Symbolic and Interval Rational Interpolation: the Problem of Unattainable Data

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Abstract. A typical problem with rational interpolation is that of a so-called unattainable point, when the interpolation condition cannot be met by the rational interpolant of the specified degree. The problem can be dealt with in at least two approaches, one of which is novel and practically oriented. We admit infinity in the independent variable as well as in the function value. Rational interpolation is solved symbolically in its full generality by Van Barel and Bultheel [9]. The authors return a parameterized set of rational interpolants of higher degree than requested but without unattainable points. In many practical applications however, observations are not exact but prone to imprecise measurements. A natural way for dealing with uncertainty in the data is by means of an uncertainty interval. It is shown in [1] how a rational function of lowest complexity can be obtained which intersects all uncertainty intervals and avoids the typical problem of unattainable data.

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RATIONAL INTERPOLATION

Given finite real data $\{(x_0, f_0), \ldots, (x_n, f_n)\}$, the classical problem of univariate rational interpolation [2, 3, 4] consists in finding an irreducible rational function $p(x)/q(x)$ that satisfies the interpolation conditions

$$\frac{p(x_i)}{q(x_i)} = f_i, \qquad i = 0, \ldots, n \tag{1}$$

and is such that the numerator and denominator polynomials satisfy the degree conditions $\partial p \le \ell$, $\partial q \le m$ with $\ell + m = n$. Non-trivial polynomials $p(x)$ and $q(x)$ of degree at most $\partial p \le \ell$ and $\partial q \le m$ that satisfy the linearized interpolation conditions

$$p(x_i) - q(x_i) f_i = 0, \qquad i = 0, \ldots, n \tag{2}$$

always exist. Indeed, when expanding $p(x)$ and $q(x)$ in terms of basis functions, for instance the monomials,

$$p(x) = \sum_{k=0}^{\ell} \alpha_k x^k, \quad q(x) = \sum_{k=0}^{m} \beta_k x^k \tag{3}$$

then (2) reduces to a homogeneous system of linear equations in the unknown coefficients of $p(x)$ and $q(x)$. Since this system has one more unknown than its total number of linear equations, the linearized interpolation problem (2) always has a non-trivial solution. Moreover, for any two non-trivial solutions $p_1(x), q_1(x)$ and $p_2(x), q_2(x)$ of (2) the polynomial $(p_1 q_2 - p_2 q_1)(x)$ of degree at most $\ell + m = n$ vanishes at $n+1$ distinct points:

$$(p_1 q_2 - p_2 q_1)(x_i) = \left[(f q_2 - p_2) q_1 - (f q_1 - p_1) q_2\right](x_i) = 0, \quad i = 0, \ldots, n.$$

Therefore it must vanish identically and we have $p_1(x) q_2(x) = p_2(x) q_1(x)$. Hence all solutions of (2) are equivalent and, up to a normalization, have the same irreducible form $r_{\ell,m}(x) = p_0(x)/q_0(x)$, which is called the *rational interpolant*.

It is well-known that the interpolating polynomial of degree at most n, interpolating $n+1$ points always exists and is unique. The condition $\ell + m = n$ is imposed in order to obtain this exact same analogy. It is true that the linearized rational interpolation problem (2) maintains the analogy to some extent, because it always has a solution and delivers the unique representation $r_{\ell,m}(x)$. However, in contrast with polynomial interpolation, the rational interpolation

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problem (1) is not always soluble and may give rise to *unattainable points* [5]. An interpolation point is called unattainable for a non-trivial solution $p(x), q(x)$ of (2) if and only if $q(x_i) = 0 = p(x_i)$ and, after cancellation of $(x - x_i)$ in $p(x)/q(x)$, in addition $p(x_i)/q(x_i) \neq f(x_i)$. A solution to the rational interpolation problem (1) exists if and only if the polynomials $p_0(x)$ and $q_0(x)$ themselves satisfy the linearized interpolation problem (2) [6].

As an example, consider the data given in Table 1, for which we compute the rational interpolant $r_{3,3}(x)$.

TABLE 1.

x_i	-1.5708	-0.9425	-0.3142	0.0000	0.3142	0.9425	1.5708
f_i	0.0000	0.5878	0.9511	1.0100	0.9511	0.5878	0.0000

Polynomials $p(x)$ and $q(x)$ that satisfy the linearized problem (2) are for instance given by

$$p(x) = -x(x+1.5708)(x-1.5708)$$
$$q(x) = 2.4661x + 0.2482x^3.$$

Clearly $p(0) = 0 = q(0)$ and after cancellation of the factor x in $p(x)/q(x)$, we have $r_{3,3}(x) = (2.4674 - x^2)/(2.4661 + 0.2482x^2)$ and $r_{3,3}(0) = 1.0005 \neq 1.01$. The rational interpolant $r_{3,3}(x)$ for these data is shown in Figure 1.

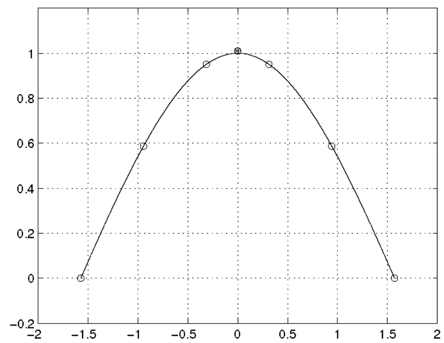

FIGURE 1. The rational interpolant $r_{3,3}(x)$ for the tabulated data (dots) which does not interpolate in 0 (star).

Hence no interpolating rational function of degree 3 in numerator and denominator exists for these data. Note that $(x_i, f_i) = (0, f_i)$ easily becomes attainable when f_i is perturbed such that $(0, f_i)$ lies on the curve of $r_{3,3}(x)$!

SYMBOLIC RATIONAL INTERPOLATION

Several numeric algorithms for rational interpolation exist that are able to detect and report unattainable points. We mention in particular the continued fraction algorithm by Werner [7] or approaches based on the barycentric formula [8]. When working in a symbolic environment, a very elegant solution to unattainability is due to Van Barel and Bultheel [9]. The authors solve this problem by returning a parameterized set of rational functions of higher degree without unattainable points. In order to increase the degree as little as possible, the degree of a polynomial couple $(p(x), q(x))$ is defined by $\max\{\partial p, \partial q\}$[1]. The goal is to find a polynomial couple $(p(x), q(x))$ that satisfies the linearized problem (2) and for which $\max\{\partial p, \partial q\}$ is minimal. It is shown in [9] that all polynomials couples of *minimal degree* that satisfy the linearized problem (2) are parameterized by a *basic pair* $(v(x), w(x))$, consisting of two polynomial couples

$$v(x) = (n_v(x), d_v(x)),$$
$$w(x) = (n_w(x), d_w(x)).$$

For an algorithm to compute a possible choice for $(v(x), w(x))$ we refer to [9]. If the rational function corresponding to $v(x)$ does not interpolate in all points (hence there are unattainable points), then it is also shown in [9] that the set

[1] A slightly more general concept of degree is used in [9] by defining a shift parameter.

given by polynomial linear combination of $v(x)$ and $w(x)$ parameterizes all interpolating rational functions of minimal degree in the following way

$$\frac{p(x)}{q(x)} = \frac{a(x)n_v(x) + b(x)n_w(x)}{a(x)d_v(x) + b(x)d_w(x)}.$$

As explained in [9], the degrees of the polynomials $a(x)$ and $b(x)$ depend on the given problem.

We illustrate this by an example. Consider again the data given in Table 1. A basic pair $(v(x), w(x))$ is given by $v(x) = (n_v(x), d_v(x))$ where

$$n_v(x) = -x(x+1.5708)(x-1.5708)$$
$$d_v(x) = 2.4661x + 0.2482x^3$$

and $w(x) = (n_w(x), d_w(x))$ with

$$n_w(x) = -2.3209 + 5.017x - 1.5268x^2 - 2.0333x^3 + x^4$$
$$d_w(x) = -2.2979 + 5.0143x - 2.9446x^2 + 0.5047x^3.$$

Clearly $v(x)$ does not solve (1). If we take for instance $v(x) - w(x)$ then the following rational function results

$$r^*_{4,3}(x) = \frac{2.3209 - 2.55x + 1.5268x^2 + 1.0333x^3 - x^4}{2.2979 - 2.5482x + 2.9446x^2 - 0.2565x^3}$$

which now, as shown in Figure 2, does interpolate all data.

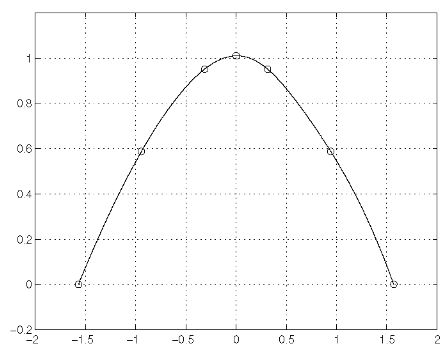

FIGURE 2. A rational function $r^*_{4,3}(x)$ which interpolates all tabulated data (dots).

INTERVAL RATIONAL INTERPOLATION

In practice, data are by nature uncertain and hence intervals. We assume that the uncertainty in the independent variable is negligible and that for each observation an uncertainty interval can be given which contains the (unknown) exact value. Hence, given vertical segments $S_n = \{(x_0, F_0), \ldots, (x_n, F_n)\}$ where $F_i = [\underline{f_i}, \overline{f_i}]$ and $\underline{f_i} < \overline{f_i}$, the goal is to determine $r_{\ell,m}(x)$ such that

$$r_{\ell,m}(x_i) \in F_i, \qquad i = 0, \ldots, n$$

and $\ell + m \ll n$ is minimal. Clearly

$$r_{\ell,m}(x_i) = \frac{p(x_i)}{q(x_i)} \in F_i \quad \Leftrightarrow \quad \underline{f_i} \leq \frac{p(x_i)}{q(x_i)} \leq \overline{f_i}, \qquad i = 0, \ldots, n. \tag{4}$$

When $q(x_i) > 0$, then (4) implies

$$\begin{cases} -p(x_i) + \overline{f_i}q(x_i) & \geq & 0 \\ p(x_i) - \underline{f_i}q(x_i) & \geq & 0 \end{cases} \quad i = 0, \ldots, n. \tag{5}$$

For $p(x)$ and $q(x)$ expanded in terms of basis functions, such as the monomials in (3), denote the vector of unknown coefficients by $\lambda = (\alpha_0, \ldots, \alpha_\ell, \beta_0, \ldots, \beta_m)$ and let

$$U = \begin{pmatrix} -x_0^0 & \ldots & -x_0^\ell & \overline{f_0}x_0^0 & \ldots & \overline{f_0}x_0^m \\ \vdots & & \vdots & \vdots & & \vdots \\ -x_n^0 & \ldots & -x_n^\ell & \overline{f_n}x_n^0 & \ldots & \overline{f_n}x_n^m \\ x_0^0 & \ldots & x_0^\ell & -\underline{f_0}x_0^0 & \ldots & -\underline{f_0}x_0^m \\ \vdots & & \vdots & \vdots & & \vdots \\ x_n^0 & \ldots & x_n^\ell & -\underline{f_n}x_n^0 & \ldots & -\underline{f_n}x_n^m \end{pmatrix}.$$

Then the solution space of (5) is given by $\mathscr{L}_{\ell,m}(S_n) = \{\lambda \in \mathbb{R}^{\ell+m+2} \mid U\lambda \geq 0\}$, which is a convex polyhedral cone with apex in the origin. If $\lambda \in \mathbb{R}^{\ell+m+2}$ is an interior point of $\mathscr{L}_{\ell,m}(S_n)$, i.e. $U\lambda > 0$, then the rational function $r_{\ell,m}(x)$ with coefficients equal to $t\lambda$, $t > 0$, solves (4). Hence no interpolation points are unattainable when restricting to the interior of $\mathscr{L}_{\ell,m}(S_n)$. It is shown in [1] that finding an interior point λ of $\mathscr{L}_{\ell,m}(S_n)$ such that the rational function $r_{\ell,m}(x)$ with coefficients equal to λ intersects the vertical segments S_n approximately at maximal distance from $\underline{f_i}$ and $\overline{f_i}$ $(i = 0, \ldots, n)$, is equivalent to solving a quadratic programming problem with a strictly convex objective function in $\mathbb{R}^{\ell+m+2}$.

If we construct vertical segments from the data in Table 1 by adding random noise from the interval $]0, 0.0623]$ to the function values, then the following rational function is obtained

$$\widetilde{r}_{2,2}(x) = \frac{p(x)}{q(x)} = \frac{1.6979 - 0.0169x - 0.6961x^2}{1.6837 - 0.0440x + 0.1565x^2}$$

which is a rational function of lowest complexity intersecting all vertical segments. The vertical segments and $\widetilde{r}_{2,2}(x)$ are illustrated in Figure 3.

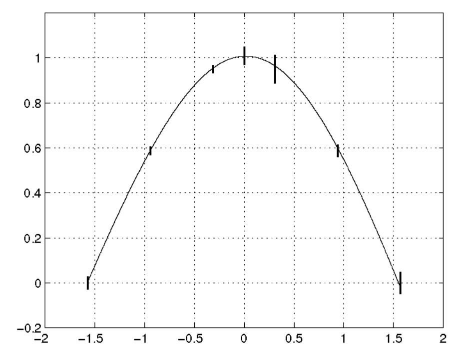

FIGURE 3. A rational function $\widetilde{r}_{2,2}(x)$ which intersects all vertical segments.

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