

# Recent Applications of Rational Approximation Theory: A Guided Tour

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Rational functions have a clear advantage compared to polynomials. They can simulate singularities and different kinds of asymptotic behaviour. These features of rational functions have proved to be very useful in a lot of applications including communication networks, EM models, shape reconstruction and the verified computation of special functions. We present a number of the more recent results.

## 1 Introduction

Rational approximation theory has been around for several centuries. To illustrate that its applications are still in demand and involve some technologically advanced problem domains, the following recently obtained results shall be discussed:

- reliable and highly efficient cell loss probability computation in the context of multiplexer models;
- adaptive multivariate rational interpolation requiring as few as possible, expensive to obtain engineering data;
- shape reconstruction from multidimensional moment information, compared to the inverse Radon transform using projections;
- a verified and fast multiprecision implementation of a class of special functions, based on the latest continued fraction results.

## 2 Cell loss probability

A statistical multiplexer is a basic model used in the design and the dimensioning of communication networks. The multiplexer model consists of a finite buffer, to store incoming packets, served by a single server with constant service time, and a more or less complicated arrival process. The aim is to determine the packet/cell loss probability (CLP) as a function of the capacity of the buffer. A variety of techniques have been developed in recent years, based on the exact analysis using matrix-analytic methods [1], fluid approximation [2] and simulation [3] to compute the CLP.

Recently another and very efficient approach to compute the CLP as a function of the system size has become available, based on the use of rational approximation [7]. The motivation behind this approach is that using matrix-analytic method, it is computationally feasible to evaluate the CLP as a function of the system size when the system size is small, and moreover it is often possible to obtain information about the asymptotic behavior [5]. In [7], the authors have employed rational approximants to compute the

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CLP, but only for models where the correlation between the cells was ignored. Considering a high degree of correlation is of major importance when the input consists of more video sources [1]. A sampling technique in combination with rational data fitting for models with more correlation between the cells will be discussed. In rare cases it happens that the poles of the computed rational function disturb the fit of the packet loss probability. A method to avoid this problem through an a priori optimal placement of poles is presented. The location of the poles will be determined, indirectly, from the system parameters.

### **3 Adaptive multivariate sampling**

During the design process of a complex physical system, computer-based simulations are often used to limit the number of expensive prototypes. However, despite the steady and continuing growth of computing speed and power, the computational cost of complex high-accuracy simulations, such as in electromagnetic modelling, can also be high. A single simulation of a design may take several minutes or hours to complete [10, 4].

We present an adaptive meta-modelling technique, in one or more dimensions, using as few data points as possible. Such a meta-model is a fast running surrogate approximation, in our case a rational approximation, of a complex time-consuming computer simulation. The sparse data used to compute the rational model are carefully selected. It is clear that the location of the free denominator zeroes of the computed model can have a lot of impact on the overall accuracy of the model. Therefore the distance of the nearest poles to the design space is monitored and influences the choice of the optimal rational model.

### **4 Shape reconstruction**

In shape reconstruction, the celebrated Fourier slice theorem plays an essential role. It allows to reconstruct the shape of a quite general object from the knowledge of its Radon transform, in other words from the knowledge of projections of the object. In case the object is a polygon [6], or when it defines a quadrature domain in the complex plane [8], its shape can also be reconstructed from the knowledge of its moments. Essential tools in the solution of the latter inverse problem are quadrature rules and formal orthogonal polynomials.

We show how shape reconstruction of general compact objects can also be realized from the knowledge of the moments. To this end we use a less known homogeneous Padé slice property. Again integral transforms, in our case the Stieltjes transform, formal orthogonal polynomials in the form of Padé denominators, and multidimensional integration formulas or cubature rules play an essential role.

### **5 Verified multiprecision function library**

The technique to provide a floating-point implementation of a function differs substantially when going from a fixed finite precision context to a finite multiprecision context. In the former, the aim is to provide an optimal mathematical model, valid on a reduced argument range and requiring as few operations as possible. Here optimal means that, with respect to the model's complexity, the truncation error is as small as it can get. The total relative error should not exceed a prescribed threshold, round-off error and argument reduction effect included. In the latter, the goal is to provide a more generic technique, from which an approximant yielding the user-defined accuracy, can be deduced at runtime. Hence best approximants are not an option since these models would have to be recomputed every time the precision is altered and a function evaluation is requested. At the same time the generic technique should propose an approximant of as low complexity as possible.

In the current approach we point out how continued fraction representations of functions can be helpful in the multiprecision context [9]. The developed generic technique is mainly based on the use of sharpened new a priori truncation error estimates. The technique is very efficient and even quite competitive when

compared to the traditional fixed precision implementations. The implementation is reliable in the sense that it allows to return a sharp interval enclosure for the requested function evaluation.

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