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Singular rules for the calculation of non-normal multivariate Padé approximants

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Abstract: Section 1 describes the univariate situation in the case of non-normal Padé approximants and Cordellier's extension of the famous five-star identity of Wynn. Section 2 repeats our definition of multivariate Padé approximants and proves a number of theorems that remain valid when going from the univariate to the multivariate case. These theorems and more new results given in Section 3, will finally also copy Cordellier's extension from the univariate to the multivariate case.

0. Motivation

Let us first sketch the univariate situation. The calculation of the function value of a Padé approximant can easily be done by means of the ε-algorithm. The interrelation between the ε-table and the Padé-table has allowed Wynn to prove the famous five-star identity bringing together five neighbouring Padé approximants. Of course this identity is only valid when we are dealing with normal Padé approximants. In [2] Cordellier described an extension of this five star identity in the case of non-normal univariate Padé approximants, while in [3] the validity of the five star identity for normal multivariate Padé approximants was proved. It is our intention now to formulate analogous multivariate theorems as the univariate ones on which Cordellier based his reasoning and thus to solve the computational problem also in case some multivariate Padé approximants are non-normal.

1. The extension of Wynn's univariate identity

We adopt some terminology introduced by Gilewicz in [7]. In this section we restrict ourselves to the univariate case. Suppose we are given the formal power series

$$f(x)=\sum_{k=0}^{\infty} c_k x^k.$$

Definition 1.1. A rational function

$$\frac{p_{[n,m]}}{q_{[n,m]}}(x)=\sum_{i=0}^{n} a_i x^i \Big/ \sum_{j=0}^{m} b_j x^j$$

is called a *Padé form* (PF) *of order* (n, m) *for* f if

$$(f \cdot q_{[n,m]} - p_{[n,m]})(x) = \sum_{k \geqslant n+m+1} d_k x^k.$$

For fixed n and m, different Padé forms $(p_{[n,m]}/q_{[n,m]})(x)$ and $(r_{[n,m]}/s_{[n,m]})(x)$ are equivalent, in other words

$$p_{[n,m]}(x) \cdot s_{[n,m]}(x) = q_{[n,m]}(x) \cdot r_{[n,m]}(x).$$

This justifies the following definition.

Definition 1.2. The irreducible form $(p_{(n,m)}/q_{(n,m)})(x)$ of a Padé form $(p_{[n,m]}/q_{[n,m]})(x)$ for f is called the *Padé approximant* (PA) *of order* (n, m) *for* f.

This irreducible form is unique up to a normalization. From now on we assume that all rational functions considered are normalized according to a fixed kind of normalization, e.g. a monic normalization of the denominator.

The PA is not necessarily a PF anymore because a polynomial in numerator and denominator of $p_{[n,m]}/q_{[n,m]}(x)$ may be cancelled. Anyhow the following properties can be proved.

Let us order the PA for different values of n and m in a table which we call the table of Padé approximants.

$$\begin{array}{cccc}
\dfrac{p_{(0,0)}}{q_{(0,0)}} & \dfrac{p_{(0,1)}}{q_{(0,1)}} & \dfrac{p_{(0,2)}}{q_{(0,2)}} & \cdots \\
\dfrac{p_{(1,0)}}{q_{(1,0)}} & \dfrac{p_{(1,1)}}{q_{(1,1)}} & \cdots & \\
\dfrac{p_{(2,0)}}{q_{(2,0)}} & \vdots & & \\
\vdots & & &
\end{array}$$

We will denote the exact degree of a polynomial by ∂.

Theorem 1.1. *Let* $(p_{(n,m)}/q_{(n,m)})(x)$ *be the* PA *of order* (n, m) *for* $f(x)$, *with* $n' = \partial p_{(n,m)}$ *and* $m' = \partial q_{(n,m)}$. *Then*

(a) $$(f \cdot q_{(n,m)} - p_{(n,m)})(x) = \sum_{k \geqslant n'+m'+t+1} d_k x^k$$

with $t \geqslant 0$ *and* $d_{n'+m'+t+1} \neq 0$.

(b) $n' \leqslant n \leqslant n' + t$ *and* $m' \leqslant m \leqslant m' + t$,

(c) $(p_{(i,j)}/q_{(i,j)})(x) = (p_{(n,m)}/q_{(n,m)})(x)$

for $n' \leqslant i \leqslant n' + t$ *and* $m' \leqslant j \leqslant m' + t$.

For the proof we refer to [8]. This theorem describes the block-structure of the table of PA and enables us to give the following definition of normality of a PA.

Definition 1.3. The Padé approximant of order (n, m) for f is called *normal* if it occurs only once in the table of PA.

For normal PA:

$$(p_{[n,m]}/q_{[n,m]})(x) = (p_{(n,m)}/q_{(n,m)})(x).$$

It is clear that in the block of size $t+1$, as showed in Fig. 1.1, all the PA belonging to the shaded triangle are also PF. The PA belonging to the other triangle have to be multiplied in numerator and denominator by a certain power of x in order to get a PF. The PF can now only be ordered in a table if they are unique. If a PF is not unique we will call it undefined. Gilewicz has proved the following [7, p. 178].

Theorem 1.2. *Let* $(p_{(n,m)}/q_{(n,m)})(x)$ *be the* PA *of order* (n, m) *for* $f(x)$ *with* $n' = \partial p_{(n,m)}$ *and* $m' = \partial q_{(n,m)}$. *Then*

(a) *the Padé form* $(p_{[i,j]}/q_{[i,j]})(x)$ *is unique for* $\min(i-n', j-m') = 0$ *and for* $\max(i-n', j-m') = t$,

(b) *the Padé form* $(p_{[i,j]}/q_{[i,j]})(x)$ *is undefined for* $n' < i < n'+t$ *and* $m' < j < m'+t$.

So the table of Padé forms has the structure, as showed in Fig. 1.2; in the shaded square all the PF are undefined. What's more: the PF $p_{[i,j]}/q_{[i,j]}$ in the shaded region of Fig. 1.3 result from the PA $(p_{(n',m')}/q_{(n',m')})$ by multiplication of numerator and denominator by the monomial $x^{\min(i-n',j-m')}$. We shall be able to generalize this to the multivariate case in Section 2.

We call two PF $(p_{[n_1,m_1]}/q_{[n_1,m_1]})$ and $(p_{[n_2,m_2]}/q_{[n_2,m_2]})$ or PA $(p_{(n_1,m_1)}/q_{(n_1,m_1)}$ and $(p_{(n_2,m_2)}/q_{(n_2,m_2)})$ neighbouring if

$$|n_1 - n_2| + |m_1 - m_2| = 1.$$

For neighbouring PF Cordellier proved the following theorem [2].

Theorem 1.3. *Two neighbouring* PF *not belonging to the same square block take different values for all nonzero arguments* x.

In fact one might say here: for all x not being the zero of a univariate homogeneous polynomial. This is clearly equivalent with the statement x being nonzero, but the formulation involving the homogeneous polynomial can be generalized to the multivariate case.

With the power series $f(x)$ we can associate its sequence of partial sums $S_n(x) = \sum_{k=0}^{n} c_k x^k$

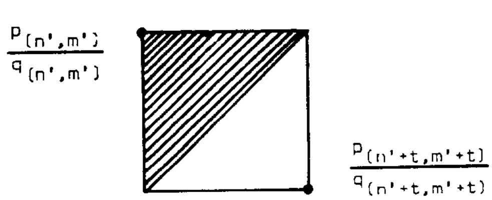


Fig. 1.1.

and introduce the Hankel determinants

$$H_k(S_n(x)) = \begin{vmatrix} S_n(x) & S_{n+1}(x) & \cdots & S_{n+k-1}(x) \\ S_{n+1}(x) & & & \\ \vdots & & & \vdots \\ S_{n+k-1}(x) & & \cdots & S_{n+2k-2}(x) \end{vmatrix}$$

and the forward differences $\Delta S_n(x) = c_{n+1}x^{n+1}$ and $\Delta^2 S_n(x) = \Delta S_{n+1}(x) - \Delta S_n(x)$. It is easy to verify that

$$p_{[n,m]} = \frac{H_{m+1}(S_{n-m}(x))}{x^{nm}} \quad \text{and} \quad q_{[n,m]} = \frac{H_m(\Delta^2 S_{n-m}(x))}{x^{nm}}$$

constitute a PF. The multivariate analogon will be given in Section 2 but first of all we shall now describe the ε-algorithm. Input of the ε-algorithm is the sequence $S_n(x)$:

$$\varepsilon_{-1}^{(j)} = 0, \quad \varepsilon_{2j}^{(-j-1)} = 0, \quad \varepsilon_0^{(j)} = S_j(x), \qquad j = 0, 1, 2, \ldots .$$

We perform the following computations:

$$\varepsilon_{j+1}^{(i)} = \varepsilon_{j-1}^{(i+1)} + \frac{1}{\varepsilon_j^{(i+1)} - \varepsilon_j^{(i)}} \qquad j = 0, 1, 2, \ldots, \quad i = -j, -j+1, \ldots .$$

The subscript j indicates a column and the superscript (i) indicates a diagonal in the ε-table

$$\begin{array}{cccc} & \varepsilon_0^{(-1)} & & \\ \varepsilon_{-1}^{(0)} & & \varepsilon_1^{(-1)} & \cdots \\ & \varepsilon_0^{(0)} & & \\ \varepsilon_{-1}^{(1)} & & \varepsilon_1^{(0)} & \cdots \\ & \varepsilon_0^{(1)} & \vdots & \\ \varepsilon_{-1}^{(2)} & \vdots & & \\ \vdots & & & \end{array}$$

It is well known that $\varepsilon_{2m}^{(n-m)} = H_{m+1}(S_{n-m}(x))/H_m(\Delta^2 S_{n-m}(x))$ [1, pp. 44–46] and thus that it is a PF. The following five star identities are respectively due to Wynn and Cordellier.

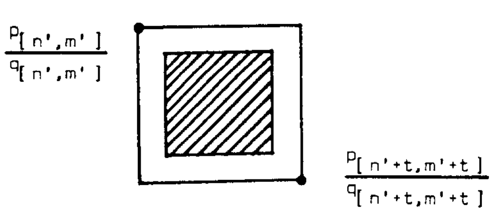


Fig. 1.2.

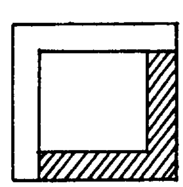


Fig. 1.3.

Theorem 1.4. (a) *If the table of* PA *is normal, then*

$$\left(\frac{p_{(i+j,j-1)}}{q_{(i+j,j-1)}}-\frac{p_{(i+j,j)}}{q_{(i+j,j)}}\right)^{-1}+\left(\frac{p_{(i+j,j+1)}}{q_{(i+j,j+1)}}-\frac{p_{(i+j,j)}}{q_{(i+j,j)}}\right)^{-1}$$
$$=\left(\frac{p_{(i+j-1,j)}}{q_{(i+j-1,j)}}-\frac{p_{(i+j,j)}}{q_{(i+j,j)}}\right)^{-1}+\left(\frac{p_{(i+j+1,j)}}{q_{(i+j+1,j)}}-\frac{p_{(i+j,j)}}{q_{(i+j,j)}}\right)^{-1}$$

for $j=0,1,2,\ldots,\ i=-j,-j+1,\ldots$;

(b) *If the table of* PA *contains a block of size* $t+1$ *with* $p_{(i+j,j)}/q_{(i+j,j)}$ *in the uppermost left corner, then*

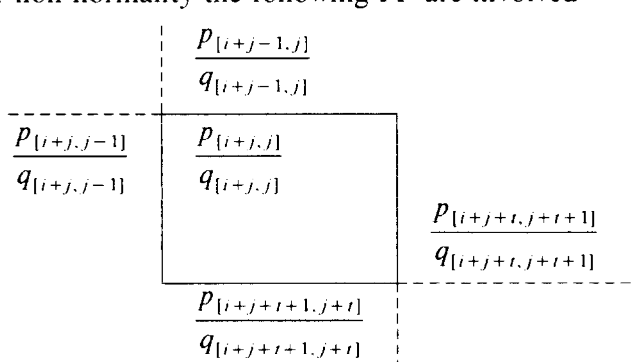
$$\left(\frac{p_{[i+j+l,j-1]}}{q_{[i+j+l,j-1]}}-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}\right)^{-1}+\left(\frac{p_{[i+j+t-l,j+t+1]}}{q_{[i+j+t-l,j+t+1]}}-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}\right)^{-1}$$
$$=\left(\frac{p_{[i+j-1,j+l]}}{q_{[i+j-1,j+l]}}-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}\right)^{-1}+\left(\frac{p_{[i+j+t+1,j+t-l]}}{q_{[i+j+t+1,j+t-l]}}-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}\right)^{-1}$$

for $l=0,\ldots,t$.

A figure will illustrate the meaning of this theorem. In case of normality the following five neighbouring PA are used

$$\begin{array}{ccc} & \dfrac{p_{(i+j-1,j)}}{q_{(i+j-1,j)}} & \\ \dfrac{p_{(i+j,j-1)}}{q_{(i+j,j-1)}} & \dfrac{p_{(i+j,j)}}{q_{(i+j,j)}} & \dfrac{p_{(i+j,j+1)}}{q_{(i+j,j+1)}} \\ & \dfrac{p_{(i+j+1,j)}}{q_{(i+j+1,j)}} & \end{array}$$

In case of non-normality the following PF are involved



These five-star identities are based on the respective identities for the ε-values [2]: if, in the ε-table, we have i, j and $t \geqslant 0$ such that

$$\varepsilon_{2j}^{(i+l)}=a,\quad \varepsilon_{2(j-1)}^{(i+l+1)}\neq a,\qquad l=0,\ldots,t,$$
$$\varepsilon_{2j}^{(i-1)}\neq a,\qquad \varepsilon_{2j}^{(i+t+1)}\neq a,$$

then

$$\left(\varepsilon_{2(j-1)}^{(i+l+1)}-a\right)^{-1}+\left(\varepsilon_{2(j+l+1)}^{(i-l-1)}-a\right)^{-1}=\left(\varepsilon_{2(j+l)}^{(i-l-1)}-a\right)^{-1}+\left(\varepsilon_{2(j+t-l)}^{(i+l+1)}-a\right)^{-1}, \quad l=0,\ldots,t. \tag{1}$$

Since the ε-algorithm is also valid for our definition of multivariate Padé approximants, this identity will also be the basis for our generalization.

2. Multivariate Padé approximants

We will describe everything in the case of two variables, but of course the statements are also valid for more than two variables. Suppose we are given a formal power series

$$f(x, y)=\sum_{i,j=0}^{\infty} c_{ij}x^iy^j=\sum_{k=0}^{\infty}\left(\sum_{i+j=k} c_{ij}x^iy^j\right)$$

which we shall denote by

$$\sum_{k=0}^{\infty} C_k(x, y)^k.$$

The $C_k(x, y)^k$ are homogeneous bivariate polynomials of degree k.

Definition 2.1. A rational function

$$\frac{p_{[n,m]}}{q_{[n,m]}}(x, y)=\sum_{i+j=nm}^{nm+n} a_{ij}x^iy^j \Big/ \sum_{i+j=nm}^{nm+m} b_{ij}x^iy^j=\sum_{i=nm}^{nm+n} A_i(x, y)^i \Big/ \sum_{i=nm}^{nm+m} B_i(x, y)^i$$

is called a *multivariate Padé form (MPF) of order* (n, m) *for* f if

$$\left(f\cdot q_{[n,m]}-p_{[n,m]}\right)(x, y)=\sum_{k\geqslant nm+n+m+1} E_k(x, y)^k.$$

If we define $S_n(x, y)=\sum_{k=0}^{n}C_k(x, y)^k$ then we can consider the Hankel determinants

$$H_k(S_n(x, y))=\begin{vmatrix} S_n(x, y) & S_{n+1}(x, y) & \cdots & S_{n+k-1}(x, y) \\ S_{n+1}(x, y) & & & \\ \vdots & & & \vdots \\ S_{n+k-1}(x, y) & & \cdots & S_{n+2k-2}(x, y) \end{vmatrix}.$$

In [5] is proved that

$$\frac{p_{[n,m]}(x, y)}{q_{[n,m]}(x, y)}=\frac{H_{m+1}(S_{n-m}(x, y))}{H_m(\Delta^2 S_{n-m}(x, y))}$$

is a MPF of order (n, m) for f. Since the quotient of these Hankel determinants can be calculated by means of the ε-algorithm, it is clear that the ε-algorithm remains valid for the calculation of our MPF.

On can also prove that another MPF $(r_{[n,m]}/s_{[n,m]})(x, y)$ of order (n, m) for $f(x, y)$ supplies an equivalent rational function:

$$p_{[n,m]}(x, y)\cdot s_{[n,m]}(x, y)=q_{[n,m]}(x, y)\cdot r_{[n,m]}(x, y)$$

and that at least one nontrivial MPF exists [4].

Hence we can define the multivariate Padé approximant

Definition 2.2. The irreducible form $(p_{(n,m)}/q_{(n,m)})(x, y)$ of a MPF$(p_{[n,m]}/q_{[n,m]})(x, y)$ for f is called the *multivariate Padé approximant (MPA) of order (n, m) for f.*

Again we assume that a normalization is prescribed so that the considered rational functions are unique.

If we order multivariate Padé approximants in a table then this table also has a block-structure [4]. We denote the order of a polynomial, which is the degree of the first nonzero term, by ∂_0. For instance in definition 2.1.:

$$\partial_0 p_{[n,m]}\geqslant nm,\qquad \partial_0 q_{[n,m]}\geqslant nm. \tag{2}$$

Theorem 2.1. *Let $(p_{(n,m)}/q_{(n,m)})(x, y)$ be the MPA of order (n, m) for $f(x, y)$ with*

$$n'=\partial p_{(n,m)}-\partial_0 q_{(n,m)}\quad\text{and}\quad m'=\partial q_{(n,m)}-\partial_0 q_{(n,m)}.$$

Then

(a) $\partial_0\left(f\cdot q_{(n,m)}-p_{(n,m)}\right)=\partial_0 q_{(n,m)}+n'+m'+t+1$ *with* $t\geqslant 0$,

(b) $n'\leqslant n\leqslant n'+t$ *and* $m'\leqslant m\leqslant m'+t$,

(c) $\dfrac{p_{(i,j)}}{q_{(i,j)}}(x, y)=\dfrac{p_{(n,m)}}{q_{(n,m)}}(x, y)$

for $n'\leqslant i\leqslant n'+t$ *and* $m'\leqslant j\leqslant m'+t$.

Proof. The proof of (a) and (b) is given in [4]. The proof of (c) was given under the condition that $\partial_0 q_{(n,m)}\leqslant n'\cdot m'$. We shall here show that this condition is in fact always satisfied for MPA. We know, by definition of n' and m', that $p_{(n,m)}$ and $q_{(n,m)}$ are of the form

$$q_{(n,m)}(x, y)=\sum_{i+j=\partial_0 q_{(n,m)}}^{\partial_0 q_{(n,m)}+m'} b_{ij}x^i y^j,\qquad p_{(n,m)}(x, y)=\sum_{i+j=\partial_0 q_{(n,m)}}^{\partial_0 q_{(n,m)}+n'} a_{ij}x^i y^j.$$

If we calculate a MPF of order (n', m') for f, we obtain $p_{[n',m']}$ and $q_{[n',m']}$ of the form

$$q_{[n',m']}(x, y)=\sum_{i+j=n'm'}^{n'm'+m'} \bar{b}_{ij}x^i y^j,\qquad p_{[n',m']}(x, y)=\sum_{i+j=n'm'}^{n'm'+n'} \bar{a}_{ij}x^i y^j.$$

Now

$$\begin{aligned}&\partial_0\left(p_{(n,m)}q_{[n',m']}-p_{[n',m']}q_{(n,m)}\right)\\&=\partial_0\left\{\left(fq_{[n',m']}-p_{[n',m']}\right)q_{(n,m)}-\left(fq_{(n,m)}-p_{(n,m)}\right)q_{[n',m']}\right\}\\&\geqslant\partial_0 q_{(n,m)}+n'm'+n'+m'+1\end{aligned}$$

while

$$\partial\left(p_{(n,m)}q_{[n',m']}-p_{[n',m']}q_{(n,m)}\right) \leqslant \partial_0 q_{(n,m)}+n'm'+n'+m'.$$

Hence

$$p_{(n,m)}q_{[n',m']}=p_{[n',m']}q_{(n,m)}.$$

Consequently

$$p_{(n',m')}/q_{(n',m')}=p_{(n,m)}/q_{(n,m)}.$$

and thus

$$\partial_0 q_{(n,m)}=\partial_0 q_{(n',m')}, \qquad \partial q_{(n,m)}=\partial q_{(n',m')} \leqslant n'm'+m',$$

which implies

$$\partial q_{(n,m)}=\partial_0 q_{(n,m)}+m'=\partial_0 q_{(n',m')}+m' \leqslant n'm'+m'$$

or $\partial_0 q_{(n,m)} \leqslant n'm'$. □

Remark that the block-structure is exactly the same as in the univariate case. Hence normality can also be defined in the same way.

Definition 2.3. The MPA $p_{(n,m)}/q_{(n,m)}$ of order (n, m) for f is called *normal* if it occurs only once in the table of MPA.

Other conditions for normality are given in [5].

For the MPF a property can be proved which is comparable with the structure of the univariate table of PF described in the Figs. 1.1, 1.2 and 1.3.

Theorem 2.2. *Let $(p_{(n,m)}/q_{(n,m)})(x, y)$ be the MPA of order (n, m) for $f(x, y)$, with $n'=\partial p_{(n,m)}-\partial_0 q_{(n,m)}$ and $m'=\partial q_{(n,m)}-\partial_0 q_{(n,m)}$. Then*

(a) $$p_{[i,j]}(x, y):=p_{(n,m)}(x, y)\cdot D_k(x, y)^k,$$

$$q_{[i,j]}(x, y):=q_{(n,m)}(x, y)\cdot D_k(x, y)^k \tag{3}$$

with $k=i\cdot j-\partial_0 q_{(n,m)}$ and $D_k(x, y)^k$ a homogeneous polynomial of degree k, constitute a MPF of order (i, j) for f, if

$$n' \leqslant i \leqslant n'+t, \qquad m' \leqslant j \leqslant m'+t, \qquad i+j \leqslant n'+m'+t;$$

(b) $$p_{[i,j]}(x, y):=p_{(n,m)}(x, y)\cdot D_k(x, y)^k,$$

$$q_{[i,j]}(x, y):=q_{(n,m)}(x, y)\cdot D_k(x, y)^k \tag{4}$$

with $k=i\cdot j-\partial_0 q_{(n,m)}+\min(i-n', j-m')$ and $D_k(x, y)^k$ a homogeneous polynomial of degree *k, constitute a MPF of order (i, j) for f, if*

$$n' \leqslant i \leqslant n' + t, \qquad m' \leqslant j \leqslant m' + t, \qquad i + j > n' + m' + t;$$

(c) *for* $\min(i - n', j - m') = 0$ *we have*

$$H_{j+1}(S_{i-j}(x, y)) = p_{(n,m)}(x, y) \cdot D_k(x, y)^k,$$

$$H_j(\Delta^2 S_{i-j}(x, y)) = q_{(n,m)}(x, y) \cdot D_k(x, y)^k$$

with $k = i \cdot j - \partial_0 q_{(n,m)}$ *and* $D_k(x, y)^k$ *a nontrivial homogeneous polynomial of degree k, and for* $\max(i - n', j - m') = t < \infty$ *we have*

$$H_{j+1}(S_{i-j}(x, y)) = p_{(n,m)}(x, y) \cdot D_k(x, y)^k,$$

$$H_j(\Delta^2 S_{i-j}(x, y)) = q_{(n,m)}(x, y) \cdot D_k(x, y)^k$$

with $k = i \cdot j - \partial_0 q_{(n,m)} + \min(i - n', j - m')$ *and* $D_k(x, y)^k$ *a nontrivial homogeneous polynomial of degree k.*

Proof. If we consider the block of equal MPA showed in Fig. 2.1. Then we will divide this block into two triangles: part (a) applies to the shaded triangle, part (b) to the other one.

(a) Since $\partial_0(fq_{(n,m)} - p_{(n,m)}) = \partial_0 q_{(n,m)} + n' + m' + t + 1$ it is obvious that

$$\partial_0\left[\left(f \cdot q_{(n,m)} - p_{(n,m)}\right) D_k\right] \geqslant ij + i + j + 1$$

if $k = ij - \partial_0 q_{(n,m)}$ and if $i + j \leqslant n' + m' + t$.

Also $\partial_0(p_{(n,m)} D_k) \geqslant ij$, $\quad \partial(p_{(n,m)} D_k) \leqslant ij + n' \leqslant ij + i$,

$\partial_0(q_{(n,m)} D_k) \geqslant ij$, $\quad \partial(q_{(n,m)} D_k) \leqslant ij + m' \leqslant ij + j$.

So $p_{(n,m)} D_k / q_{(n,m)} D_k$ satisfies Definition 2.1.

(b) Now

$$\partial_0\left[\left(f \cdot q_{(n,m)} - p_{(n,m)}\right) D_k\right] \geqslant ij + i + j + 1$$

because

$$\partial_0 q_{(n,m)} + n' + m' + t + 1 + \left(ij - \partial_0 q_{(n,m)}\right) + \min(i - n', j - m') \geqslant ij + i + j + 1.$$

The order and degree of numerator and denominator also satisfy the necessary conditions so that we have again a MPF.

(c) We are especially interested in MPF lying in the shaded region of Fig. 2.2. For the first row and the first column in the block we have $\min(i - n', j - m') = 0$ and MPF given by Definition

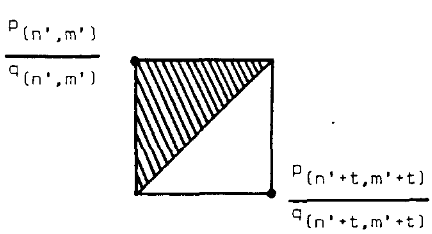


Fig. 2.1.

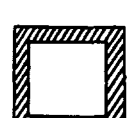


Fig. 2.2.

2.1 can only be of the form (3). In [6] we proved that for $\min(i-n', j-m')=0$ the determinant $H_j(\Delta S_{i-1})$ is nontrivial. Since this is the term of lowest degree in $q_{[i,j]}=H_j(\Delta^2 S_{i-j})$ it is obvious that D_k is nontrivial. For the last row and the last column in the block the proof goes as follows. Consider a MPF $p_{[i,j]}/q_{[i,j]}$. Then there is a polynomial $T(x, y)$ such that $p_{[i,j]}=p_{(n,m)}\cdot T$ and $q_{[i,j]}=q_{(n,m)}\cdot T$. If we write $t_0=\partial_0 T$ and $T(x, y)=\Sigma\partial_{k=t_0}^{T} T_k(x, y)^k$ then the following conclusion can be drawn [4]: $p_{(n,m)}\cdot T_{t_0}/q_{(n,m)}\cdot T_{t_0}$ is a MPF of order (i, j)

$$\partial_0\{(fq_{(n,m)}-p_{(n,m)})T_{t_0}\}=\partial_0 q_{(n,m)}+n'+m'+t+1+t_0 \quad \text{because of Theorem 2.1 (a).}$$

$$\partial_0\{(fq_{(n,m)}-p_{(n,m)})T_{t_0}\}\geqslant ij+i+j+1 \quad \text{because of Definition 2.1.}$$

$$\partial T=\partial q_{[i,j]}-\partial q_{(n,m)}\leqslant ij+j-\partial_0 q_{(n,m)}-m'.$$

Take for instance the case $i=n'+t$. Then

$$\partial_0 q_{(n,m)}+n'+m'+t+1+t_0\geqslant ij+n'+t+j+1$$

which implies

$$t_0\geqslant ij-\partial_0 q_{(n,m)}+j-m'.$$

Also

$$\partial T\leqslant ij-\partial_0 q_{(n,m)}+j-m'.$$

Hence $p_{[i,j]}/q_{[i,j]}$ is of the form (4). To prove the nontriviality of $T(x, y)=D_k(x, y)^k$ we take a look at the linear systems of equations satisfied by a MPF of order (i, j): if $p_{[i,j]}/q_{[i,j]}(x, y)=\sum_{l=0}^{i}A_l(x, y)^{ij+l}/\sum_{l=0}^{j}B_l(x, y)^{ij+l}$ where A_l and B_l are homogeneous polynomials of degree $(ij+1)$, then

$$\begin{cases} C_0\cdot B_0(x, y)^{ij}=A_0(x, y)^{ij}, \\ C_1(x, y)\cdot B_0(x, y)^{ij}+C_0\cdot B_1(x, y)^{ij+1}=A_1(x, y)^{ij+1}, \\ \vdots \\ C_i(x, y)^i\cdot B_0(x, y)^{ij}+\cdots+C_0\cdot B_i(x, y)^{ij+i}=A_i(x, y)^{ij+i} \end{cases} \tag{5a}$$

with $B_l(x, y)^{ij+l}\equiv 0$ if $l>ij+j$, and

$$\begin{cases} C_{i+1}(x, y)^{i+1}\cdot B_0(x, y)^{ij}+\cdots+C_{i+1-j}(x, y)^{i+1-j}\cdot B_j(x, y)^{ij+j}=0, \\ \vdots \\ C_{i+j}(x, y)^{i+j}\cdot B_0(x, y)^{ij}+\cdots+C_i(x, y)^i\cdot B_j(x, y)^{ij+j}=0 \end{cases} \tag{5b}$$

with $C_l(x, y)^l\equiv 0$ if $l<0$. Consider the case $i=n'+t$. If $q_{[i,j]}(x, y)=H_j(\Delta^2 S_{i-j}(x, y))\equiv 0$ then for all (x, y) in $\mathbb{R}^2$ the rank of the homogeneous system is less than m. So an additional equation

$$C_{i+j+1}(x, y)^{i+j+1}\cdot B_0(x, y)^{ij}+\cdots+C_{i+1}(x, y)^{i+1}\cdot B_j(x, y)^{ij+j}=0 \tag{5c}$$

can be added and one can see [3] that the enlarged homogeneous system still has a nontrivial solution. This enables us to construct nontrivial polynomials $p_{[i+1,j]}=\sum_{l=0}^{i+1}A_l(x, y)^{(i+1)j+l}$ and

$q_{[i+1,j]} = \sum_{l=0}^{j} B_l(x, y)^{(i+1)j+l}$ for which the equations (5a), (5b), and (5c) can be rewritten as:

$$\begin{cases} C_0 \cdot B_0(x, y)^{(i+1)j} = A_0(x, y)^{(i+1)j}, \\ \vdots \\ C_i(x, y)^i \cdot B_0(x, y)^{(i+1)j} + \cdots + C_0 \cdot B_i(x, y)^{(i+1)j+i} = A_i(x, y)^{(i+1)j+i}, \\ C_{i+1}(x, y)^{i+1} \cdot B_0(x, y)^{(i+1)j} + \cdots + C_{i+1-j}(x, y)^{i+1-j} \cdot B_j(x \cdot y)^{(i+1)j+j} \\ \quad = A_{i+1}(x, y)^{(i+1)j+i+1} \end{cases}$$

with $A_{i+1}(x, y)^{(i+1)j+i+1} \equiv 0$, and

$$\begin{cases} C_{i+2}(x, y)^{i+2} \cdot B_0(x, y)^{(i+1)j} + \cdots + C_{i+2-j}(x, y)^{i+2-j} \cdot B_j(x, y)^{(i+1)j+j} = 0, \\ \vdots \\ C_{i+j+1}(x, y)^{i+j+1} \cdot B_0(x, y)^{(i+1)j} + \cdots + C_{i+1}(x, y)^{i+1} \cdot B_j(x, y)^{(i+1)j+j} = 0. \end{cases}$$

Consequently $p_{(i+1,j)}/q_{(i+1,j)} = p_{(i,j)}/q_{(i,j)} = p_{(n',m')}/q_{(n',m')}$ which contradicts the definition of the block-size.

In case $j = m' + t$ we can prove in an analogous way that $p_{(i,j+1)}/q_{(i,j+1)} = p_{(i,j)}/q_{(i,j)} = p_{(n',m')}/q_{(n',m')}$ which is again a contradiction. Hence $D_k(x, y)^k$ is nontrivial. □

The formulas (2), (3) and (4) clearly indicate that in the multivariate case it is not possible to have unicity of the Padé forms: there is a tremendous choice for the homogeneous forms D_k. In the univariate case there's only one homogeneous polynomial of degree k to multiply numerator and denominator of the PA with in order to get a PF. However, the problem of unicity of the Padé form is eliminated if we consider the table of the $(p_{[n,m]}/q_{[n,m]})(x, y) = \{ H_{m+1}(S_{n-m}(x, y))/H_m(\Delta^2 S_{n-m}(x, y))\}$, i.e.,

$$\begin{array}{cccc} \dfrac{H_1(S_0)}{H_0(\Delta^2 S_0)} & \dfrac{H_2(S_{-1})}{H_1(\Delta^2 S_{-1})} & \dfrac{H_3(S_{-2})}{H_2(\Delta^2 S_{-2})} & \cdots \\ \dfrac{H_1(S_1)}{H_0(\Delta^2 S_1)} & \dfrac{H_2(S_0)}{H_1(\Delta^2 S_0)} & \cdots & \\ \dfrac{H_1(S_2)}{H_0(\Delta^2 S_2)} & \vdots & & \\ \vdots & & & \end{array} \tag{6}$$

Let us now try to establish five-star identities for these quantities.

3. Multivariate five-star identities

First of all we generalize Theorem 1.3.

Theorem 3.1. *Two neighbouring MPF in* (6) *of which the MPA do not belong to the same block, take different values for all arguments* (x, y) *not being the zero of a certain bivariate homogeneous polynomial.*

Proof: Consider the Padé forms $p_{[n_1,m_1]}/q_{[n_1,m_1]}$ and $p_{[n_2,m_2]}/q_{[n_2,m_2]}$. Because of Theorem 2.2(c) they are well defined, i.e. the denominators are nontrivial.

Without loss of generality we can assume that $n_2 \geqslant n_1$ and $m_2 \geqslant m_1$. Now

$$\begin{aligned}&\partial_0\left(p_{[n_1,m_1]}q_{[n_2,m_2]} - p_{[n_2,m_2]}q_{[n_1,m_1]}\right)\\ &= \partial_0\left(\left(fq_{[n_2,m_2]} - p_{[n_2,m_2]}\right)q_{[n_1,m_1]} - \left(fq_{[n_1,m_1]} - p_{[n_1,m_1]}\right)q_{[n_2,m_2]}\right)\\ &\geqslant n_1m_1 + n_2m_2 + n_1 + m_1 + 1,\end{aligned}$$

because $n_1 + m_1 \leqslant n_2 + m_2$ while

$$\begin{aligned}&\partial\left(p_{[n_1,m_1]}q_{[n_2,m_2]} - p_{[n_2,m_2]}q_{[n_1,m_1]}\right)\\ &\leqslant n_1m_1 + n_2m_2 + \max(n_1 + m_2,\ n_2 + m_1) \leqslant n_1m_1 + n_2m_2 + n_1 + m_1 + 1.\end{aligned}$$

So

$$\left(p_{[n_1,m_1]}q_{[n_2,m_2]} - p_{[n_2,m_2]}q_{[n_1,m_1]}\right)(x,\ y) = D_k(x,\ y)^k$$

with $k = n_1m_1 + n_2m_2 + n_1 + m_1 + 1$ and D_k homogeneous. Hence $(p_{[n_1,m_1]}/q_{[n_1,m_1]})(x,\ y) = (p_{[n_2,m_2]}/q_{[n_2,m_2]})(x,\ y)$ implies $D_k(x,\ y)^k = 0$. □

An important consequence of this theorem is that neighbouring MPF in (6) of which the MPA do not belong to the same block, take different values on a dense set in $\mathbb{R}^2$.

As we already remarked, the ε-algorithm remains valid for the calculation of the function value of a MPA: with $\varepsilon_0^{(j)} = S_j(x,\ y)$ we have

$$\varepsilon_{2m}^{(n-m)} = H_{m+1}(S_{n-m}(x,\ y))/H_m(\Delta^2 S_{n-m}(x,\ y)).$$

Also relation (1) remains valid since the multivariate ε-algorithm is performed exactly in the same way as the univariate ε-algorithm; only the starting values $\varepsilon_0^{(j)}$ are multivariate partial sums instead of univariate ones. So we have the following generalization of Theorem 1.4.

Theorem 3.2. *For*

$$\frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x,\ y) = \frac{H_{j+1}(S_i(x,\ y))}{H_j(\Delta^2 S_i(x,\ y))}$$

the following identities hold:

(a) *If the table of MPA is normal, then*

$$\begin{aligned}&\left(\frac{p_{[i+j,j-1]}}{q_{[i+j,j-1]}}(x,\ y) - \frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x,\ y)\right)^{-1} + \left(\frac{p_{[i+j,j+1]}}{q_{[i+j,j+1]}}(x,\ y) - \frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x,\ y)\right)^{-1}\\ &= \left(\frac{p_{[i+j-1,j]}}{q_{[i+j-1,j]}}(x,\ y) - \frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x,\ y)\right)^{-1} + \left(\frac{p_{[i+j+1,j]}}{q_{[i+j+1,j]}}(x,\ y) - \frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x,\ y)\right)^{-1}\end{aligned}$$

for $j = 0, 1, 2, \ldots, i = -j, -j+1, \ldots$.

(b) *If the table of MPA contains a block of size* $t+1$ *with* $(p_{(i+j,j)}/q_{(i+j,j)})(x, y)$ *in the uppermost left corner, then*

$$\left(\frac{p_{[i+j+l,j-1]}}{q_{[i+j+l,j-1]}}(x, y)-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x, y)\right)^{-1}+\left(\frac{p_{[i+j+t-l,j+t+1]}}{q_{[i+j+t-l,j+t+1]}}(x, y)-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x, y)\right)^{-1}$$
$$=\left(\frac{p_{[i+j-1,j+l]}}{q_{[i+j-1,j+l]}}(x, y)-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x, y)\right)^{-1}+\left(\frac{p_{[i+j+t+1,j+t-l]}}{q_{[i+j+t+1,j+t-l]}}(x, y)-\frac{p_{[i+j,j]}}{q_{[i+j,j]}}(x, y)\right)^{-1}$$

for $l=0,\ldots,t$.

Proof. The proof of (a) was given in [3]. The proof of (b) is performed in the same way as in [2]. □

References

[1] C. Brezinski, *Accélération de la Convergence en Analyse Numérique*, Lecture Notes in Math. **584** (Springer, Berlin, 1977).

[2] F. Cordellier, Démonstration algébrique de l'extension de l'identité de Wynn aux tables de Padé non normales, In: [9] 36–60.

[3] A. Cuyt, *Padé Approximants for Operators: Theory and Applications*, Lecture Notes in Math. **1065** (Springer, Berlin, 1984).

[4] A. Cuyt, Multivariate Padé approximants, *Journ. Math. Anal. Appl.* **96**(1) (1983) 283–293.

[5] A. Cuyt, Regularity and normality of abstract Padé approximants, projection property and product property, *J. Approx. Theory* **35**(1) (1982) 1–11.

[6] A. Cuyt, H. Werner and L. Wuytack, On the continuity of the multivariate Padé operator, *J. Comput. Appl. Math.* **10**(3) (1984) 95–102.

[7] J. Gilewicz, *Approximants de Padé*, Lecture Notes in Math. **667** (Springer, Berlin, 1978).

[8] H. Padé, Sur la représentation approchée d'une fonction par des fractions rationelles, *Ann. Ecole Norm. Sup.* **9** (1892) 3–93.

[9] L. Wuytack, Padé Approximation and its Applications, Lecture Notes in Math. **765** (Springer, Berlin, 1979).