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THE QD-ALGORITHM FOR PADÉ-APPROXIMANTS IN OPERATOR THEORY*

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Abstract. It is well known that the quotient-difference algorithm can be used to construct univariate Padé-approximants. In this paper we see that the Padé-approximants for nonlinear operators $F: X \to Y$ where X is a Banach space and Y a commutative Banach algebra, introduced by the author, can also be obtained by means of the QD-algorithm and can consequently be obtained as convergents of a continued fraction, if the scalar QD-algorithm is reformulated as in §1. The definition of abstract Padé-approximants will be repeated in §2, while the operator QD-algorithm will be treated in §3.

1. The scalar QD-scheme. Let us consider a nonlinear real-valued function f of one real variable x, analytic at the origin

$$f(x)=\sum_{k=0}^{\infty} c_k x^k \quad \text{with } c_k=\frac{1}{k!}f^{(k)}(0).$$

We will present the QD-algorithm in a slightly different way than usual, but the two approaches are equivalent. The advantage of this approach is that it can be generalized to the case where F is a nonlinear operator from a Banach space X to a commutative Banach algebra Y.

Let the series f be normal, i.e.,

$$\begin{vmatrix} c_n x^n & c_{n+1}x^{n+1} & \cdots & c_{n+k-1}x^{n+k-1} \\ c_{n+1}x^{n+1} & & & \\ \vdots & & & \vdots \\ c_{n+k-1}x^{n+k-1} & & & c_{n+2k-2}x^{n+2k-2} \end{vmatrix} \neq 0$$

for $n=0,1,2,\cdots$ and $k=1,2,\cdots$. This determinant is a monomial of degree $k(n+k-1)$ in the variable x. Demanding that this monomial be nontrivial is equivalent to demanding that this determinant evaluated at $x=1$ be nonzero.

For a normal series we can construct a double entry table of numbers $q_k^{(n)}$ and $e_k^{(n)}$ defined as follows:

$$\begin{aligned} &e_0^{(n)}=0, && n=0,1,\cdots, \\ &q_1^{(n)}=\frac{c_{n+1}x^{n+1}}{c_n x^n}, && n=0,1,\cdots, \\ &e_k^{(n)}=q_k^{(n+1)}+e_{k-1}^{(n+1)}-q_k^{(n)}, && n=0,1,2,\cdots, \quad k=1,2,\cdots, \\ &q_{k+1}^{(n)}=\frac{q_k^{(n+1)}e_k^{(n+1)}}{e_k^{(n)}}, && n=0,1,2,\cdots, \quad k=1,2,\cdots. \end{aligned}$$

From this QD-algorithm we can obtain Padé-approximants to the function f as follows.

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The (l,m) Padé-approximant (numerator of degree l and denominator of degree m) for $l \geq m$ is equal to the $(2m)$th convergent K_{2m} of the continued fraction

$$c_0+c_1x+\cdots+c_{l-m}x^{l-m}+\frac{c_{l-m+1}x^{l-m+1}|}{|1}-\frac{q_1^{(l-m+1)}|}{|1}-\frac{e_1^{(l-m+1)}|}{|1}-\frac{q_2^{(l-m+1)}|}{|1}-\frac{e_2^{(l-m+1)}|}{|1}-\cdots$$

if $K_0=\sum_{k=0}^{l-m}c_kx^k$, and to the $(2m+1)$th convergent K_{2m+1} of the continued fraction

$$c_0+c_1x+\cdots+c_{l-m-1}x^{l-m-1}+\frac{c_{l-m}x^{l-m}|}{|1}-\frac{q_1^{(l-m)}|}{|1}-\frac{e_1^{(l-m)}|}{|1}-\frac{q_2^{(l-m)}|}{|1}-\frac{e_2^{(l-m)}|}{|1}-\cdots$$

if $K_0=\sum_{k=0}^{l-m-1}c_kx^k$[1].

The terms $q_k^{(n)}$ and $e_k^{(n)}$ each contain a factor x now because of the definition of $q_1^{(n)}$. Since the series f is normal the Padé-approximants are also normal [1].

2. Abstract Padé-approximants for operators. We briefly repeat the definition of Padé-approximant in operator theory and a determinantal formula for the calculation. More details can be found in [3]. Let X be a Banach space and Y a commutative algebra (0 denotes the unit for addition and I the unit for multiplication). Let $F: X \to Y$ be analytic in the open ball $B(0,r)$ with centre $0 \in X$ and radius $r>0$ [5, p. 113]

$$F(x)=\sum_{k=0}^{\infty}\frac{1}{k!}F^{(k)}(0)x^k \quad \text{for } \|x\|<r,$$

where $F^{(k)}(0)$ is the kth Fréchet-derivative of F in 0 and thus a symmetric k-linear bounded operator, and where $(1/0!)F^{(0)}(0)x^0=F(0)$.

DEFINITION 2.1. $F(x)=O(x^k)\ (k\in\mathbb{N})$ if nonnegative constants $r<1$ and K exist such that $\|F(x)\|\leq K\|x\|^k$ for $\|x\|<r$.

Write $D(F)=\{x\in X \mid F(x)$ is regular in Y, i.e., there exists $y\in Y: F(x)\cdot y=I\}$.

DEFINITION 2.2. An *abstract polynomial* is a nonlinear operator $P: X\to Y$ with $P(x)=A_nx^n+A_{n-1}x^{n-1}+\cdots+A_0$, where A_i is a symmetric i-linear bounded operator $(i=0,\cdots,n)$ [5, p. 194].

When we have two abstract polynomials P and Q, we can construct an abstract rational operator $Q^{-1}\cdot P$ where $Q^{-1}(x)$ is the inverse element of $Q(x)$ for the multiplication in the Banach algebra Y. Of course division by $Q(x)$ can only be performed when x is in $D(Q)$.

DEFINITION 2.3. The pair of abstract polynomials $(P(x),Q(x))=(\sum_{i=0}^{l}A_{lm+i}x^{lm+i},\sum_{j=0}^{m}B_{lm+j}x^{lm+j})$ such that the abstract power series $(F\cdot Q-P)(x)=O(x^{lm+l+m+1})$ is called a *solution of the Padé-approximation problem of order* (l,m).

The shift of degrees by $l\cdot m$ provides us with many nice properties [2], [3] and will also provide us with an abstract QD-scheme.

Let us denote by $Q_\triangle^{-1}\cdot P_\triangle$ a reduced form of the abstract rational operator $Q^{-1}\cdot P$; in other words $P=P_\triangle\cdot T$ and $Q=Q_\triangle\cdot T$ and we have cancelled this abstract polynomial T in both numerator and denominator. Different solutions of the Padé-approximation problem and different reduced forms are equivalent (denoted by $\simeq$); i.e., they satisfy the relation

$$(P,Q)\simeq(R,S)\Leftrightarrow P(x)\cdot S(x)=Q(x)\cdot R(x) \quad \forall x\in X.$$

DEFINITION 2.4. The *abstract Padé-approximant of order* (l,m) *for* F is the equivalence class containing all the pairs (P,Q) satisfying Definition 2.3 and all the pairs $(P_\triangle, Q_\triangle)$ which are the numerator and denominator of a reduced form of $Q^{-1}\cdot P$.

Let us write $C_k=(1/k!)F^{(k)}(0)$. We call the series F normal if there exists x in X such that

$$H_k(C_n)=\begin{vmatrix} C_n x^n & C_{n+1}x^{n+1} & \cdots & C_{n+k-1}x^{n+k-1} \\ \vdots & & & \vdots \\ C_{n+k-1}x^{n+k-1} & & & C_{n+2k-2}x^{n+2k-2} \end{vmatrix}$$

is regular in Y for $n=0,1,2,\cdots$ and $k=1,2,\cdots$. When the series

$$C_0+\sum_{k=1}^{\infty}\left(C_k x^k - C_{k-1}x^{k-1}\right)$$

is normal, a representation of $P(x)$ and $Q(x)$ satisfying Definition 2.3 is given by

$$P(x)=\begin{vmatrix} F_l(x) & F_{l-1}(x) & \cdots & F_{l-m}(x) \\ C_{l+1}x^{l+1} & & & C_{l-m+1}x^{l-m+1} \\ \vdots & & & \vdots \\ C_{l+m}x^{l+m} & & & C_l x^l \end{vmatrix},$$

$$Q(x)=\begin{vmatrix} I & I & \cdots & I \\ C_{l+1}x^{l+1} & \cdots & & C_{l-m+1}x^{l-m+1} \\ \vdots & & & \vdots \\ C_{l+m}x^{l+m} & \cdots & & C_l x^l \end{vmatrix},$$

where $F_l(x)=\sum_{k=0}^{l}C_k x^k$ [3, p. 25].

From now on we shall denote these determinants by $P_{[l,m]}(x)$ and $Q_{[l,m]}(x)$, respectively. The pair $(P_{[l,m]}, Q_{[l,m]})$ can be considered as a representative of the abstract Padé-approximant of order (l,m) for F. If we introduce the notation

$$\Delta C_k x^k = C_{k+1}x^{k+1} - C_k x^k$$

then normality of the series $C_0+\sum_{k=1}^{\infty}(C_k x^k - C_{k-1}x^{k-1})$ is equivalent to $H_k(\Delta C_n)$ being regular in Y for some x in X and for all $n=0,1,2,\cdots$ and $k=1,2,\cdots$. So normality of the series $C_0+\sum_{k=0}^{\infty}\Delta C_k x^k$ implies regularity of $Q_{[l,m]}(x)=H_m(\Delta C_{l-m+1})$ for some x and thus existence of $Q_{[l,m]}^{-1}\cdot P_{[l,m]}$.

3. The abstract QD-scheme. For a normal series F we can define the abstract QD-scheme as follows:

$$E_0^{(n)}=0, \qquad n=0,1,\cdots,$$

$$Q_1^{(n)}=\left(C_{n+1}x^{n+1}\right)\cdot\left(C_n x^n\right)^{-1}, \qquad n=0,1,\cdots,$$

$$E_k^{(n)}=Q_k^{(n+1)}+E_{k-1}^{(n+1)}-Q_k^{(n)}, \qquad n=0,1,\cdots, \quad k=1,2,\cdots,$$

$$Q_{k+1}^{(n)}=Q_k^{(n+1)}\cdot E_k^{(n+1)}\cdot\left(E_k^{(n)}\right)^{-1}, \qquad n=0,1,\cdots, \quad k=1,2,\cdots.$$

The existence of all the $E_k^{(n)}$ and $Q_k^{(n)}$ is proved as in [4, pp. 610–611]. Let us construct the following continued fractions in the Banach algebra Y:

$$(1)\qquad \cfrac{\sum_{k=0}^{l-m} C_k x^k + C_{l-m+1}x^{l-m+1}}{I-\cfrac{Q_1^{(l-m+1)}}{I-\cfrac{E_1^{(l-m+1)}}{I-\cfrac{Q_2^{(l-m+1)}}{I-\cfrac{E_2^{(l-m+1)}}{I-\dots}}}}}$$

and

$$(2)\qquad \cfrac{\sum_{k=0}^{l-m-1} C_k x^k + C_{l-m}x^{l-m}}{I-\cfrac{Q_1^{(l-m)}}{I-\cfrac{E_1^{(l-m)}}{I-\cfrac{Q_2^{(l-m)}}{I-\cfrac{E_2^{(l-m)}}{I-\dots}}}}}$$

where division means multiplication by the inverse element for multiplication in Y.

We shall now prove that these continued fractions are of the same form as in the univariate case where only a factor x remains in $q_k^{(n)}$ and $e_k^{(n)}$ after division of their numerator by denominator and we shall also prove that the convergents of these continued fractions yield our abstract Padé-approximants.

THEOREM 1. *If we write* $Q_k^{(n)}=N_{q,k,n}/D_{q,k,n}$ *and* $E_k^{(n)}=N_{e,k,n}/D_{e,k,n}$ *then* $\partial N_{q,k,n}=\partial D_{q,k,n}+1$ *and* $\partial N_{e,k,n}=\partial D_{e,k,n}+1$, *where* ∂ *indicates the degree of the abstract polynomial.*

Proof. The proof is by induction. For $k=1$ we have

$$N_{q,k,n}=C_{n+1}x^{n+1},\qquad D_{q,k,n}=C_n x^n,$$
$$N_{e,k,n}=\left(C_{n+1}x^{n+1}\right)^2-C_nx^n\cdot C_{n+2}x^{n+2},\qquad D_{e,k,n}=C_nx^n\cdot C_{n+1}x^{n+1},$$

so that

$$\partial N_{q,k,n}=n+1=\partial D_{q,k,n}+1,\qquad \partial N_{e,k,n}=2n+2=\partial D_{e,k,n}+1.$$

Suppose the theorem holds for $Q_1^{(n)},\cdots,Q_k^{(n)}$, $E_1^{(n)},\cdots,E_k^{(n)}$; we shall prove it then for $Q_{k+1}^{(n)}$ and $E_{k+1}^{(n)}$.

Since $Q_{k+1}^{(n)}=Q_k^{(n+1)}\cdot E_k^{(n+1)}\cdot(E_k^{(n)})^{-1}$, we have

$$Q_{k+1}^{(n)}=\frac{N_{q,k,n+1}\cdot N_{e,k,n+1}\cdot D_{e,k,n}}{N_{e,k,n}\cdot D_{q,k,n+1}\cdot D_{e,k,n+1}}=\frac{N_{q,k+1,n}}{D_{q,k+1,n}}.$$

Thus $\partial N_{q,k+1,n}=\partial N_{q,k,n+1}+\partial N_{e,k,n+1}+\partial D_{e,k,n}=\partial D_{q,k+1,n}+1$. For $E_{k+1,n}$ the proof is analogous.

Consider the following descending staircase:

$$\begin{array}{lll} P_{[l-m,0]}(x)\cdot Q^{-1}_{[l-m,0]}(x) & & \\ P_{[l-m+1,0]}(x)\cdot Q^{-1}_{[l-m+1,0]}(x) & P_{[l-m+1,1]}(x)\cdot Q^{-1}_{[l-m+1,1]}(x) & \\ & P_{[l-m+2,1]}(x)\cdot Q^{-1}_{[l-m+2,1]}(x) & \cdots \\ & & \vdots \end{array}$$

THEOREM 2. *$P_{[l,m]}(x)\cdot Q^{-1}_{[l,m]}(x)$ is the $(2m)$th convergent of the continued fraction* (1).

Proof. Let on the above staircase $P_{[l-m+i,j]}(x)\cdot Q^{-1}_{[l-m+i,j]}(x)$ be denoted by K_{i+j}, $i+j=0,1,\cdots$.

Regularity of the $H_k(C_n)$ and the $H_k(\Delta C_n)$ implies that [3, pp. 38–39]

$$\begin{aligned} K_{2i+1}-K_{2i}&=(-1)^i H_{i+1}(C_{l-m+i+1})H_i(C_{l-m+i+1})H_i^{-1}(\Delta C_{l-m+i})H_i^{-1}(\Delta C_{l-m+i+1}),\\ K_{2i}-K_{2i-1}&=(-1)^{i-1}H_i(C_{l-m+i+1})H_i(C_{l-m+i})H_i^{-1}(\Delta C_{l-m+i})H_{i-1}^{-1}(\Delta C_{l-m+i}),\\ K_{i+j}-K_{i+j-2}&=(-1)^{j-1}\left[H_j(C_{l-m+i})\right]^2 H_j^{-1}(\Delta C_{l-m+i})H_{j-1}^{-1}(\Delta C_{l-m+i+1}) \end{aligned}$$

are regular.

So it is possible to construct the continued fraction

$$\text{(3)}\qquad \sum_{k=0}^{l-m} C_k x^k + \cfrac{C_{l-m+1}x^{l-m+1}}{I+\cfrac{\frac{K_1-K_2}{K_2-K_0}}{I+\sum_{n=3}^{\infty}\left|\frac{\frac{(K_{n-1}-K_n)(K_{n-2}-K_{n-3})}{(K_n-K_{n-2})(K_{n-1}-K_{n-3})}}{I}\right.}}$$

with convergents K_0, K_1, $K_2,\cdots$, where division again means multiplication by the inverse element for multiplication defined in Y. It is easy to verify that

$$\frac{K_1-K_2}{K_2-K_0}=Q_1^{(l-m+1)} \quad\text{and}\quad \frac{(K_2-K_3)(K_1-K_0)}{(K_3-K_1)(K_2-K_0)}=E_1^{(l-m+1)},$$

using the representation of $P_{[l-m,0]}(x)$, $Q_{[l-m,0]}(x)$, $P_{[l-m+1,0]}(x)$, $Q_{[l-m+1,0]}(x),\cdots$ given in the previous section.

Let us denote

$$\frac{(K_{n-1}-K_n)(K_{n-2}-K_{n-3})}{(K_n-K_{n-2})(K_{n-1}-K_{n-3})}$$

by $A_{n/2}^{(l-m+1)}$ if n is even and by $B_{(n-1)/2}^{(l-m+1)}$ if n is odd. We write also $A_1^{(l-m+1)}=Q_1^{(l-m+1)}$.

If we write down the continued fraction that is the even contraction of (3) (i.e., a continued fraction having as convergents K_{2n} for $n=0,1,2,\cdots$), we get

$$\sum_{k=0}^{l-m} C_k x^k + \cfrac{C_{l-m+1}x^{l-m+1}}{I-A_1^{(l-m+1)}-\cfrac{A_1^{(l-m+1)}B_1^{(l-m+1)}}{I-B_1^{(l-m+1)}-A_2^{(l-m+1)}-\dots}} \tag{4}$$

If we write down the continued fraction that is the odd contraction of (3) with $l-m$ replaced by $l-m-1$ (i.e, a continued fraction having as convergents the $P_{[l-m,0]}(x)\cdot Q_{[l-m,0]}^{-1}(x)$, $P_{[l-m+1,1]}(x)\cdot Q_{[l-m+1,1]}^{-1}(x),\cdots$ on the descending staircase (6)), we get

$$\sum_{k=0}^{l-m-1} C_k x^k + \cfrac{C_{l-m}x^{l-m}A_1^{(l-m)}}{I-A_1^{(l-m)}-B_1^{(l-m)}-\cfrac{B_1^{(l-m)}A_2^{(l-m)}}{I-A_2^{(l-m)}-B_2^{(l-m)}-\dots}} \tag{5}$$

Because (4) and (5) have the same convergents, we have

$$A_k^{(l-m+1)}B_k^{(l-m+1)}=B_k^{(l-m)}A_{k+1}^{(l-m)},\quad B_{k-1}^{(l-m+1)}+A_k^{(l-m+1)}=B_k^{(l-m)}+A_k^{(l-m)},\qquad k=1,2,\cdots,$$

if we put $B_0^{(l-m+1)}=0$. So

$$A_k^{(l-m+1)}=Q_k^{(l-m+1)},\quad B_k^{(l-m+1)}=E_k^{(l-m+1)},\qquad k=1,2,\cdots.$$

This completes the proof.

Analogously we can formulate and prove the next theorem.

THEOREM 3. *$P_{[l,m]}(x)\cdot Q_{[l,m]}^{-1}(x)$ is the $(2m+1)$th convergent of the continued fraction* (2).

This can easily be seen by writing down the continued fraction (3) with $l-m$ replaced by $l-m-1$; the convergents of this continued fraction are the abstract Padé-approximants on the following descending staircase:

$$\begin{array}{lll} P_{[l-m-1,0]}(x)\cdot Q_{[l-m-1,0]}^{-1}(x) & & \\ P_{[l-m,0]}(x)\cdot Q_{[l-m,0]}^{-1}(x) & P_{[l-m,1]}(x)\cdot Q_{[l-m,1]}^{-1}(x) & \\ & P_{[l-m+1,1]}(x)\cdot Q_{[l-m+1,1]}^{-1}(x) & \cdots \\ & & \vdots \end{array} \tag{6}$$

We illustrate Theorems 2 and 3 by means of a simple example. Consider

$$F: C'([0,T])\to C([0,T]): x(t)\to e^{x(t)}\frac{dx}{dt}-(1+d).$$

The unit in the Banach algebra $C([0,T])$ is the constant function $x(t)=1$, so we shall write $I=1$.

The Taylor series development of F around $x(t)=0$, is

$$F(x)=-(1+d)+\frac{dx}{dt}\sum_{k=0}^{\infty}\frac{1}{k!}x^k(t).$$

Let us calculate, for instance, the $(l,2)$ abstract Padé-approximants for $l\geq 0$. If we use the determinantal representation of $P_{[l,2]}(x)$ and $Q_{[l,2]}(x)$ we find that

$$P_{[l,2]}(x)=\left(\frac{dx}{dt}\right)^2\frac{x^{2l-2}(t)}{l!(l-1)!}\left[\sum_{k=0}^{l}C_kx^k-\frac{2x(t)}{l+1}\sum_{k=0}^{l-1}C_kx^k+\frac{x^2(t)}{l(l+1)}\sum_{k=0}^{l-2}C_kx^k\right],$$

$$Q_{[l,2]}(x)=\left(\frac{dx}{dt}\right)^2\frac{x^{2l-2}(t)}{l!(l-1)!}\left[1-\frac{2x(t)}{l+1}+\frac{x^2(t)}{l(l+1)}\right].$$

Now $P_{[l,2]}(x)\cdot Q_{[l,2]}^{-1}(x)$ is the 4th convergent of the continued fraction (1). We calculate the necessary elements in the QD-table

$$Q_1^{(n)}=\frac{C_{n+1}x^{n+1}}{C_nx^n}=\frac{x(t)}{n},$$

$$E_1^{(n)}=Q_1^{(n+1)}-Q_1^{(n)}=\frac{-x(t)}{n(n+1)},$$

$$Q_2^{(n)}=\frac{Q_1^{(n+1)}\cdot E_1^{(n+1)}}{E_1^{(n)}}=x(t)\frac{n}{(n+1)(n+2)}.$$

Note that in $Q_1^{(n)}$ the quotient of an $(n+1)$-linear operator by an n-linear operator is a linear operator, which is not true in general but which simplifies the calculations a lot.

It is easy to check that

$$P_{[l,2]}(x)\cdot Q_{[l,2]}^{-1}(x)=\sum_{k=0}^{l-2}C_kx^k+\cfrac{C_{l-1}x^{l-1}}{1-\cfrac{Q_1^{(l-1)}}{1-\cfrac{E_1^{(l-1)}}{1-\cfrac{Q_2^{(l-1)}}{1}}}}$$

where the division is here a division of continuous functions. Analogously we can see that $P_{[l,2]}(x)\cdot Q_{[l,2]}^{-1}(x)$ is also the 5th convergent of the continued fraction (2).

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