Combining the Radon, Markov and Stieltjes transforms for object reconstruction

Annie CUYT and Brigitte VERDONK

Dept of Mathematics and Computer Science, University of Antwerp
Middelheimlaan 1, B–2020 Antwerpen, Belgium
{annie.cuyt, brigitte.verdonk}@ua.ac.be

Abstract. In shape reconstruction, the celebrated Fourier slice theorem plays an essential role. By virtue of the relation between the Radon transform, the Fourier transform and the 2-dimensional inverse Fourier transform, the shape of an object can be reconstructed from the knowledge of the object's Radon transform. Unfortunately, a discrete implementation requires the use of interpolation techniques, such as in the filtered back projection.
We show how the need for interpolation can be overcome by using the relationship between the Radon transform, the Markov transform and the 2-dimensional Stieltjes transform. When combining the knowledge of an object's Radon transform for discrete angles θ, with the less well-known Padé slice theorem, the object under consideration can be reconstructed from the solution of a linear least squares problem.
The new technique is applicable in all higher dimensions. Here we illustrate it through the reconstruction of some interesting two-dimensional objects.

1 The Radon, Markov and Stieltjes integral transforms

The Radon transform $R_{\vec{\xi}}\,(u)$ of a square-integrable n-variate function $f(\vec{x})$ with $\vec{x} = (x_1, \ldots, x_n)$ is defined as

$$R_{\vec{\xi}}\,(u) = \int_{\mathbb{R}^n} f(\vec{x})\ \delta(\vec{\xi}\vec{x} - u)\ d\vec{x} \qquad d\vec{x} = dx_1 \ldots dx_n$$

with $||\vec{\xi}|| = 1$ and $\vec{\xi}\vec{x} = u$ an $(n-1)$-dimensional manifold orthogonal to $\vec{\xi}$. When $n = 2$, $\vec{\xi}$ is fully determined by an angle θ and

$$R_\theta(u) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t,s)\,\delta(t\cos\theta + s\sin\theta - u)\,dt\,ds$$

For $n = 3$, $\vec{\xi}$ is determined by angles θ and ϕ and

$$R_{\theta,\phi}(u) = \int_{\mathbb{R}^3} f(t,s,v)\,\delta(t\cos\phi\cos\theta + s\cos\phi\sin\theta + v\sin\phi - u)\,dt\,ds\,dv$$

In the sequel of the text, to simplify notation, we mainly focus on the two-dimensional case, without loss of generality. Let the square-integrable function $f(t,s)$ be defined in a compact region A of the first quadrant $t \geq 0,\ s \geq 0$ of the plane. According to a fundamental property of the Radon transform $R_\theta(u)$ of $f(t,s)$ [7], the following relation holds for any square-integrable function $F(u)$:

$$\int_{-\infty}^{+\infty} R_\theta(u) F(u)\, du = \int_0^\infty \int_0^\infty f(t,s) F(t\cos\theta + s\sin\theta)\, dt\, ds \tag{1}$$

If we take $F(u) = 1/(1+zu)$, then

$$g_\theta(z) = \int_{-\infty}^{+\infty} \frac{R_\theta(u)}{1+zu}\, du = \int_0^\infty \int_0^\infty \frac{f(t,s)}{1+(t\cos\theta + s\sin\theta)z}\, dt\, ds \tag{2}$$

A Markov function is defined to be a function with an integral representation

$$g(z) = \int_a^b \frac{f(u)}{1+zu}\, du \qquad -\infty < a \leq 0 \leq b < +\infty \tag{3}$$
$$z \notin]-\infty, -1/b] \cup [-1/a, +\infty[$$

where $f(u)$ is non-trivial and positive and the moments

$$c_i = \int_a^b u^i f(u)\, du \qquad i = 0, 1, \ldots \tag{4}$$

are finite. If f is nonzero in $[a,b]$ with $0 < a < b$ then (3) is considered on $[0,b]$. If f is nonzero in $[a,b]$ with $a < b < 0$, then (3) is considered on $[a,0]$. A Markov series is defined to be a series

$$\sum_{i=0}^{\infty} (-1)^i c_i z^i \tag{5}$$

which is derived by a formal expansion of (3). The Markov function $g(z)$ is also called the Markov transform of the function $f(u)$. Furthermore, in case (5) is the formal series expansion of a Markov function with a nonzero radius of convergence, the Markov moment problem, in which one reconstructs $f(u)$ from the moments c_i, is determinate. In other words, given the moments c_i, a function $f(u)$ exists which allows the representation of c_i by (4) and this function $f(u)$ is uniquely determined.

A bivariate Stieltjes function $g(z,w)$ is defined by the integral representation

$$g(z,w) = \int_0^\infty \int_0^\infty \frac{f(t,s)}{1+(zt+ws)}\, dt\, ds \tag{6}$$

where $f(t,s)$ is non-trivial and positive. Its finite real-valued moments are given by

$$c_{ij} = \int_0^\infty \int_0^\infty t^i s^j f(t,s)\, dt\, ds$$

A formal expansion of (6) provides the bivariate Stieltjes series

$$\sum_{i,j=0}^{\infty} \binom{i+j}{i} (-1)^{i+j} c_{ij} z^i w^j \tag{7}$$

The function $g(z, w)$ is also called the bivariate Stieltjes transform of $f(t, s)$.

Now let us have another look at (2) and identify our object under reconstruction with its characteristic function. If $f(t, s)$ is the characteristic function of a compact set A lying in the first quadrant, then $g_\theta(z)$ is a Markov function, because $R_\theta(u)$ is zero outside a region of compact support. Furthermore, since $g_\theta(z) = g(z\cos\theta, z\sin\theta)$, there is a close link between the bivariate Stieltjes transform of the characteristic function of A and the Markov transform of its Radon transform. In order to translate these properties into an algorithm for the reconstruction of A from the knowledge of its Radon transform $R_\theta(u)$, we need to show that its Markov transform is easy to compute.

2 The Padé slice property

Given a series of the form (5), one constructs Padé approximants of this series as follows. With the moments c_i introduced in (4), the coefficients $a_0, \ldots, a_{m+k}$ and $b_0, \ldots, b_m$ are computed such that for

$$p_{m+k,m}(z) = \sum_{i=0}^{m+k} a_i z^i \qquad q_{m+k,m}(z) = \sum_{i=0}^{m} b_i z^i$$

the series expansion of $(gq_{m+k,m} - p_{m+k,m})(z)$ satisfies

$$\sum_{i=0}^{\infty} d_i z^i = \left(\sum_{i=0}^{\infty} (-1)^i c_i z^i\right) q_{m+k,m}(z) - p_{m+k,m}(z) = O(z^{2m+k+1}) \tag{8}$$

In other words, the $2m + k + 2$ coefficients $a_0, \ldots, a_{m+k}$ and $b_0, \ldots, b_m$ are determined from the $2m + k + 1$ conditions $d_0 = 0, \ldots, d_{2m+k} = 0$ and an additional normalization condition for $p_{m+k,m}(z)/q_{m+k,m}(z)$. The irreducible form of $p_{m+k,m}(z)/q_{m+k,m}(z)$ is denoted by $r_{m+k,m}(z)$ and is called the $(m + k, m)$ Padé approximant. It is usually normalized by putting the constant term in the denominator equal to 1. The following theorems play a crucial role in our novel object reconstruction technique.

Theorem 1. [1, p. 228] *For the Markov function* (3)*, each sequence of Padé approximants* $\{r_{m+k,m}(z)\}_{m\in\mathbb{N}}$ *with* $k \geq -1$ *converges to* (3) *for* $z \notin]-\infty, -1/b] \cup [-1/a, +\infty[$*. The rate of convergence is governed by*

$$\limsup_{m\to\infty} |g(z) - r_{m+k,m}(z)|^{1/m} \leq \left| \frac{\sqrt{1/z + b} - \sqrt{1/z + a}}{\sqrt{1/z + b} + \sqrt{1/z + a}} \right|$$

Padé approximants have been generalized to higher dimensions by several authors in different ways. For an overview and comparison of these definitions the reader is referred to [5]. For our purpose the definition given in [4, 3] is most useful. Again without loss of generality, we repeat it only for bivariate functions, but it can be defined in any number of variables.

Given the moments c_{ij}, one can compute an $(m+k, m)$ homogeneous bivariate Padé approximant of (7) as follows. First, we introduce the homogeneous expressions

$$A_\ell(z,w) = \sum_{i+j=\ell} a_{ij} z^i w^j \qquad B_\ell(z,w) = \sum_{i+j=\ell} b_{ij} z^i w^j$$

to define the polynomials

$$p_{m+k,m}(z,w) = \sum_{\ell=(m+k)m}^{(m+k)(m+1)} A_\ell(z,w)$$
$$q_{m+k,m}(z,w) = \sum_{\ell=(m+k)m}^{(m+k+1)m} B_\ell(z,w)$$

Second, we introduce the notation

$$C_\ell(z,w) = \sum_{i+j=\ell} \binom{\ell}{i} c_{ij} z^i w^j \tag{9}$$

and write down the homogeneous accuracy-through-order conditions

$$\begin{aligned}\sum_{i,j=0}^{\infty} d_{ij} z^i w^j &= \left(\sum_{\ell=0}^{\infty} (-1)^\ell C_\ell(z,w)\right) q_{m+k,m}(z,w) - p_{m+k,m}(z,w) \\ &= O\left(z^i w^j, i+j \geq (m+k+2)m+k+1\right)\end{aligned} \tag{10}$$

It has been shown [4, pp. 60–61] that a nontrivial solution of these conditions can always be computed. Moreover, all solutions $p_{m+k,m}(z,w)/q_{m+k,m}(z,w)$ deliver the same unique irreducible form $r_{m+k,m}(z,w)$ which is called the homogeneous Padé approximant of (7). A proper normalization of $r_{m+k,m}(z,w)$ can still be chosen, but differs most of the times from the univariate normalization $q_{m+k,m}(0) = 1$ since the denominator of $r_{m+k,m}(z,w)$ need not start with a constant term. It starts with a homogeneous expression in z and w of as low degree as possible.

This homogeneous generalization of the Padé approximant is the only one to satisfy the following powerful slice property, formulated in Theorem 2. The Markov function given in (2),

$$g_\theta(z) = \int_0^\infty \int_0^\infty \frac{f(t,s)}{1+(t\cos\theta + s\sin\theta)z}\, dt\, ds \tag{11a}$$
$$= g(z\cos\theta, z\sin\theta) \qquad -\pi/2 < \theta \leq \pi/2 \tag{11b}$$

is viewed here as a particular slice of the bivariate Stieltjes transform. Let us denote the univariate $(m+k,m)$ Padé approximant of $g_\theta(z)$ by $r_{m+k,m}^{(g_\theta)}(z)$. Then the following projection property holds, even for a more general class of functions than the Stieltjes transform $g(z,w)$.

Theorem 2. [8, 2] *Let $g(z,w)$ be holomorphic in the origin and let $g_\theta(z) = g(z\cos\theta, z\sin\theta)$. The homogeneous Padé approximant $r_{m+k,m}(z,w)$ of $g(z,w)$ satisfies*

$$r_{m+k,m}(z\cos\theta, z\sin\theta) = r_{m+k,m}^{(g_\theta)}(z) \qquad -\pi/2 < \theta \leq \pi/2$$

where $r_{m+k,m}^{(g_\theta)}(z)$ is the univariate Padé approximant of $g_\theta(z)$.

In other words, restricting the homogeneous Padé approximant of $g(z,w)$ to the slice

$$S_\theta = \{(z\cos\theta, z\sin\theta) \mid z \in \mathbb{R}\} \tag{12}$$

is equivalent to computing the univariate Padé approximant of the slice function $g_\theta(z)$. Because of the link between the Markov, the Radon and the Stieltjes transform expressed in (2) and (11), we have that

$$C_\ell(z\cos\theta, z\sin\theta) = \sum_{i=0}^{\ell} \binom{\ell}{i} c_{i,\ell-i} \cos^i\theta \sin^{\ell-i}\theta \qquad \ell = 0, 1, 2, \ldots \tag{13}$$

are the univariate moments of the Radon transform $R_\theta(u)$, which we denote in the sequel by $C_\ell^{(\theta)}$.

3 Algorithm for use with discrete angles $\boldsymbol{\theta}$.

It is now easy to reconstruct the characteristic function $f(t,s)$ of a compact set A lying in the first quadrant, from its Radon transform. With the tools discussed in Section 1 and 2, we can formulate the following reconstruction algorithm. In order to have $-1 \leq a(\theta) \leq b(\theta) \leq 1$ in Theorem 1, we further assume that A is also lying within the unit circle. This is only a matter of scaling.

- Input of the algorithm is some indirect information that is available on the object A, either its Radon transform for a discrete number of angles θ_n (bivariate case) or θ_n and ϕ_k (trivariate case). If the univariate moments $C_\ell^{(\theta)}$ of the Radon transform or the multivariate moments c_{ij} of $f(t,s)$ (bivariate) or c_{ijk} of $f(t,u,v)$ (trivariate) are given instead, one skips the first, respectively the first two steps of the algorithm.
- Compute the moments

$$C_\ell^{(\theta)} = \int_{a(\theta)}^{b(\theta)} u^\ell R_\theta(u)\, du$$

for a discrete number of angles $\theta = \theta_n$ with $0 \le n \le N$. From the parameterized moments $C_\ell^{(\theta_n)}$ the bivariate moments $c_{i,\ell-i}$ can be computed by solving (13), possibly in the least squares sense. To this end, the Radon transform must be available for a sufficient number of angles θ.

- With the moments c_{ij} one computes, for successive m, the homogeneous Padé approximant $r_{m-1,m}(z,w)$ of the Stieltjes transform $g(z,w)$. Increasing m to $m+1$, implies adding the moments $c_{i,2m-i}$ and $c_{i,2m+1-i}$ to the data. The latter may imply an increase of the number of angles θ for which (13) can be written down. Theorem 1 used in conjunction with Theorem 2 guarantees that on each slice S_{θ_n} the $\{r_{m-1,m}(z)\}_{m\in\mathbb{N}}$ converge rapidly to $g(z,w)$ restricted to that slice. The correct relationship between m and the number N of angles θ is as follows:
  1. Start with $m = 1$ and compute $r_{0,1}(z,w)$ from c_{00}, c_{10} and c_{01}, or equivalently from $C_0^{(\theta_0)}$ and $C_1^{(\theta_n)}$ for $n = 0, 1$. The latter two parameterized moments allow to obtain the two bivariate moments c_{10} and c_{01}.
  2. As long as $||r_{m-2,m-1} - r_{m-1,m}||/||r_{m-1,m}||$ is not small enough, increase m to $m+1$ and compute $r_{m,m+1}(z,w)$. To this end we need to know the moments $c_{i,2m-i}$ and $c_{i,2m+1-i}$. These can be computed from $C_{2m}^{(\theta_0)}, \ldots, C_{2m}^{(\theta_{2m})}$ and $C_{2m+1}^{(\theta_0)}, \ldots, C_{2m+1}^{(\theta_{2m+1})}$ using (13).
  3. In the end, for the last computed $r_{m-1,m}(z,w)$, we have $N = 2m+1$. Since the sequence $\{r_{m-1,m}(z,w)\}_m$ converges quite rapidly to $g(z,w)$, N is usually not very large.
- At the same time, for each $-\pi/2 < \theta_n \le \pi/2$ and each $0 \le z_j \le 1$, the value of the Stieltjes transform $g(z,w)$ evaluated at $(z_j \cos\theta_n, z_j \sin\theta_n)$ can be approximated to high accuracy by a cubature formula

$$\sum_{i=1}^{L} \frac{\omega_i}{1 + z_j(t_i \cos\theta_n + s_i \sin\theta_n)} f(t_i, s_i) \qquad n = 0, 1, \ldots, \qquad j = 0, 1, \ldots$$

  with weights ω_i and nodes (t_i, s_i). Subsequently the values $f(t_i, s_i)$ are computed from the least squares problem

$$\sum_{i=1}^{L} \frac{\omega_i}{1 + z_j(t_i \cos\theta_n + s_i \sin\theta_n)} f(t_i, s_i) \approx g(z_j \cos\theta_n, z_j \sin\theta_n) \tag{14a}$$

$$= \lim_{m\to\infty} r_{m-1,m}(z_j \cos\theta_n, z_j \sin\theta_n) \tag{14b}$$

- The reconstruction of A is identified with

$$A \approx \{(t_i, s_i) \mid f(t_i, s_i) \ge 0.5\}$$

  The threshold 0.5 is chosen because for the original shape $f(t,s) = 1$ inside A and $f(t,s) = 0$ outside A.

Note that the homogeneous Padé approximant $r_{m+k,m}(z,w)$ is expressed in cartesian coordinates for its computation and that we switch to polar coordinates to invoke Theorem 1 and 2 and write down the linear system (14). Since

$r_{m+k,m}(z)$ is a bivariate rational function and not a set of discrete data, this does not create a problem at all.
Since the homogeneous Padé approximant can be defined analogously in any number of variables, the procedure for three-dimensional shape reconstruction is entirely similar. Then we restrict ourselves to the slices

$$S_{\theta,\phi} = \{(z\cos\phi\cos\theta, z\cos\phi\sin\theta, z\sin\phi) \mid z \in \mathbb{R}\}$$

and the 3-dimensional version of (1) and the Theorems 1 and 2 yields

$$\int_{-\infty}^{+\infty} \frac{R_{\theta,\phi}(u)}{1+zu} du = \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(t,u,v)}{1+(t\cos\phi\cos\theta + u\cos\phi\sin\theta + v\sin\phi)z}\, dt\, du\, dv$$
$$= \lim_{m\to\infty} r_{m+k,m}(z\cos\phi\cos\theta, z\cos\phi\sin\theta, z\sin\phi)$$

Note that the vector $(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ generating the 1-dimensional slice $S_{\theta,\phi}$ belongs to the 3-dimensional unit sphere.

4 Numerical illustration

Without loss of generality we only give 2-dimensional numerical examples. Within the set of interesting objects A we present a non-convex example (reconstruction of the lemniscate in Figure 1) and an example with non-connected boundary (reconstruction of the ellips with hole in Figure 2).
In the visualization of the reconstructed planar object, we delimit the original shape in black and show the reconstructed area in grey. With the reconstruction of the object, we also list the number of angles θ_n and the number of radial points z_j used in the least squares formulation (14), the degree m of the Padé denominator and the relative error $\epsilon = \max_{x^2+y^2\le 1} |r_{m-2,m-1} - r_{m-1,m}|/|r_{m-1,m}|$ in the computation of the Padé approximant. The value ϵ is an estimate of the relative error present in the right hand side of the linear least squares problem (14).
The least squares problem (14), which is an inverse problem, is in general ill-conditioned and therefore a regularization technique must be applied. In all of the following examples we have found the technique known as truncated SVD [6] to do an excellent job.
For the approximation of $g(z_j\cos\theta_n, z_j\sin\theta_n)$ we use the simple compound 4-point Gauss-Legendre product rule [9, pp. 230–231]

$$\int_a^{a+h} \int_b^{b+k} \frac{f(t,s)}{1+z_j(t\cos\theta_n + s\sin\theta_n)} dt\, ds \approx hk \sum_{i=1}^{4} \frac{f(t_i,s_i)}{1+z_j(t_i\cos\theta_n + s_i\sin\theta_n)}$$
$$(t_i,s_i) = (a + \frac{3\pm\sqrt{3}}{6}h, b + \frac{3\pm\sqrt{3}}{6}k)$$

with $-1 \le a < a+h \le 1, -1 \le b < b+k \le 1$. For Figure 1, $h = 1/16 = k$ and for Figure 2, $h = 1/32 = k$.

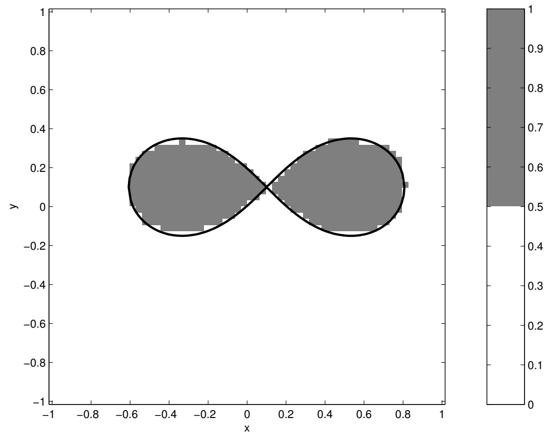

Fig. 1. $A = \{(t,u) \mid \big((t-0.1)^2 + (u-0.1)^2 + 1/4\big)^2 - (t-0.1)^2 = 1/16\}$
$\#\theta_n = 80, \#z_j = 60, h = k = 1/16, m = 10, \epsilon = 5.3 \times 10^{-7}$

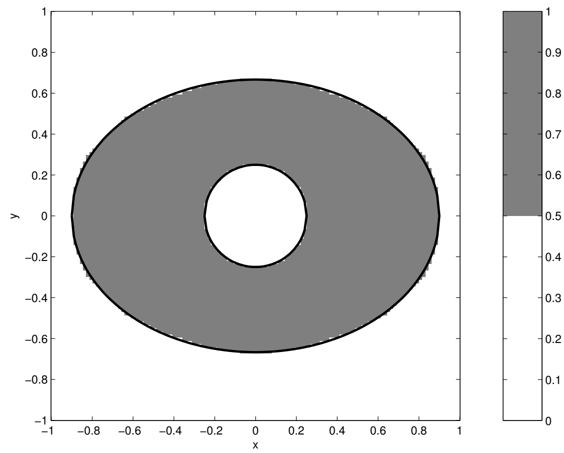

Fig. 2. $A = \{(t,u) \mid 81t^2/100 + 4u^2/9 \leq 1\} \setminus \{(t,u) \mid t^2 + u^2 < 1/16\}$
$\#\theta_n = 25, \#z_j = 15, h = k = 1/32, m = 10, \epsilon = 1.2 \times 10^{-4}$

References

1. G.A. Baker, Jr. and P. Graves-Morris. *Padé approximants (2nd Ed.)*. Cambridge University Press, 1996.
2. C. Chaffy. Interpolation polynomiale et rationnelle d'une fonction de plusieurs variables complexes. Thèse, Inst. Polytech. Grenoble, 1984.
3. A. Cuyt. A comparison of some multivariate Padé approximants. *SIAM J. Math. Anal.*, 14:195–202, 1983.
4. A. Cuyt. *Padé approximants for operators: theory and applications*. LNM 1065, Springer Verlag, Berlin, 1984.
5. A. Cuyt. How well can the concept of Padé approximant be generalized to the multivariate case. *J. Comput. Appl. Math.*, 105:25–50, 1999.
6. P.C. Hansen. The truncated SVD as a method for regularization. *BIT*, 27:543–553, 1987.
7. A.C. Kak and M. Slaney. *Principles of Computerized Tomographic Imaging*. IEEE Press, 1988.
8. J. Karlsson. Private communication. 1981.
9. A.H. Stroud. *Approximate calculation of multiple integrals*. Prentice-Hall, 1971.