}$$

$$z_3^{(3)}(\lambda, \mu) = +\frac{\sqrt{3 - \sqrt{5}}}{2\sqrt{2}}$$

$$z_4^{(4)}(\lambda, \mu) = +\frac{\sqrt{3 + \sqrt{5}}}{2\sqrt{2}}$$

$$A_i^{(4)}(\lambda, \mu) = \text{independent of } (\lambda, \mu)$$

Illustration



$z_1^{(4)}(\lambda, \mu)$, $z_2^{(4)}(\lambda, \mu)$, $z_3^{(4)}(\lambda, \mu)$, $z_4^{(4)}(\lambda, \mu)$

Illustration

disk: $\|(x, y)\|_2 \leq 1, \ \lambda = \cos\theta, \ \mu = \sin\theta, \ -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$

$$\iint_{\|(x,y)\|_2 \leq 1} p_{2m-1}(x\cos\theta + y\sin\theta) w(x, y)\, dx\, dy$$

independent of θ

$$\Downarrow$$

symbolic 4-curve Gaussian cubature rule

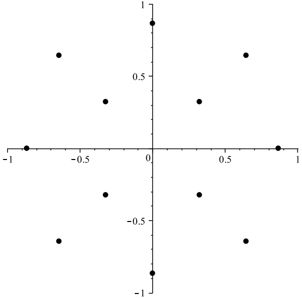
$$\Leftrightarrow$$

numeric 12-point minimal cubature rule

Illustration

$w(z) = 1,\ m = 4,\ 2m-1 = 7$

$$\iint_{\|(x,y)\|_2\le 1} f(x,y)\,dx\,dy \approx \sum_{i=1}^{12} B_i^{(4)} f(x_i,y_i)$$



$$B_i^{(4)} = \frac{(551 \pm 41\sqrt{29})\pi}{6264}\ \text{(rectangles)},\ \frac{2\pi}{27}\ \text{(axes)}$$

Illustration

square: $\|(x,y)\|_\infty \le 1,\ w(z) = 1,\ V_3(x,y) = \mathcal{L}_3(\lambda,\mu;z)$

$$z_1^{(3)}(\lambda,\mu) = -\sqrt{\frac{3(\lambda^4+\mu^4)+10\lambda^2\mu^2}{5(\lambda^2+\mu^2)}}$$

$$z_2^{(3)}(\lambda,\mu) = 0$$

$$z_3^{(3)}(\lambda,\mu) = \sqrt{\frac{3(\lambda^4+\mu^4)+10\lambda^2\mu^2}{5(\lambda^2+\mu^2)}}$$

$$A_1^{(3)}(\lambda,\mu) = A_3^{(3)}(\lambda,\mu) = \frac{2(\lambda^2+\mu^2)}{3\left(z_3^{(3)}(\lambda,\mu)\right)^2}$$

$$A_2^{(3)}(\lambda,\mu) = 4\left(1 - \frac{\lambda^2+\mu^2}{3\left(z_3^{(3)}(\lambda,\mu)\right)^2}\right)$$

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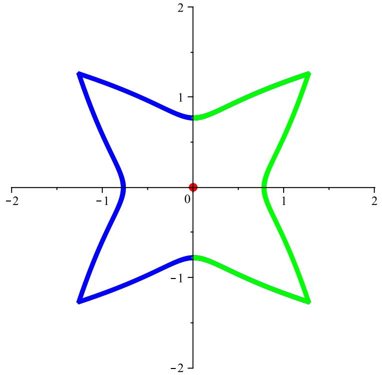
Illustration



$z_1^{(3)}(\lambda, \mu)$, $z_2^{(3)}(\lambda, \mu)$, $z_3^{(3)}(\lambda, \mu)$

Illustration

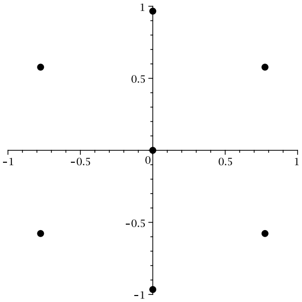
$$x(1-x)\frac{\partial^2 F_2}{\partial x^2} - xy\frac{\partial^2 F_2}{\partial x \partial y} + (2-(\alpha+2)x)\frac{\partial F_2}{\partial x} - y\frac{\partial F_2}{\partial y} - \alpha F_2 = 0$$

$$\begin{aligned} F_2(\alpha, 1, 1; 2, 2; \lambda, \mu) \\ &= \int_0^1 \int_0^1 (1-\lambda x - \mu y)^{-\alpha}\, dx\, dy \\ &= \int_{-1}^1 \int_{-1}^1 \frac{1}{4}\left(1 - \frac{\lambda+\mu+\lambda x+\mu y}{2}\right)^{-\alpha} dx\, dy \\ &\approx G_3(\lambda,\mu) := \sum_{i=1}^{3} \frac{A_i^{(3)}(\lambda,\mu)}{4}\left(1 - \frac{\lambda+\mu+z_i^{(3)}(\lambda,\mu)}{2}\right)^{-\alpha} \end{aligned}$$

Illustration

compare to Radon's 7-point order 6 approximation R_3 [Radon 1948]

$$F_2(\alpha, 1, 1; 2, 2; \lambda, \mu) \approx R_3 := \sum_{i=1}^{7} \frac{B_i^{(3)}}{4} \left(1 - \frac{\lambda + \mu + \lambda x_i + \mu y_i}{2}\right)^{-\alpha}$$



$$B_i^{(3)} = \frac{5}{9} \text{ (rectangle)}, \frac{20}{63} \text{ (axis)}, \frac{8}{7} \text{ (origin)}$$

$F_2(\log 2, 1, 1; 2, 2; r\cos j\pi/k, r\sin j\pi/k)$

r	j	k	F_2	$F_2 - G_3$	$F_2 - R_3$
1/2	0	1	$1.24878542341637 \times 10^0$	2.0×10^{-5}	2.0×10^{-5}
1/2	1	2	$1.24878542341637 \times 10^0$	2.0×10^{-5}	-6.3×10^{-6}
1/2	1	1	$8.63536332176800 \times 10^{-1}$	5.8×10^{-7}	5.8×10^{-7}
1/2	3	4	$1.01276420123863 \times 10^0$	8.9×10^{-6}	7.1×10^{-6}
1	0	1	$3.25889070928323 \times 10^0$	9.8×10^{-1}	9.8×10^{-1}
1	1	2	$3.25889070928323 \times 10^0$	9.8×10^{-1}	2.3×10^{-1}
1	1	1	$7.72377702823892 \times 10^{-1}$	1.2×10^{-5}	1.2×10^{-5}
1	2	3	$1.26581509450621 \times 10^0$	4.9×10^{-3}	4.8×10^{-3}
1	5	4	$7.02555439039178 \times 10^{-1}$	1.7×10^{-5}	1.3×10^{-5}
1	7	4	$1.05976599939755 \times 10^0$	9.9×10^{-4}	8.2×10^{-4}
5	5	4	$3.77604178657645 \times 10^{-1}$	7.6×10^{-4}	6.3×10^{-4}
10	5	4	$2.60473326815451 \times 10^{-1}$	1.5×10^{-3}	1.2×10^{-3}

$F_2(1/8, 1, 1; 2, 2; r\cos j\pi/k, r\sin j\pi/k)$

r	j	k	F_2	$F_2 - G_3$	$F_2 - R_3$
1/2	0	1	$1.03941973413604 \times 10^{0}$	1.0×10^{-6}	1.0×10^{-6}
1/2	1	2	$1.03941973413604 \times 10^{0}$	1.0×10^{-6}	-3.3×10^{-7}
1/2	1	1	$9.73416542767547 \times 10^{-1}$	4.1×10^{-8}	4.1×10^{-8}
1/2	3	4	$1.00150741213994 \times 10^{0}$	5.5×10^{-7}	4.4×10^{-7}
1	0	1	$1.14285714496901 \times 10^{0}$	1.1×10^{-2}	1.1×10^{-2}
1	1	2	$1.14285714496901 \times 10^{0}$	1.1×10^{-2}	-3.9×10^{-3}
1	1	1	$9.53152098789471 \times 10^{-1}$	9.5×10^{-7}	9.5×10^{-7}
1	2	3	$1.03657602454864 \times 10^{0}$	2.3×10^{-4}	2.2×10^{-4}
1	5	4	$9.37292569668350 \times 10^{-1}$	1.4×10^{-6}	1.1×10^{-6}
1	7	4	$1.00667220285694 \times 10^{0}$	5.6×10^{-5}	4.6×10^{-5}
5	5	4	$8.34745461087668 \times 10^{-1}$	9.8×10^{-5}	8.1×10^{-5}
10	5	4	$7.78937606474670 \times 10^{-1}$	2.4×10^{-4}	2.1×10^{-4}

Bivariate orthogonal basis

$$\iint_{\|(x,y)\|_p \le 1} (\lambda_k x + \mu_k y)^i \mathcal{V}_m(\lambda_k, \mu_k; \lambda_k x + \mu_k y) w(z)\, dx\, dy = 0, \quad i < m$$

Bivariate orthogonal basis

$$\iint_{\|(x,y)\|_p \le 1} (\lambda_k x + \mu_k y)^i \mathcal{V}_m(\lambda_k, \mu_k; \lambda_k x + \mu_k y) w(z)\, dx\, dy = 0, \qquad i < m$$

$\{\mathcal{V}_m(\lambda_k, \mu_k; \lambda_k x + \mu_k y), 0 \le k \le m\}$ bivariate basis if

$$\begin{vmatrix} \lambda_0^m & \lambda_0^{m-1}\mu_0 & \dots & \lambda_0\mu_0^{m-1} & \mu_0^m \\ \lambda_1^m & \lambda_1^{m-1}\mu_1 & \dots & \lambda_1\mu_1^{m-1} & \mu_1^m \\ \vdots \\ \lambda_m^m & \lambda_m^{m-1}\mu_m & \dots & \lambda_m\mu_m^{m-1} & \mu_m^m \end{vmatrix} \ne 0$$

Bivariate orthogonal basis

$$w_\nu(z) = (1-z^2)^{\nu-\frac{1}{2}}$$

in addition to

$$\iint_{\|(x,y)\|_p \le 1} (\lambda_k x + \mu_k y)^i \mathcal{V}_m(\lambda_k, \mu_k; \lambda_k x + \mu_k y) w_\nu(z)\, dx\, dy = 0,$$

$$i < m$$

on the Euclidean unit disk $\|(x,y)\|_2 \le 1$ also holds

$$\iint_{\|(x,y)\|_2 \le 1} (\lambda_k x + \mu_k y)^i \mathcal{V}_m(\lambda_j, \mu_j; \lambda_j x + \mu_j y) w_\nu(z)\, dx\, dy = 0,$$

$$\lambda_k \neq \lambda_j,\, \mu_k \neq \mu_j,\, i < m$$

Bivariate orthogonal basis

for $\nu = 1/2$ and $\lambda_k = \cos\left(\frac{k\pi}{m+1}\right)$, $\mu_k = \sin\left(\frac{k\pi}{m+1}\right)$ even the following

$$\iint_{\|(x,y)\|_2 \le 1} (\lambda_k x + \mu_k y)^m \mathcal{V}_m(\lambda_j, \mu_j; \lambda_j x + \mu_j y)\, dx\, dy = 0, \qquad j \neq k$$

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$$\mathcal{V}_m(\lambda, \mu; z) = \mathcal{L}_m(\lambda, \mu; z) = U_m(z), \text{ Chebyshev second kind}$$

$$\left\{U_m\left(x\cos\left(\frac{k\pi}{m+1}\right) + y\sin\left(\frac{k\pi}{m+1}\right)\right), 0 \le k \le m\right\} \text{ fully orthogonal basis}$$

Bivariate orthogonal basis

$$m = 0 : U_0 = 1$$

$$m = 1 : U_1(x) = 2x$$
$$U_1(y) = 2y$$

$$m = 2 : U_2(x) = 4x^2 - 1$$
$$U_2\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) = x^2 + 2xy\sqrt{3} + 3y^2 - 1$$
$$U_2\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) = x^2 - 2xy\sqrt{3} + 3y^2 - 1$$
$$\text{a.o.} : \iint_{\|(x,y)\|_2 \le 1} U_2\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) U_2\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) dx dy = 0$$

Bivariate orthogonal basis

$$m = 3 : U_3(x) = 8x^3 - 4x$$

$$U_3(y) = 8y^3 - 4y$$

$$U_3\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) = 2\sqrt{2}x^3 + 6\sqrt{2}x^2y + 6\sqrt{2}xy^2 + 2\sqrt{2}y^3 - 2\sqrt{2}x - 2\sqrt{2}y$$

$$U_3\left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) = -2\sqrt{2}x^3 + 6\sqrt{2}x^2y - 6\sqrt{2}xy^2 + 2\sqrt{2}y^3 - 2\sqrt{2}x - 2\sqrt{2}y$$

$$\text{a.o.} : \iint_{\|(x,y)\|_2 \le 1} U_3\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) U_3\left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) dx\,dy = 0$$

Lebesgue constants

polynomial interpolation:

$$p_m(x,y) = \sum_{i=0}^{M} f_i \ell_i(x,y), \qquad M+1 = \frac{(m+1)(m+2)}{2}$$

$$(x_i, y_i) \in \{\|(x,y)\|_2 \le 1\}$$

$$\varphi_m(x,y) = (x^0y^0 \dots x^{j_1}y^{j_2} \dots x^0y^m), \quad 0 \le j_1 + j_2 \le m$$

$$\tau_m((x_0,y_0),\dots,(x_M,y_M)) = \begin{bmatrix} \varphi_m(x_0,y_0) \\ \vdots \\ \varphi_m(x_M,y_M) \end{bmatrix}$$

Lebesgue constants

$$\ell_i(x, y) = \frac{\det \tau_m \left((x_0, y_0), \ldots, (x_{i-1}, y_{i-1}), (x, y), (x_{i+1}, y_{i+1}), (x_M, y_M)\right)}{\det \tau_m \left((x_0, y_0), \ldots, (x_M, y_M)\right)},$$

$$\det(\tau_m) \neq 0$$

Lebesgue constants

$$\ell_i(x,y) = \frac{\det \tau_m\left((x_0,y_0),\ldots,(x_{i-1},y_{i-1}),(x,y),(x_{i+1},y_{i+1}),(x_M,y_M)\right)}{\det \tau_m\left((x_0,y_0),\ldots,(x_M,y_M)\right)},$$

$$\det(\tau_m) \neq 0$$

optimize:

$$\Lambda_m\left((x_0,y_0),\ldots,(x_M,y_M)\right) = \max_{\|(x,y)\|_2 \leq 1} \sum_{i=0}^{M} |\ell_i(x,y)|$$

Lebesgue constants

unisolvence for several configurations of interpolation points on concentric circles in the unit disk [Bojanov and Xu, 2003]:

$$n_1 + \cdots + n_k = \left\lfloor \frac{m}{2} \right\rfloor + 1$$

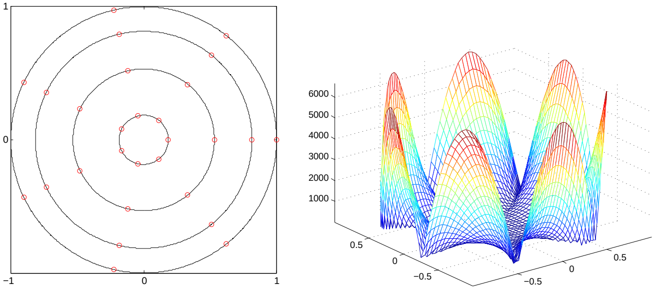
$$\begin{aligned} m_1 &= m - n_1 + 1 \\ m_2 &= m - 2n_1 - n_2 + 1 \\ &\vdots \\ m_k &= m - 2n_1 - \cdots - 2n_{k-1} - n_k + 1 \end{aligned}$$

$j = 1, \ldots, k$:

take $2m_j + 1$ equidistant points on the n_j circles with radii $r_1^{(j)}, \ldots, r_{n_j}^{(j)}$

Example

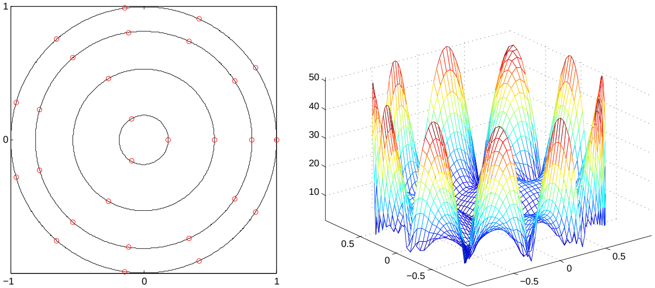
$m = 6$,
$k = 1$,
$n_1 = 4(m_1 = 3)$: 4 concentric circles with 7 points each



28 interpolation points and the Lebesgue function

Example

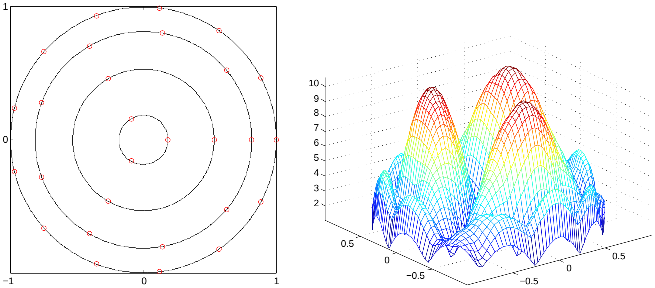
$m = 6$,
$k = 2$,
$n_1 = 1(m_1 = 6)$, $n_2 = 3(m_2 = 2)$: 1 circle with 13 points and 3 concentric circles with 5 points each



28 interpolation points and the Lebesgue function

Example

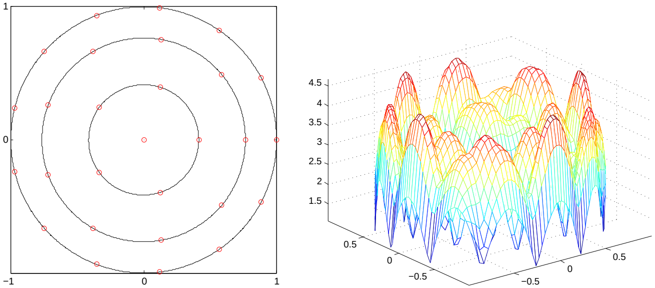
$m = 6$,
$k = 3$,
$n_1 = 1(m_1 = 6)$, $n_2 = 2(m_2 = 3)$, $n_3 = 1(m_3 = 0)$: 1 circle with 13 points, 2 concentric circles with 7 points each, and the origin



28 interpolation points and the Lebesgue function

Example

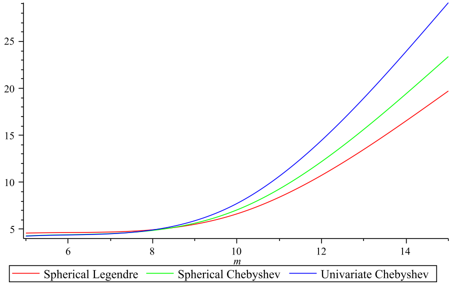
$m = 6$,
$k = 4$,
$n_1 = 1(m_1 = 6)$, $n_2 = 1(m_2 = 4)$, $n_3 = 1(m_3 = 2)$, $n_4 = 1(m_4 = 0)$: 3 concentric circles with 13, 9, 5 points respectively and the origin



28 interpolation points and the Lebesgue function

Example

m; $k := \lfloor \frac{m}{2} \rfloor + 1$; $n_1 = \cdots = n_k = 1$;
$m_1 = 2m + 1$, $m_{j+1} = m_j - 4, j = 1, \ldots, k - 1$;
radii $r_1^{(j)}$, $j = 1, \ldots, k$, zeros of orthogonal polynomials



growth of Lebesgue constant on the unit disk