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Regularity and Normality of Abstract Padé Approximants Projection-Property and Product-Property

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Padé approximants in one variable have proved to be very useful in numerical analysis, especially in the presence of singularities: in the solution of nonlinear equations and ordinary and partial differential equations, in numerical integration, etc. The concept of Padé approximant has been generalized to operator theory starting from power series expansions as is done in the classical theory. We repeat the definition of abstract Padé approximant in Section 2. The generalization is such that the classical Padé approximant is a special case of the theory and a lot of of its interesting properties remain valid for the abstract approximants. We prove some of those properties in Section 4 and formulate a projection-and a product-property in Section 5.

1. INTRODUCTION

Let $X \supsetneqq \{0\}$ be a Banach space and $Y \supseteq \{0, I\}$ be a commutative Banach algebra where 0 is the unit for addition and I is the unit for multiplication (X and Y have the same scalar field Λ, where Λ is $\mathbb{R}$ or $\mathbb{C}$). For every positive integer k we consider the spaces $L(X^k, Y) = \{L \mid L$ is a k-linear bounded operator $L: X \to L(X^{k-1}, Y)\}$, where $L(X^0, Y) = Y$. So $Lx_1 \cdots x_k = (Lx_1) x_2 \cdots x_k$ with $(x_1, ..., x_k)$ in X^k and Lx_1 in $L(X^{k-1}, Y)$. The operator L in $L(X^k, Y)$ is called symmetric if $Lx_1 \cdots x_k = Lx_{i_1} \cdots x_{i_k}$ for all $(x_1, ..., x_k)$ in X^k and all permutations $(i_1, ..., i_k)$ of $(1, ..., k)$ [4, pp. 100–103].

DEFINITION 1.1. An *abstract polynomial* is a nonlinear operator $P: X \to Y$ such that $P(x) = A_n x^n + \cdots + A_0 \in Y$ with $A_i \in L(X^i, Y)$ and A_i symmetric, $i = 0, 1, ..., n$. The degree of $P(x)$ is n.

The notation for the exact degree of $P(x)$ is ∂P (the largest integer k with $A_k x^k \not\equiv 0$) and the notation for the order of $P(X)$ is $\partial_0 P$ (the smallest integer k with $A_k x^k \not\equiv 0$).

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Abstract polynomials are differentiated (we use Fréchet derivatives) as in elementary calculus. We assume that the notion of abstract analyticity is known to the reader [4, p. 113].

Let $F: X \to Y$. We write $D(F) = \{x \in X \mid F(x)$ is regular in Y, i.e., $\exists y \in Y$: $F(x) \cdot y = I = y \cdot F(x)\}$. If F is continuous then $D(F)$ is an open set in X.

2. Definition

Let $F: X \to Y$ be analytic in 0. Then there exists an open ball $B(0, r)$ with centre 0 in X and radius $r > 0$,

$$F(x) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0)\, x^k \qquad \text{for} \qquad \|x\| < r. \tag{1}$$

where $F^{(k)}(0)$, the kth derivative of F in 0, is a symmetric k-linear operator and $(1/o!)\, F^{(0)}(0)\, x^0 = F(0)$.

Definition 2.1. We say that $F(x) = O(x^j)$ if there exist $J \in \mathbb{R}_0^+$ and an open ball $B(0, r)$ with $0 < r < 1$ such that for all x in $B(0, r)$: $\|F(x)\| \leqslant J \cdot \|x\|^j$ $(j \in \mathbb{N})$.

Write $(1/k!)\, F^{(k)}(0) = C_k \in L(X^k, Y)$.

Definition 2.2. The couple of abstract polynomials $(P(x), Q(x)) = (A_{n\cdot m+n} x^{n\cdot m+n} + \cdots + A_{n\cdot m} x^{n\cdot m},\ B_{n\cdot m+m} x^{n\cdot m+m} + \cdots + B_{n\cdot m} x^{n\cdot m})$ is called *a solution of the Padé approximation problem of order* (n, m) if the abstract power series $(F \cdot Q - P)(x) = O(x^{n\cdot m+n+m+1})$. (The choice of $(P(x), Q(x))$ is justified in [1].)

The condition in Definition 2.2 is equivalent to (1a) and (1b):

$$\left\{\begin{array}{ll} C_0 \cdot B_{n\cdot m} x^{n\cdot m} = A_{n\cdot m} x^{n\cdot m}, & \forall x \in X, \\ C_1 x \cdot B_{n\cdot m} x^{n\cdot m} + C_0 \cdot B_{n\cdot m+1} x^{n\cdot m+1} = A_{n\cdot m+1} x^{n\cdot m+1}, & \forall x \in X, \\ \vdots & \\ C_n x^n \cdot B_{n\cdot m} x^{n\cdot m} + \cdots + C_0 \cdot B_{n\cdot m+n} x^{n\cdot m+n} = A_{n\cdot m+n} x^{n\cdot m+n}, & \forall x \in X, \end{array}\right. \tag{1a}$$

with $B_j \equiv 0 \in L(X^j, Y)$ if $j > n \cdot m + m$,

$$\left\{\begin{array}{ll} C_{n+1} x^{n+1} \cdot B_{n\cdot m} x^{n\cdot m} + \cdots + C_{n+1-m} x^{n+1-m} \cdot B_{n\cdot m+m} = 0, & \forall x \in X, \\ \vdots & \\ C_{n+m} x^{n+m} \cdot B_{n\cdot m} x^{n\cdot m} + \cdots + C_n x^n \cdot B_{n\cdot m+m} x^{n\cdot m+m} = 0, & \forall x \in X, \end{array}\right. \tag{1b}$$

with $C_k \equiv 0 \in L(X^k, Y)$ if $k < o$.

3. Existence and Unicity of a Solution

For every n and m a solution of (1b) and (1a) exists. It can often be computed by a method given in [1] and repeated partly in Section 4. After calculating the solution of (1a) and (1b) we are going to look for an irreducible rational approximant.

We define $1/Q: D(Q) \to Y: x \to [Q(x)]^{-1}$ (inverse element of $Q(x)$ for the multiplication in Y).

Definition 3.1. Let P and Q be two abstract polynomials. We call $1/Q \cdot P$ *reducible* if there exist abstract polynomials T, R, S such that $P = T \cdot R$ and $Q = T \cdot S$ and $\partial T \geqslant 1$.

From now on we suppose that the considered Banach space X and the commutative Banach algebra Y are such that an irreducible form $1/Q \cdot P$ with $D(P) \neq \phi$ or $D(Q) \neq \phi$ exists and is unique, and that Y contains no nilpotent elements. This is the case, for instance, when $X = \mathbb{R}^p$ and $Y = \mathbb{R}^q$.

For more general spaces conditions to get a unique irreducible rational approximant are described in [2].

In a commutative Banach algebra Y without nilpotent elements we can prove that for abstract polynomials R and T with $D(T) \neq \varnothing$,

$$\partial R \leqslant \partial(R \cdot T) - \partial_0 T. \tag{2}$$

Definition 3.2. Let (P, Q) be a couple of abstract polynomials satisfying Definition 2.2 and suppose $D(Q) \neq \phi$ or $D(P) \neq \phi$. Possibly $1/Q \cdot P$ is reducible. Let $1/Q_* \cdot P_*$ be the irreducible form of $1/Q \cdot P$ such that $0 \in D(Q_*)$ and $Q_*(0) = I$, if it exists. We then call $1/Q_* \cdot P_*$ *the abstract Padé approximant* (*APA*) *of order* (n, m) *for* F.

Definition 3.3. If for all the solutions (P, Q) of (1a) and (1b) with $D(P) \neq \phi$ or $D(Q) \neq \phi$ the irreducible form $1/Q_* \cdot P_*$ is such that $0 \notin D(Q_*)$, then we call $1/Q_* \cdot P_*$ *the abstract rational approximant* (*ARA*) *of order* (n, m) *for* F.

Unicity of the *APA* and *ARA* is based on the equivalence-property of solutions of (1a) and (1b): if the couples (P, Q) and (R, S) of abstract polynomials both satisfy Definition 2.2, then for all x in X: $P(x) \cdot S(x) = R(x) \cdot Q(x)$.

We give the following example. Consider $F: \mathbb{R}^2 \to \mathbb{R}: \binom{x}{y} \to 1 + x/(0.1 - y) + \sin(xy)$.

$$\text{The } (1, 1)\text{-}APA \text{ in } \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1 + 10x - 10.1y}{1 - 10.1y};$$

$$\text{the } (1, 2)\text{-}ARA \text{ in } \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{x - 1.01y + 10x^2 + 10y^2 - 20.2xy}{x - 1.01y + 10y^2 - 10.1xy + 2.01xy^2}.$$

In both cases the solution (P, Q) supplied a reducible rational approximant $1/Q \cdot P$; but in the second case $Q_*(0)$ is still not regular in Y. If for all the solutions (P, Q) of (1a) and (1b), $0 \notin D(Q_*)$ or $D(Q) = \phi = D(P)$, we shall call the abstract Padé approximant *undefined.*

From now on, when mentioning abstract Padé approximants, we consider only the abstract Padé approximants that are not undefined. When $X = \mathbb{R} = Y$ ($A = \mathbb{R}$), then the definition of APA is precisely the classical definition of Padé approximant and none of the abstract Padé approximants are undefined. Let (P, Q) be a solution of (1a) and (1b). Because of Definition 3.2 it is possible that (P_*, Q_*) itself does not satisfy Definition 2.2. However, the following result holds, which is a more accurate formulation than the result mentioned in [1]. We denote by T the abstract polynomial that is cancelled in numerator and denominator of $1/Q \cdot P$ to get $1/Q_* \cdot P_*$.

THEOREM 3.1. *Let* $1/Q_* \cdot P_*$ *be the* (n, m)*-APA for* F. *Then* $n' = \partial P_* \leqslant n$ *and* $m' = \partial Q_* \leqslant m$ *and there exists an integer* s, $o \leqslant s \leqslant \max(n, m)$, *such that for* $T(x) = \sum_k T_k x^k$, $\partial T = n \cdot m + s$, $\partial_0 T \geqslant n \cdot m$, $D(T) \neq \phi$ *and* $(P_* \cdot T, Q_* \cdot T)$ *satisfies Definition* 2.2. *If*

$$\partial_0 T = n \cdot m + r \qquad \text{and} \qquad D(T_{n \cdot m + r}) \neq \phi, \tag{3}$$

then $(P_* \cdot T_{n \cdot m + r}, Q_* \cdot T_{n \cdot m + r})$ *also satisfies Definition* 2.2 *and* $o \leqslant r \leqslant \min(n - n', m - m')$.

4. Properties of the Abstract Padé Table

We repeat the notion of abstract Padé table. Let $R_{n,m}$ denote the (n, m)-APA for F if it is not undefined. The elements $R_{n,m}$ can be arranged in a table

$$\begin{array}{cccc} R_{0,0} & R_{0,1} & R_{0,2} & \cdots \\ R_{1,0} & R_{1,1} & R_{1,2} & \cdots \\ R_{2,0} & R_{2,1} & R_{2,2} & \cdots \\ R_{3,0} & \vdots & \vdots & \\ R_{4,0} & & & \\ \vdots & & & \end{array}$$

Gaps can occur in this Padé table because of undefined elements.

DEFINITION 4.1. The (n, m)-*APA* $1/Q_* \cdot P_*$ for F is called *regular* if $(F \cdot Q_* - P_*)(x) = O(x^{n+m+1})$.

DEFINITION 4.2. The (n, m)-*APA* $1/Q_* \cdot P_*$ for F is called *normal* if it occurs only once in the abstract Padé table.

An important property of the table is Theorem 4.1; the proof is given in [1].

THEOREM 4.1. *Let* $1/Q_* \cdot P_* = R_{n,m}$ *be the abstract Padé approximant of order* (n, m) *for* F. *Let* (3) *be satisfied. Then*

(a) $\partial_0(F \cdot Q_* - P_*) = n' + m' + t + 1$, *with* $t \geqslant o$;

(b) $n \leqslant n' + t$ *and* $m \leqslant m' + t$;

(c) *for all* k *and* l *such that* $n' \leqslant k \leqslant n' + t$ *and* $m' \leqslant l \leqslant m' + t$, $R_{k,l} = R_{n,m}$;

(d) $R_{n,m}$ *is normal if and only if* $n' = n$ *and* $m' = m$ *and* $\partial_0(F \cdot Q_* - P_*) = n + m + 1$.

Clearly the elements in the first column of the abstract Padé table are regular because the (n, o)-*APA* is the nth partial sum of the Taylor series development of F and $(F \cdot Q - P)(x) = F(x) \cdot I - \sum_{i=0}^{n} C_i x^i = O(x^{n+1})$.

If C_0 is regular in Y then also the first row of the abstract Padé table consists of regular abstract Padé approximants. (Do not confuse regularity in Y with the notion of regularity of an *APA*.)

If we have the abstract Padé table and want to recover the operator F which has been approximated, we merely have to look at the first column of the table to reconstruct the Taylor series expansion of the operator F. These results agree with the classical theory of Padé approximations (in the classical theory, regularity of C_0 is equivalent to its being a nonzero real number).

DEFINITION 4.3. For an operator $F: X \to Y$ we can define $\Delta_{m,n} \in L(X^{n \cdot (m+1)}, Y)$ as

$$\Delta_{m,n} := \sum_{i_0=0}^{m} \cdots \sum_{i_m=0}^{m} \left(\varepsilon_{i_0 \cdots i_m} \bigotimes_{j=0}^{m} C_{n-j+i_j} \right),$$

$$\Delta_{-1,n} := I,$$

where $\varepsilon_{i_0 \cdots i_m} = +1$ when $i_0 \cdots i_m$ is an even permutation of $o \cdots m$, $\varepsilon_{i_0 \cdots i_m} = -1$ when $i_0 \cdots i_m$ is an odd permutation of $o \cdots m$, and $\varepsilon_{i_0 \cdots i_m} = 0$ elsewhere, and the tensor product $\otimes$ is as in [3, p. 318]. We could introduce the notion of *abstract determinant* in Y, analogous to the notion of determinant in the field Λ. Then

$$\Delta_{m,n} x^{n\cdot(m+1)} = \begin{vmatrix} C_n x^n & \cdots & C_{n-m} x^{n-m} \\ \vdots & C_n x^n \quad \ddots & \vdots \\ C_{n+m} x^{n+m} & \cdots & C_n x^n \end{vmatrix}.$$

We remark that for the procedure of calculating the solution $(P, Q) = (\sum_{i=n\cdot m}^{n\cdot m+n} A_i x^i, \sum_{j=n\cdot m}^{n\cdot m+m} B_j x^j)$ of (1a) and (1b), described in [1], which we are going to call the *abstract determinant procedure* (*AD procedure*), we have

$$B_{n\cdot m+m} x^{n\cdot m+m} = (-1)^m \Delta_{m-1,n+1} x^{n\cdot m+m},$$

$$A_{n\cdot m+n} x^{n\cdot m+n} = \Delta_{m,n} x^{n\cdot m+n},$$

$$B_{n\cdot m} x^{n\cdot m} = \Delta_{m-1,n} x^{n\cdot m},$$

$$(F\cdot Q - P)(x) = (-1)^m \Delta_{m,n+1} x^{(n+1)\cdot(m+1)} + \cdots,$$

and

$$Q(x) = \begin{vmatrix} I & \cdots & & I \\ C_{n+1} x^{n+1} & C_n x^n & \cdots & C_{n-m+1} x^{n-m+1} \\ C_{n+2} x^{n+2} & & & \\ \vdots & & & \vdots \\ C_{n+m} x^{n+m} & \cdots & & C_n x^n \end{vmatrix},$$

$$P(x) = \begin{vmatrix} F_n(x) & F_{n-1}(x) & \cdots & F_{n-m}(x) \\ C_{n+1} x^{n+1} & \cdots & & C_{n+1-m} x^{n+1-m} \\ \vdots & & & \vdots \\ C_{n+m} x^{n+m} & \cdots & & C_n x^n \end{vmatrix},$$

$$(F\cdot Q - P)(x) = \begin{vmatrix} \bar{F}_{n+m}(x) & \bar{F}_{n+m-1}(x) & \cdots & \bar{F}_n(x) \\ C_{n+1} x^{n+1} & \cdots & & C_{n+1-m} x^{n+1-m} \\ \vdots & & & \vdots \\ C_{n+m} x^{n+m} & \cdots & & C_n x^n \end{vmatrix},$$

where $F_k(x) = \sum_{i=0}^{k} C_i x^i$ and $\bar{F}_k(x) = F(x) - F_k(x)$.

THEOREM 4.2. *If the* (n, m)*-APA* $R_{n,m} = 1/Q_* \cdot P_*$ *for* F *exists and can* *be obtained via the AD procedure and if $T(x)=\Delta_{m-1,n}x^{n\cdot m}$ then the (n,m)-APA for F is normal if and only if*

$$\Delta_{m-1,n}x^{n\cdot m}\not\equiv 0,$$
$$\Delta_{m-1,n+1}x^{(n+1)\cdot m}\not\equiv 0,$$
$$\Delta_{m,n}x^{n\cdot(m+1)}\not\equiv 0,$$
$$\Delta_{m,n+1}x^{(n+1)\cdot(m+1)}\not\equiv 0.$$

Proof. Let (P,Q) denote the solution of (1a) and (1b) obtained via the AD procedure. Since $P=P_*\cdot T$ and $Q=Q_*\cdot T$ with $D(T)\neq\phi$, already $\Delta_{m-1,n}x^{n\cdot m}\not\equiv 0$. Suppose $\Delta_{m,n}x^{n\cdot m+n}\equiv 0$. Then $\partial P_*\leqslant\partial P-n\cdot m<n$ because of (2). Suppose $\Delta_{m-1,n+1}x^{n\cdot m+m}\equiv 0$. Then $\partial Q_*\leqslant\partial Q-n\cdot m<m$ because of (2). These conclusions contradict the normality of $R_{n,m}$. Because $D(\Delta_{m-1,n})\neq\phi$, $\partial_0(F\cdot Q-P)=\partial_0[(F\cdot Q_*-P_*)\cdot\Delta_{m-1,n}]=\partial_0(F\cdot Q_*-P_*)+n\cdot m$ and so $\Delta_{m,n+1}x^{(n+1)\cdot(m+1)}\not\equiv 0$.

(P,Q) still denotes the solution of (1a) and (1b) obtained via the AD-procedure. Since $\partial P=n\cdot m+n$ and $P=P_*\cdot\Delta_{m-1,n}$, $\partial P_*=n$. Since $\partial Q=n\cdot m+m$ and $Q=Q_*\cdot\Delta_{m-1,n}$, $\partial Q_*=m$. Because $D(\Delta_{m-1,n})\neq\varnothing$, $\partial_0(F\cdot Q_*-P_*)=\partial_0(F\cdot Q-P)-n\cdot m=n+m+1$. So $R_{n,m}$ is normal.

THEOREM 4.3. *The (n,m)-APA for F is regular if (3) is satisfied with $r=O$.*

Proof. If $R_{n,m}=1/Q_*\cdot P_*$ and $(P,Q)=(P_*\cdot T,Q_*\cdot T)$ satisfies (1a) and (1b), then $\partial_0(F\cdot Q-P)\geqslant n\cdot m+n+m+1$. Because $D(T_{n\cdot m})\neq\phi$, $\partial_0(F\cdot Q-P)=\partial_0(F\cdot Q_*-P_*)+n\cdot m$ and so $\partial_0(F\cdot Q_*-P_*)\geqslant n+m+1$.

Let us now illustrate these results by an example.

$$\text{Take } F:\mathbb{R}^2\to\mathbb{R}:\begin{pmatrix}x\\y\end{pmatrix}\to 1+\frac{x}{0.1-y}+\sin(xy)$$
$$=1+10x+101xy+\sum_{k=3}^{\infty}10^kxy^{k-1}+\sum_{k=1}^{\infty}(-1)^k\frac{(xy)^{2k+1}}{(2k+1)!}.$$

The (1, 1)-*APA* in $\begin{pmatrix}0\\0\end{pmatrix}$ is $\dfrac{1+10x-10.1y}{1-10.1y}$

$$\text{and}\begin{cases}(F\cdot Q_*-P_*)\begin{pmatrix}x\\y\end{pmatrix}=O(xy^2)=O\left(\begin{pmatrix}x\\y\end{pmatrix}^3\right)\text{ exactly;}\\ \partial P_*=1\\ \partial Q_*=1\end{cases}$$

in other words $R_{1,1}$ is normal.

The (3, 1)-*APA* in $\begin{pmatrix}0\\0\end{pmatrix}$ is $\dfrac{1+10x-10y+xy-10xy^2}{1-10y}$

$$\text{and}\begin{cases}(F\cdot Q_*-P_*)\begin{pmatrix}x\\y\end{pmatrix}=O(x^3y^3)=O\left(\begin{pmatrix}x\\y\end{pmatrix}^6\right) \text{ exactly;}\\ \partial P_*=3\\ \partial Q_*=1\end{cases}$$

so $R_{3,1}$ is regular and we have the following square of equal elements: $R_{3,1}=R_{3,2}=R_{4,1}=R_{4,2}$.

5. Projection- and Product-Property of Abstract Padé Approximants

Consider Banach spaces X_i, $i=1,\dots,p$. The space $X=\prod_{i=1}^p X_i$, normed by one of the following Minkowski norms,

$$\|x\|_q=\left(\sum_{i=1}^p \|x_i\|_{(i)}^q\right)^{1/q}$$

or

$$\|x\|_1=\sum_{i=1}^p \|x_i\|_{(i)}$$

or

$$\|x\|_\infty=\max(\|x_1\|_{(1)},\dots,\|x_p\|_{(p)}),$$

where $\|x_i\|_{(i)}$ is the norm of x_i in X_i and $x=(x_1,\dots,x_p)$, is also a Banach space.

Let Y still be a commutative Banach algebra. We introduce the notations

$$^j\tilde{x}=(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_p),$$
$$x_{'j'}=(x_1,\dots,x_{j-1},x_{j+1},\dots,x_p).$$

THEOREM 5.1. *Let* $X=\prod_{i=1}^p X_i$ *and* $R_{n,m}(x)=(1/Q_*\cdot P_*)(x)$ *be the* (n,m)*-APA for* $F: X\to Y$ *and* $j\in\{1,\dots,p\}$. *Let* (3) *be satisfied. If*

$$S(x_{'j'}):=S_*(^j\tilde{x}),$$
$$R(x_{'j'}):=P_*(^j\tilde{x}),$$
$$G(x_{'j'}):=F(^j\tilde{x}),$$

then the irreducible form $(1/S_* \cdot R_*)(x_{'j'})$ *of* $(1/S \cdot R)(x_{'j'})$, *such that* $S_*(0) = I$, *is the* (n, m)-*APA for* $G(x_{'j'})$.

Proof. First we remark that for $L \in L(X^k, Y)$, if $M(x_{'j'})^k = M(x_1, ..., x_{j-1}, x_{j+1}, ..., x_p)^k := L({}^j\tilde{x})^k$, then $M \in L((\prod_{i=1, i \neq j}^{p} X_i)^k, Y)$. Let $n' = \partial P_*$ and $m' = \partial Q_*$.

Since $R_{n,m}(x)$ is the (n, m)-*APA* for F, there is $t \geqslant o$,

$$(F \cdot Q_* - P_*)(x) = O(x^{n' + m' + t + 1}),$$

$$n' \leqslant n \leqslant n' + t,$$

$$m' \leqslant m \leqslant m' + t,$$

according to Theorem 4.1. Using one of the three norms given above $(1 \leqslant q \leqslant \infty)$: $\|{}^j\tilde{x}\|_q = \|(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_p)\|_q$ in $\prod_{i=1}^{p} X_i$ equals $\|x_{'j'}\|_q = \|(x_1, ..., x_{j-1}, x_{j+1}, ..., x_p)\|_q$ in $\prod_{i=1, i \neq j}^{p} X_i$.

Thus $(F \cdot Q_* - P_*)({}^j\tilde{x}) = (G \cdot S - R)(x_{'j'}) = O((x_{'j'})^{n' + m' + t + 1})$. Take $s = n \cdot m + \max(o, n + m - (n' + m' + t))$ and $D_s \in L((\prod_{i=1, i \neq j}^{p} X_i)^s, Y)$ with $D(D_s) \cap D(R) \neq \phi$ or $D(D_s) \cap D(S) \neq \phi$ (it is easy to prove that D_s exists). Now

$$\begin{aligned}
[(G \cdot S - R) \cdot D_s](x_{'j'}) &= O((x_{'j'})^{n \cdot m + \max(n' + m' + t + 1, n + m + 1)}) \\
&= O((x_{'j'})^{n \cdot m + n + m + 1}) \qquad \text{since } n + m + 1 \\
&\leqslant \max(n' + m' + t + 1, n + m + 1),
\end{aligned}$$

$$\begin{aligned}
\partial(S \cdot D_s) &\leqslant n \cdot m + \max(o, n + m - (n' + m' + t)) + m' \\
&= n \cdot m + \max(m', m + (n - n' - t)) \\
&\leqslant n \cdot m + m \qquad \text{since } n \leqslant n' + t \text{ and } m' \leqslant m,
\end{aligned}$$

$$\begin{aligned}
\partial(R \cdot D_s) &\leqslant n \cdot m + \max(o, n + m - (n' + m' + t)) + n' \\
&= n \cdot m + \max(n', n + (m - m' - t)) \\
&\leqslant n \cdot m + n \qquad \text{since } n' \leqslant n \text{ and } m \leqslant m' + t.
\end{aligned}$$

The irreducible form of $1/(S \cdot D_s) \cdot (R \cdot D_s)$ is the irreducible form of $1/S \cdot R$ and $S(0) = Q_*(0) = I$.

We searched for a product-property of the following kind. Let X_1, X_2 be Banach spaces and Y a commutative Banach algebra. If $(1/Q_1 \cdot P_1)(x_1)$ is the (n, m)-*APA* for the operator $F_1 : X_1 \to Y$ in x_{01}, and $(1/Q_2 \cdot P_2)(x_2)$ is the (n, m)-*APA* for the operator $F_2 : X_2 \to Y$ in x_{02}, is then

$$(1/Q \cdot P)(x_1, x_2) := \frac{1}{Q_1(x_1) \cdot Q_2(x_2)} \cdot (P_1(x_1) \cdot P_2(x_2))$$

the (n,m)-*APA* for $F: X_1 \times X_2 \to Y: (x_1, x_2) \to F_1(x_1) \cdot F_2(x_2)$ in (x_{01}, x_{02})? In fact it is not at all natural to have a property like this; the following counterexample proves it.

Let $F_1: \mathbb{R} \to \mathbb{R}: x \to e^x$, $F_2: \mathbb{R} \to \mathbb{R}: y \to e^y$, then $F: \mathbb{R}^2 \to \mathbb{R}: (x, y) \to e^x \cdot e^y = e^{x+y}$. Take $n = 1$ and $m = 2$ and $x_{01} = 0 = x_{02}$. The $(1, 2)$-*APA* for F_1 is

$$\left(\frac{1}{Q_1} \cdot P_1\right)(x) = \frac{1 + \frac{1}{3}x}{1 - \frac{2}{3}x + \frac{1}{6}x^2},$$

for F_2 is

$$\left(\frac{1}{Q_2} \cdot P_2\right)(y) = \frac{1 + \frac{1}{3}y}{1 - \frac{2}{3}y + \frac{1}{6}y^2},$$

for F is

$$\frac{1 + \frac{1}{3}(x+y)}{1 - \frac{2}{3}(x+y) + \frac{1}{6}(x+y)^2} \neq \frac{1}{Q_1(x) \cdot Q_2(y)} \cdot (P_1(x) \cdot P_2(y)).$$

Let X be a Banach space and Y_i commutative Banach algebras. We consider nonlinear operators $F_i: X \to Y_i$, $i = 1,..., q < \infty$ and $F: X \to \prod_{i=1}^q Y_i: x \to (F_i(x), i = 1,..., q)$ where $\prod_{i=1}^q Y_i$ is a commutative Banach algebra with component-wise multiplication and normed by one of the Minkowski norms $\|(y_1,..., y_q)\|_p$, $1 \leqslant p \leqslant \infty$. (We can obtain, by renorming, that $\|(I_1,..., I_q)\|_p = 1$, where I_i is the unit for the multiplication in Y_i.)

THEOREM 5.2. *Let* $\bigcap_{i=1}^q D(P_i) \neq \phi$ *or* $\bigcap_{i=1}^q D(Q_i) \neq \phi$ *for the solutions* (P_i, Q_i) *of* (1a) *and* (1b) *for* F_i. *Then* $1/Q_{*i} \cdot P_{*i}$ *is the* (n,m)-*APA for* F_i, $i = 1,..., q$ *if and only if* $1/Q_* \cdot P_* = (1/Q_{*i} \cdot P_{*i}, i = 1,..., q)$ *is the* (n,m)-*APA for* F.

Proof. First we remark that if $L_i \in L(X^j, Y_i)$ for $i = 1,..., q$, then $Lx^j := (L_i x^j, i = 1,..., q) \in L(X^j, \prod_{i=1}^q Y_i)$. For each $i = 1,..., q$ there is a polynomial T_i with $D(T_i) \neq \phi$ such that $(F \cdot Q_i - P_i)(x) = [(F \cdot Q_{*i} - P_{*i}) \cdot T_i](x) = O(x^{n \cdot m + n + m + 1})$. So there exists $K_i \in \mathbb{R}_0^+$ and an open ball $B(0, r_i)$ with $0 < r_i < 1$: $\|(F \cdot Q_i - P_i)(x)\| \leqslant K_i \cdot \|x\|^{n \cdot m + n + m + 1}$ for $i = 1,..., q$. We use the Minkowski norm $\| \|_p$ in $\prod_{i=1}^q Y_i$ for some p with $1 \leqslant p \leqslant \infty$. Then for $p = 1$: let $K = \sum_{i=1}^q K_i$; for $p = \infty$: let $K = \max_i K_i$; for $1 < p < \infty$: let $K = (\sum_i K_i^p)^{1/p}$; and we find $\|((F \cdot Q_i - P_i)(x), i = 1,..., q)\| \leqslant K \cdot \|x\|^{n \cdot m + n + m + 1}$ for $\|x\| < \min_i r_i$. Thus $((P_i, Q_i), i = 1,..., q) = ((P_{*i} \cdot T_i, Q_{*i} \cdot T_i), i = 1,..., q)$ satisfies (1a) and (1b) for F.

Now the irreducible form of $(1/(Q_{*i} \cdot T_i) \cdot P_{*i} \cdot T_i,\ i=1,..., q)$ is $(1/Q_{*i} \cdot P_{*i},\ i=1,..., q)$ since $(Q_{*i} \cdot T_i,\ i=1,..., q) = (Q_{*i},\ i=1,..., q) \cdot (T_i,\ i=1,..., q)$ and $(P_{*i} \cdot T_i,\ i=1,..., q) = (P_{*i},\ i=1,..., q). (T_i,\ i=1,..., q)$. Also $Q_*(0) := (Q_{*i}(0),\ i=1,..., q) = (I_1,..., I_q)$ unit for the multiplication in $\prod_{i=1}^q Y_i$.

Since $1/Q_* \cdot P_*$ is the (n, m)-*APA* for F, there exists a polynomial T with $D(T) \neq \phi$ such that $[(F \cdot Q_* - P_*) \cdot T](x) = O(x^{n \cdot m+n+m+1})$. We write $(T)_i$ for the ith component-operator of T. The proof is based on the fact that $\|[(F_i \cdot Q_{*i} - P_{*i}) \cdot (T)_i](x)\| \leqslant \|[(F \cdot Q_* - P_*) \cdot T](x)\|$ for $i=1,..., q$ and for whatever Minkowski norm used in $\prod_{i=1}^q Y_i$. So $(P_{*i} \cdot (T)_i,\ Q_{*i} \cdot (T)_i)$ satisfies (1a) and (1b) for F_i and the irreducible form of $1/(Q_{*i}(T)_i) \cdot P_{*i}(T)_i$ is $1/Q_{*i} \cdot P_{*i}$. Also $Q_{*i}(0) = I_i$ in Y_i.

Theorems 5.1 and 5.2 are illustrated by the following numerical examples. Take $G: \mathbb{R}^2 \to \mathbb{R}: \binom{x}{y} \to \frac{1}{2}(1 + e^{x+y})$. The $(1, 1)$-*APA* for G in $\binom{0}{0}$ is $1/(1 - \frac{1}{2}(x + y))$. For $j = 1$, $x = 0$,

$$G_1: \mathbb{R} \to \mathbb{R}: y \to \tfrac{1}{2}(1 + e^y);$$

for $j = 2$, $y = 0$,

$$G_2: \mathbb{R} \to \mathbb{R}: x \to \tfrac{1}{2}(1 + e^x).$$

And indeed the $(1, 1)$-*APA* for G_1 in 0 equals $1/(1 - \frac{1}{2}y)$ and for G_2 in 0 equals $1/(1 - \frac{1}{2}x)$.

We already considered $F: \mathbb{R}^2 \to \mathbb{R}: \binom{x}{y} \to 1 + x/(0.1 - y) + \sin(xy)$ with the following $(1, 1)$-*APA* in $\binom{0}{0}$: $(1 + 10x - 10.1y)/(1 - 10.1y)$. Now the $(1, 1)$-*APA* in $\binom{0}{0}$ for $\binom{F}{G}: \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 + \dfrac{x}{0.1 - y} + \sin(xy) \\ \tfrac{1}{2}(1 + e^{x+y}) \end{pmatrix} \quad \text{is} \quad \frac{\begin{pmatrix} 1 + 10x - 10.1y \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 - 10.1y \\ 1 - \frac{1}{2}(x + y) \end{pmatrix}},$$

as claimed in Theorem 5.2.

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